First Order Logic:
3. Normal Forms

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Equivalences of Quantified Formulae

change of variable

$$\underset{x}{\forall}\varphi \equiv \underset{y}{\forall}\varphi_{x\to y} \qquad \underset{x}{\exists}\varphi \equiv \underset{y}{\exists}\varphi_{x\to y}$$

deMorgan

$$\neg\underset{x}{\forall}\varphi \equiv \underset{x}{\exists}\neg\varphi \qquad \neg\underset{x}{\exists}x\varphi \equiv \underset{x}{\forall}\neg\varphi$$

commuting

$$(\underset{x}{\forall}\varphi)\wedge(\underset{x}{\forall}\psi) \equiv \underset{x}{\forall}(\varphi\wedge\psi) \qquad (\underset{x}{\exists}\varphi)\vee(\underset{x}{\exists}\psi) \equiv \underset{x}{\exists}(\varphi\vee\psi)$$

$$(\underset{x}{\forall}\varphi)\vee(\underset{x}{\forall}\psi) \not\equiv \underset{x}{\forall}(\varphi\vee\psi) \qquad (\underset{x}{\exists}\varphi)\wedge(\underset{x}{\exists}\psi) \not\equiv \underset{x}{\exists}(\varphi\wedge\psi)$$

$$\Rightarrow \not\Leftarrow \qquad \Leftarrow \not\Rightarrow$$

if x does not occur in ψ

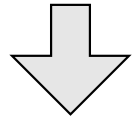
$$(\underset{x}{\forall}\varphi)\vee\psi \equiv \underset{x}{\forall}(\varphi\vee\psi) \qquad (\underset{x}{\exists}\varphi)\wedge\psi \equiv \underset{x}{\exists}(\varphi\wedge\psi)$$

Normal Forms

original formula

Has arbitrary structure and all logical symbols.

$(\ldots \wedge \ldots \forall \ldots) \Rightarrow \ldots \exists \ldots \neg(\ldots \Leftrightarrow \ldots (\ldots \vee \ldots)) \ldots$



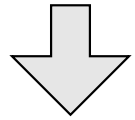
replace $\Rightarrow, \Leftrightarrow$

sink $\neg$ by deMorgan, etc.

formula in NNF: negation normal form

$\neg$ occurs in literals only, no $\Rightarrow, \Leftrightarrow$.

$(\ldots \wedge \overline{P}[\ldots] \ldots \forall \ldots) \vee \ldots \exists \ldots (\ldots \wedge \ldots (\ldots \overline{Q}[\ldots] \vee \ldots)) \ldots$

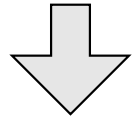


float $\forall, \exists$

formula in PNF: prenex normal form

All quantifiers at the beginning, the rest is the matrix of the formula.

$\forall\forall\exists\forall\exists\exists \ldots ((\ldots \wedge \overline{P}[\ldots] \ldots) \vee \ldots\ldots (\ldots \wedge \ldots (\ldots \overline{Q}[\ldots] \vee \ldots)) \ldots)$



distributivity of $\vee$ over $\wedge$

formula in PNF and CNF: conjunctive normal form

The matrix is a conjuctive set of clauses (disjunctive sets of literals).

$\forall\forall\exists\forall\exists\exists \ldots ((\ldots \wedge \overline{P}[\ldots] \wedge \ldots) \ldots \wedge \ldots (\ldots \vee \overline{Q}[\ldots] \vee \ldots))$

