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Linz, August 2012

Clemens G. Raab

Kurzzusammenfassung

Das Ziel dieser Doktorarbeit ist die Weiterentwicklung von Computeralgebramethoden zur Berechnung von definiten Integralen. Eine Art den Wert eines definiten Integrals zu berechnen führt über das Auswerten einer Stammfunktion des Integranden. Im neunzehnten Jahrhundert war Joseph Liouville einer der ersten die die Struktur elementarer Stammfunktionen von elementaren Funktionen untersuchten. Im frühen zwanzigsten Jahrhundert wurden Differentialkörper als algebraische Strukturen zur Modellierung der differentiellen Eigenschaften von Funktionen eingeführt. Mit deren Hilfe hat Robert H. Risch im Jahr 1969 einen vollständigen Algorithmus für transzendente elementare Integranden veröffentlicht. Seither wurde dieses Resultat von Michael F. Singer, Manuel Bronstein und einigen anderen auf bestimmte andere Klassen von Integranden erweitert. Andererseits können, für den Fall dass keine Stammfunktion in geeigneter Form verfügbar ist, basierend auf dem Prinzip der parametrischen Integration (oft *creative telescoping* genannt) lineare Relationen gefunden werden, welche vom Parameterintegral erfüllt werden.

Das Hauptresultat dieser Doktorarbeit erweitert das oben Erwähnte zu einem vollständigen Algorithmus für elementare parametrische Integration einer bestimmten Funktionenklasse, welche den Großteil der in der Praxis auftretenden speziellen Funktionen abdeckt, z.B. orthogonale Polynome, Polylogarithmen, Besselfunktionen, etc. Es wird auch eine Methode zur Modellierung dieser Funktionen mittels geeigneter Differentialkörper angegeben. Für Liouville'sche Integranden weist dieser Algorithmus eine deutlich verbesserte Effizienz gegenüber dem entsprechenden von Singer et al. 1985 präsentierten Algorithmus auf. Zusätzlich wird auch eine Verallgemeinerung von Czichowskis Algorithmus zur Berechnung des logarithmischen Teils des Integrals dargelegt. Überdies werden auch teilweise Erweiterungen des Integrationsalgorithmus auf weitere Funktionen behandelt.

Als Teilprobleme des Integrationsalgorithmus müssen auch Lösungen bestimmten Typs von linearen gewöhnlichen Differentialgleichungen gefunden werden. Auch hierzu werden Beiträge geleistet, wobei jene die sich mit der direkten Lösung von Differentialgleichungssystemen befassen auf eine Zusammenarbeit mit Moulay A. Barkatou zurückgehen.

Für Liouville'sche Integranden wurde der Algorithmus in Form des *Mathematica*-Pakets `Integrator` implementiert. Teile davon können auch mit allgemeineren Funktionen umgehen. Diese Methoden können auf einen Großteil der indefiniten wie definiten Integrale aus Integraltafeln angewandt werden. Zusätzlich wurden mit dem Paket auch interessante Integrale erfolgreich behandelt, die nicht in Tabellen aufscheinen bzw. bei welchen derzeitige Computeralgebrasysteme wie *Mathematica* oder *Maple* nicht zum Ziel führen. Außerdem zeigen wir wie Parameterintegrale aus der Arbeit anderer Forscher mit dem Paket gelöst werden, z.B. ein Integral aus der Untersuchung der Entropie bestimmter Prozesse.

Abstract

The general goal of this thesis is to investigate and develop computer algebra tools for the simplification resp. evaluation of definite integrals. One way of finding the value of a definite integral is via the evaluation of an antiderivative of the integrand. In the nineteenth century Joseph Liouville was among the first who analyzed the structure of elementary antiderivatives of elementary functions systematically. In the early twentieth century the algebraic structure of differential fields was introduced for modeling the differential properties of functions. Using this framework Robert H. Risch published a complete algorithm for transcendental elementary integrands in 1969. Since then this result has been extended to certain other classes of integrands as well by Michael F. Singer, Manuel Bronstein, and several others. On the other hand, if no antiderivative of suitable form is available, then linear relations that are satisfied by the parameter integral of interest may be found based on the principle of parametric integration (often called *differentiating under the integral sign* or *creative telescoping*).

The main result of this thesis extends the results mentioned above to a complete algorithm for parametric elementary integration for a certain class of integrands covering a majority of the special functions appearing in practice such as orthogonal polynomials, polylogarithms, Bessel functions, etc. A general framework is provided to model those functions in terms of suitable differential fields. If the integrand is Liouvillian, then the present algorithm considerably improves the efficiency of the corresponding algorithm given by Singer et al. in 1985. Additionally, a generalization of Czichowski's algorithm for computing the logarithmic part of the integral is presented. Moreover, also partial generalizations to include other types of integrands are treated.

