Thinking Programs: Exercises

Chapter 4: Building Models

Please ensure that your definitions of constants, functions, and numbers are indeed well-formed according to the criteria stated in this book.

1. Formally define the following notions (constants, functions, predicates) over the set \( \mathbb{N} = \{0, 1, 2, \ldots\} \) of natural numbers:

   - Number \( m \) divides \( n \) if \( m \) multiplied with \( p \) gives \( n \), for some number \( p \). Numbers \( q \) and \( r \) are the quotient and remainder of \( m \) divided by \( n \) if \( m = n \cdot q + r \) and \( r < n \). The quotient \( q \) of \( m \) divided by \( n \) is that number \( q \) such that \( q \) and \( r \) are the quotient and remainder of \( m \) divided by \( n \) for some number \( r \). The remainder \( r \) of \( m \) divided by \( n \) is that number \( r \) such that such that \( q \) and \( r \) are the quotient and remainder of \( m \) divided by \( n \) for some number \( q \).

   - The predecessor of \( n \) is that number \( m \) such that its successor \( m + 1 \) is \( n \) (if such a number exists). The natural difference of \( m \) and \( n \) is that number \( d \) that added to \( n \) yields \( m \) (if such a number exists). The natural square root of \( n \) is the largest natural number whose square is less than equal \( n \) (which always exists).

   - Number \( p \) is a prime number if it is greater equal 2 and the only divisors of \( p \) are 1 and \( p \) itself. \( p \) is a prime factor of \( n \) if \( p \) is prime and divides \( n \). The prime factor set of \( n \) is the set of all prime factors of \( n \). The prime set up to \( n \) is the set of all prime numbers less than equal \( n \). The next prime number after \( n \) is the smallest prime number greater equal \( n \). Numbers \( m \) and \( n \) are relatively prime if they have no common prime factors. The prime factorization of \( n \) is that (non-strictly) increasing sequence of prime numbers whose product is \( n \).

   - The decimal representation of \( n \) is the shortest sequence of decimal digits that in the decimal number system represents \( n \). The value of a sequence \( s \) of numerical digits in base \( b \) is the number denoted by \( s \) in the positional number system with that base (lookup the detailed mathematical definitions in the literature).

2. Let \( S \) be some finite set. We define a set \( \text{Graph} \) of labeled tuples as follows:

   \[
   \text{Graph} := \{ \langle v, e \rangle \mid V \subseteq S \land E \subseteq V \times V \}
   \]

   We call every element \( G \in \text{Graph} \) a (directed) graph with vertices \( G.v \) and edges \( G.e \). Based on this, formalize the following definitions of some basic notions (constants, functions, predicates) in the theory of directed graphs:

   - Let \( \text{empty} \) be that graph that has no vertices and no edges. The size of a graph \( G \) is the sum of the number of its vertices and the number of its edges. \( G \) is symmetric if it contains for every edge also the corresponding inverse edge.
3. Formally define the following notions (constants, functions, predicates) over the theory of (set-theoretic) functions (in these definitions you may omit all type signatures):

- In graph $G$, a vertex $v_1$ is directly connected to vertex $v_2$ if there is in $G$ an edge from $v_1$ to $v_2$. Vertices $v_1$ and $v_2$ are adjacent, if $v_1$ is directly connected to $v_2$ or $v_2$ is directly connected to $v_1$.

- In graph $G$, the successor set of vertex $v$ is the set of all vertices to which $v$ is directly connected. The outdegree of $v$ is the size of the successor set of $v$. The predecessor set of $v$ is the set of all vertices that are directly connected to $v$. The indegree of $v$ is the size of the predecessor set of $v$. The neighbor set of $v$ is the set of all vertices adjacent to $v$. The degree of $v$ is the size of its neighbor set (or, equivalently the sum of its indegree and its outdegree). Graph $g$ is regular if all its nodes have the same degree.

- A sequence $p \in S^*$ is a path in $G$, if every element in $p$ is a vertex of $G$ and every successive pair of elements in $p$ is connected by an edge of $G$. $p$ is a path from $v_1$ to $v_2$ if $p$ is a path whose first element is $v_1$ and whose last element is $v_2$. In graph $G$, a vertex $v_1$ is connected to a vertex $v_2$ if there is some path in $G$ from $v_1$ to $v_2$. Graph $G$ is (strongly) connected if all pairs of nodes in $G$ are connected.

- In graph $G$, sequence $p$ is a cycle if $p$ is a path in $G$ that has at least two vertices where its first and its last vertex are the same. Graph $G$ is acyclic if it does not contain any cycles. Graph $G$ is a tree if it is connected and acyclic.

3. Formally define the following notions (constants, functions, predicates) over the theory of (set-theoretic) functions (in these definitions you may omit all type signatures):

- $f$ is a partial function from $A$ to $B$ (written as $f : A \rightarrow B$) if $f$ is a subset of $A \times B$ that does not contain two different tuples with the same first component. The domain of $f$ is the set with every value $a$ such that $f$ contains a tuple with $a$ as its first component. The range of $f$ is the set with every value $b$ such that $f$ contains a tuple with $b$ as the second component. apply$(f, a)$ is that value $b$ such that the tuple with first component $a$ and second component $b$ is in $f$ (if such a tuple exists). $f$ is a total function from $A$ to $B$ (written as $f : A \rightarrow B$) if $f$ is a partial function from $A$ to $B$ whose domain is $A$.

- $f$ is injective from $A$ to $B$ if it is a total function from $A$ to $B$ such that $f$ does not contain two tuples with the same second argument. $f$ is surjective from $A$ to $B$ if it is a total function from $A$ to $B$ such that $f$ contains for every element in $B$ a tuple with that element as second component. $f$ is bijective from $A$ to $B$ if it is both injective and surjective from $A$ to $B$.

- $f$ is an identity if it contains for every value $a$ in its domain a tuple with first component $a$ and also second component $a$. $f$ is the composition of $g$ and $h$ if $f$ contains a tuple $(a, b)$ if and only if $g$ contains a tuple $(a, c)$ and $h$ contains a tuple $(c, b)$ for some value $b$. The composition of $g$ and of $h$ is that function $f$ that satisfies above criterion.

- The image under $f$ of a set $A$ is that set that contains every value $b$ such that $f$ contains a tuple with first component $a$ and second component $b$ for some value $a$ in $A$. The preimage under $f$ of a set $B$ is that set that contains every value $a$ such that $f$ contains a tuple with first component $a$ and second component $b$ for some value $b$ in $B$. 