Chapter 5: Recursion

In some of the following exercises, we use the function \( \text{Fin}(A) \) to denote the set of all finite subsets of set \( A \) and \( \text{Inf}(A) \) to denote the set of all infinite subsets of \( A \).

1. Consider the following grammar:

\[
N \in \text{Nat} \\
N ::= 0 \mid s(N)
\]

Intuitively, every value \( s^n(0) \in \text{Nat} \) (the \( n \)-fold application of constructor \( s \) to \( 0 \)) represents the natural number \( n \in \mathbb{N} \). Define by primitive (structural) recursion the function \( +: \text{Nat} \times \text{Nat} \to \text{Nat} \) such that \( m + n \) returns the “sum” of its arguments and the predicate \( \leq \subseteq \text{Nat} \times \text{Nat} \) that determines whether the first argument is “less than or equal” the second one. Based upon \( m + n \), define a corresponding “multiplication” function \( m \cdot n \) from which you then define a corresponding “power” function \( m^n \).

2. Consider your definition of \( m + n \) from Exercise 1. First prove by structural induction \( \forall n \in \text{Nat}. \ 0 + n = n + 0 \). Then prove \( \forall m \in \text{Nat}, n \in \text{Nat}. \ s(m + n) = m + s(n) \). Based upon these results, prove the commutativity property \( \forall m \in \text{Nat}, n \in \text{Nat}. \ m + n = n + m \).

3. Define by an inductive definition for finite sequences \( s, t \in \mathbb{N}^* \) of equal length the relation \( s \leq t \) that is true if every element of \( s \) is less than or equal the corresponding element of \( t \) (you may assume the existence of operations head and tail over finite sequences). Transform the definitions into the “rule-oriented” style. Verify that your definitions are indeed well-formed by checking that the defining formulas satisfy the syntactic criteria required for upward continuity.

4. Repeat Exercise 3 but for infinite sequences \( s, t \in \mathbb{N}^{\omega} \) by using coinductive definitions. Verify that your definitions are indeed well-formed by checking that the defining formulas satisfy the syntactic criteria required for downward continuity.

5. Define by an inductive definition the relation “\( A \) contains only prime numbers” for \( A \in \text{Fin}(\mathbb{N}) \). Define by a coinductive definition the same relation for \( A \in \text{Inf}(\mathbb{N}) \). Transform the definitions into the “rule-oriented” style. Verify that your definitions are indeed well-formed by checking that the defining formulas satisfy the syntactic criteria required for upward respectively downward continuity.

6. Define by an inductive definition the subset relation \( A \subseteq B \) for finite sets \( A \in \text{Fin}(\mathbb{N}) \) and \( B \in \text{Fin}(\mathbb{N}) \). Transform the definition into the “rule-oriented” style. Verify that your definition is indeed well-formed by checking that the defining formula satisfies the syntactic criteria required for upward continuity.
7. Define by a coinductive definition the subset relation \( A \subseteq B \) for infinite sets \( A \in \text{Inf}(\mathbb{N}) \) and \( B \in \text{Inf}(\mathbb{N}) \). Transform the definition into the “rule-oriented” style. Verify that your definition is indeed well-formed by checking that the defining formula satisfies the syntactic criteria required for downward continuity.

8. Define by an inductive function definition the union function \( A \cup B \) for finite sets \( A \in \text{Fin}(\mathbb{N}) \) and \( B \in \text{Fin}(\mathbb{N}) \). Verify that your definition is indeed well-formed by checking that the defining formula satisfies the syntactic criteria required for upward continuity.

9. Define by a coinductive function definition the union function \( A \cup B \) for infinite sets \( A \in \text{Inf}(\mathbb{N}) \) and \( B \in \text{Inf}(\mathbb{N}) \) (please note that this definition must iterate over “both” sets in order to add elements from both sets to the result). Verify that your definition is indeed well-formed by checking that the defining formula satisfies the syntactic criteria required for downward continuity.

10. Repeat Exercises 8 and 9 by defining the intersection function \( A \cap B \) over finite respectively infinite sets \( A \) and \( B \).

11. Let \( A \in \text{Fin}(\mathbb{N}) \) and \( n \in \mathbb{N} \). Introduce by a (possibly rule-oriented) inductive definition over \( A \) a predicate \( A \geq n \) that states “every element of \( A \) is greater than or equal \( n \)”. Prove by the principle of induction for properties defined as least fixed points (applied to the unary relation \( \cdot \geq n \)) that \( A \geq n \Rightarrow (\forall a \in A. a \geq n) \) holds.

12. Repeat Exercise 11 but apply the proof principle of “fixed point induction”. To justify the application of this principle, verify that the formula defining \( A \geq n \) indeed satisfies the syntactic constraints of an “inclusive formula”.

13. Let \( A \) be an infinite subset of \( \mathbb{N} \) and \( n \in \mathbb{N} \). Introduce by a (possibly rule-oriented) coinductive definition over \( A \) a predicate \( A \geq n \) that states “every element of \( A \) is greater than or equal \( n \)”. Furthermore, define by coinduction a function \( A + n \) whose result is the set derived from \( A \) by adding \( n \) to each of its elements. Prove by the principle of coinduction for properties defined as greatest fixed points (applied to the unary relation \( \cdot \geq n \)) that \( A + n \geq n \) holds.

14. Let \( A \in \text{Inf}(\mathbb{N}) \). Consider the operation \( A + n \) introduced in Exercise 13. Prove by the principle of coinduction that \( A + 0 \sim A \) holds (here \( \sim \) is the bisimilarity relation introduced by the definition of \( A + n \)).