Alternative: PNF and DNF: disjunctive normal form

Normal Forms: Example Transformation

$$\underset{x}{\forall}((R[x] \wedge \neg Q[x]) \Rightarrow \underset{y}{\exists}(T[x,y] \wedge S[y]))$$

$$\underset{x}{\forall}(\neg(R[x] \wedge \neg Q[x]) \vee \underset{y}{\exists}(T[x,y] \wedge S[y]))$$

$$\underset{x}{\forall}((\overline{R}[x] \vee Q[x]) \vee \underset{y}{\exists}(T[x,y] \wedge S[y]))$$

$$\underset{x}{\forall}\ \underset{y}{\exists}(\overline{R}[x] \vee Q[x] \vee (T[x,y] \wedge S[y]))$$

$$\underset{x}{\forall}\ \underset{y}{\exists}((\overline{R}[x] \vee Q[x] \vee T[x,y]) \wedge (\overline{R}[x] \vee Q[x] \vee S[y]))$$

Normal Forms: Example Proof

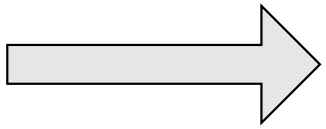
Proof of equivalence using conjunctive normal form.

$$
\begin{array}{lcl}
\left(\underset{x}{\forall} P[x]\right) \Rightarrow Q & \equiv & \underset{x}{\exists}\,(P[x] \Rightarrow Q) \\
\neg\left(\underset{x}{\forall} P[x]\right) \vee Q & & \underset{x}{\exists}\,(\neg P[x] \vee Q) \\
\left(\underset{x}{\exists} \neg P[x]\right) \vee Q & & \underset{x}{\exists}\,\left(\overline{P}[x] \vee Q\right) \\
\left(\underset{x}{\exists} \overline{P}[x]\right) \vee Q & & \\
\underset{x}{\exists}\,\left(\overline{P}[x] \vee Q\right) & &
\end{array}
$$

Skolem Transformation (New Constant)

"Skolemization": $\exists_x \varphi \longrightarrow \varphi_{x \to a}$ (a: constant, "new" – does not occur in φ)

Not equivalent transformation! $\exists_x \varphi \not\equiv \varphi_{x \to a}$ But preserves satisfiability.



$\exists_x P[x]$		$P[a]$
satisfiable	$\longleftarrow$ iff $\longrightarrow$	satisfiable
(assume)	proof $\longrightarrow$	(prove)
Exists I: $I \models \exists_x P[x]$		Exists I: $I \models P[a]$
Take I: $I \models \exists_x P[x]$		
$I = \begin{cases} D_I \\ P_I : D_I \to \{\mathbb{T}, \mathbb{F}\} \end{cases}$		
Exists $d \in D_I : P_I[d] = \mathbb{T}$		
Take $d \in D_I : P_I[d] = \mathbb{T}$	$\longrightarrow$	Construct J: $J \models P[a]$
		$J = \begin{cases} D_J = D_I \\ P_J = P_I : D_J \to \{\mathbb{T}, \mathbb{F}\} \\ a_J = d \end{cases}$
		Thus $P_J[a_J] = \mathbb{T}$.

proof $\longleftarrow$: suggested exercise

Skolem Transformation (New Function)

$\underset{x}{\forall}\,\underset{y}{\exists}\varphi \longrightarrow \underset{x}{\forall}\varphi_{y\to f[x]}$ (f: function, "new" – does not occur in φ)

Not equivalent transformation, but preserves satisfiability.

$\underset{x}{\forall}\,\underset{y}{\exists}P[x,y]$ satisfiable $\longleftarrow$ iff $\longrightarrow$ $\underset{x}{\forall}P[x,f[x]]$ satisfiable

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(prove) proof $\longleftarrow$ (assume)

Exists I: $I \models \underset{x}{\forall}\,\underset{y}{\exists}P[x,y]$

Exists I: $I \models \underset{x}{\forall}P[x,f[x]]$

Take I: $I \models \underset{x}{\forall}P[x,f[x]]$

$$I = \begin{cases} D_I \\ f_I : D_I \to D_I \\ P_I : D_I \to \{\mathbb{T},\mathbb{F}\} \end{cases}$$

For each $d \in D_I$: $P_I[d, f_I[d]]$

Construct J: $J \models \underset{x}{\forall}\,\underset{y}{\exists}P[x,y]$ $\longleftarrow$

$$J = \begin{cases} D_J = D_I \\ P_J = P_I : D_J \to \{\mathbb{T},\mathbb{F}\} \end{cases}$$

Prove: for each $d \in D_J$, exists $d' \in D_J$, such that $P_J[d,d'] = \mathbb{T}$

Take $d_0 \in D_J$ (arbitrary but fixed) , take witness for d' : $f_I[d_0]$, and obtain

$$P_J[d_0, f_J[d_0]] = P_I[d_0, f_J[d_0]] = \mathbb{T}.$$

proof $\longrightarrow$: suggested exercise

Skolem Normal Form

General Skolem transformation:

$$\underset{x_1}{\forall} \ldots \underset{x_n}{\forall} \underset{y}{\exists} \varphi \longrightarrow \underset{x_1}{\forall} \ldots \underset{x_n}{\forall} \varphi_{y \to f[x_1,\ldots,x_n]},$$

where f is a new function symbol (constant if $n = 0$).

The transformation preserves satisfiabiliy, as well as inconsistency.

By repeated application on a formula which is already in prenex normal form (PNF), all existential quantifiers are eliminated, thus we obtain a formula in Skolem normal form:

$$\underset{x_1}{\forall} \ldots \underset{x_n}{\forall} \varphi,$$

where φ is a quantifier-free formula.
If φ is in conjunctive normal form (CNF), then we have a formula in Skolem standard form. This is typically represented without any logical symbols:

- each clause is set of literals, implicitely a disjunction;
- the whole formula is a set of clauses, implicitely a conjunction;
- all variables are implicitely universally quantified (there must be a clear distinction between constants and variables).

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Skolem Standard Form: Examples 1, 2

$$\underset{x}{\forall}\ \underset{y}{\exists}\ ((\overline{R}[x] \vee Q[x] \vee T[x, \mathrm{y}]) \wedge (\overline{R}[x] \vee Q[x] \vee S[\mathrm{y}]))$$

$$\Downarrow\ y \to f[x]$$

$$\underset{x}{\forall}((\overline{R}[x] \vee Q[x] \vee T[x, \mathrm{f[x]}]) \wedge (\overline{R}[x] \vee Q[x] \vee S[\mathrm{f[x]}]))$$

$$\underset{x}{\forall}\ \underset{y}{\exists}\ \underset{z}{\exists}\ \Big((\neg P[x, \mathrm{y}] \ \wedge\ Q[x, \mathrm{z}]) \ \vee\ R[x, \mathrm{y}, \mathrm{z}]\Big)$$

$$\Downarrow\ y \to f[x], z \to g[x]$$

$$\underset{x}{\forall}\ \Big((\neg P[x, \mathrm{f[x]}] \ \wedge\ Q[x, \mathrm{g[x]}]) \ \vee\ R[x, \mathrm{f[x]}, \mathrm{g[x]}]\Big)$$

$$\Downarrow\ \text{distributivity}$$

$$\underset{x}{\forall}\ \Big((\overline{P}[x, \mathrm{f[x]}] \vee R[x, \mathrm{f[x]}, \mathrm{g[x]}]) \wedge (Q[x, \mathrm{g[x]}] \vee R[x, \mathrm{f[x]}, \mathrm{g[x]}])\Big)$$

Skolem Standard Form: Example 3

$$\underset{x}{\forall}\,\underset{y}{\exists}\left(\underset{z}{\exists}\left(P[x,z]\wedge P[y,z]\right)\;\Rightarrow\;\underset{u}{\exists}Q[x,y,u]\right)$$

$\Downarrow$ eliminate $\Rightarrow$

$$\underset{x}{\forall}\,\underset{y}{\exists}\left(\neg\underset{z}{\exists}\left(P[x,z]\wedge P[y,z]\right)\;\vee\;\underset{u}{\exists}Q[x,y,u]\right)$$

$\Downarrow$ sink $\neg$

$$\underset{x}{\forall}\,\underset{y}{\exists}\left(\underset{z}{\exists}\neg\left(P[x,z]\wedge P[y,z]\right)\;\vee\;\underset{u}{\exists}Q[x,y,u]\right)$$

$\Downarrow$ sink $\neg$

$$\underset{x}{\forall}\,\underset{y}{\exists}\left(\underset{z}{\forall}\left(\overline{P}[x,z]\vee\overline{P}[y,z]\right)\;\vee\;\underset{u}{\exists}Q[x,y,u]\right)$$

$\Downarrow$ prenex

$$\underset{x}{\forall}\,\underset{y}{\exists}\,\underset{z}{\forall}\,\underset{u}{\exists}\left(\overline{P}[x,z]\vee\overline{P}[y,z]\vee Q[x,y,u]\right)$$

$\Downarrow$ $y\to f[x],\;\; u\to g[x,z]$

$$\underset{x}{\forall}\,\underset{z}{\forall}\left(\overline{P}[x,z]\vee\overline{P}[f[x],z]\vee Q[x,f[x],g[x,z]]\right)$$

Skolem Standard Form: Representation

$$\underset{x}{\forall}((\overline{R}[x] \vee Q[x] \vee T[x, f[x]]) \wedge (\overline{R}[x] \vee Q[x] \vee S[f[x]]))$$
$$\{\{\overline{R}[x],\ Q[x],\ T[x, f[x]]\},\ \ \{\overline{R}[x],\ Q[x],\ S[f[x]]\}\}$$

$$\underset{x}{\forall}\ \left((\overline{P}[x, f[x]] \vee R[x, f[x], g[x]]) \wedge (Q[x, g[x]] \vee R[x, f[x], g[x]])\right)$$
$$\{\{\overline{P}[x, f[x]],\ R[x, f[x], g[x]]\},\ \ \{Q[x, g[x]],\ R[x, f[x], g[x]]\}\}$$

$$\underset{x}{\forall}\ \underset{z}{\forall}\left(\overline{P}[x, z] \vee \overline{P}[f[x], z] \vee Q[x, f[x], g[x, z]]\right)$$
$$\{\{\overline{P}[x, z],\ \overline{P}[f[x], z],\ Q[x, f[x], g[x, z]]\}\}$$

Refutation

The refutation method:

$$\varphi_1, \ldots, \varphi_n \models \psi \text{ iff } \wedge\{\varphi_1, \ldots, \varphi_n, \neg\psi\} \text{ is inconsistent.}$$

For transforming this formula in normal form, it is sufficient to transform each individual conjunct.

Example:

$$\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4 \models \psi$$

$$\varphi_1 : \quad \underset{x}{\forall}\,\underset{y}{\forall}\,\underset{z}{\exists} P[x, y, z]$$

$$\varphi_2 : \quad \underset{x}{\forall}\,\underset{y}{\forall}\,\underset{z}{\forall}\,\underset{u}{\forall}\,\underset{v}{\forall}\,\underset{w}{\forall}(((P[x, y, u] \wedge P[y, z, v] \wedge P[u, z, w]) \Rightarrow P[x, v, w]) \wedge ((P[x, y, u] \wedge P[y, z, v] \wedge P[x, v, w]) \Rightarrow P[u, z, w]))$$

$$\varphi_3 : \quad \underset{x}{\forall}(P[x, e, x] \wedge P[e, x, x])$$

$$\varphi_4 : \quad \underset{x}{\forall}(P[x, i[x], e] \wedge P[i[x], x, e])$$

$$\psi : \quad \left(\underset{x}{\forall} P[x, x, e]\right) \Rightarrow \underset{u}{\forall}\,\underset{v}{\forall}\,\underset{w}{\forall}(P[u, v, w] \Rightarrow P[v, u, w])$$