As subproblems of the integration algorithm one also has to find solutions of linear ordinary differential equations of a certain type. Some contributions are also made to solve those problems in our setting, where the results directly dealing with systems of differential equations have been joint work with Moulay A. Barkatou.

For the case of Liouvillian integrands we implemented the algorithm in form of our *Mathematica* package `Integrator`. Parts of the implementation also deal with more general functions. Our procedures can be applied to a significant amount of the entries in integral tables, both indefinite and definite integrals. In addition, our procedures have been successfully applied to interesting examples of integrals that do not appear in these tables or for which current standard computer algebra systems like *Mathematica* or *Maple* do not succeed. We also give examples of how parameter integrals coming from the work of other researchers can be solved with the software, e.g., an integral arising in analyzing the entropy of certain processes.

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Manuel Kauers deserves special thanks for many helpful discussions where he gave an overview of existing algorithms in the context of this thesis and suggested some problems to look at. He also provided me with his experimental implementation of the procedures given in Bronstein's book [Bro], which our package `Integrator` now is based on.

Being in the DK also provided me with many opportunities (along with generous financial support) to attend international conferences, get in touch with many researchers, and spend several months abroad, which I am very grateful for. In particular I profited a lot from the extensive research stays where I learned a lot each time. The following saying certainly applies: "We are like dwarfs standing on the shoulders of giants."

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Contents

Chapter 1

Introduction

Integration of functions can be done in two variants: indefinite and definite integration, which are closely related via the fundamental theorem of calculus. On the one hand, an indefinite integrals still is a function in the variable of integration and is nothing else than the antiderivative of a given function $f(x)$. On the other hand, a definite integral is the value

$$\int_a^b f(x)\,dx$$

resulting from integrating the function $f(x)$ over the given interval (a, b). Another difference between the two is that in general it is easy to verify an indefinite integral just by differentiating it, whereas in general it is hard to verify the result of a definite integral without recomputing it.

In earlier times large tables of integrals were compiled by hand. Nowadays, computer algebra tools play an important role in the evaluation of definite integrals and we will mention some approaches below. Tables of integrals are even used in modern software as well. Algorithms for symbolic integration in general proceed in three steps. First, in computer algebra the functions typically are modeled by algebraic structures. Then, the computations are done in the algebraic framework and, finally, the result needs to be interpreted in terms of functions again. Some considerations concerning the first step, i.e., algebraic representation of functions, will be part of Chapter 2 and we also refer to the appendix for this purpose. A brief overview of some approaches and corresponding algorithms will be given below in this chapter. In the thesis we will focus entirely on the approach using differential fields and Manuel Bronstein's book on symbolic integration [Bro] will be our main reference. Algorithms for this setting will be presented in Chapters 3 and 4. The subtle issues of the last step, i.e., translating the algebraic result to a valid statement in the world of functions, will not be dealt with in detail. A more detailed overview of the contents of this thesis can be found at the end of this chapter.

Definite integration

For the evaluation of definite integrals many tools may be applied to transform them to simpler integrals which are known or can be evaluated easily: change of variable, series expansion of the integrand, integral transforms, etc. As mentioned above by the fundamental theorem of calculus it is obvious that we can use indefinite integrals for the evaluation of definite integrals. It is well known that for a function $g(x)$ with $g'(x) = f(x)$ we have

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

This fact has also been exploited in order to evaluate definite integrals for which a corresponding indefinite integral is not available in nice form. We give an overview of this method, which will be the main focus for computing definite integrals in this thesis. If the integral depends on a parameter, we can differentiate the parameter integral with respect to this parameter and obtain an integral that might be evaluated more easily. Under suitable assumptions on the integrand we have

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{df}{dy}(x,y)\,dx,$$

which is called *differentiating under the integral sign*. A related paradigm, known as *creative telescoping*, is used in symbolic summation to compute recurrences for parameter dependent sums. Based on these two principles Almkvist and Zeilberger [AZ90] were the first to propose a completely systematic way for treating parameter integrals by differentiating under the integral sign by giving an algorithm to compute differential equations for parameter integrals with holonomic integrands. They gave a fast variant of it for hyperexponential integrands, which may also be used for computing recurrences for such parameter integrals. From a very general point of view the underlying principle might be understood as combination of the fundamental theorem of calculus and the linearity of the integral in the following way. If for integrable functions $f_0(x), \dots, f_m(x)$ and constants $c_0, \dots, c_m$ the function $g(x)$ is an antiderivative such that

$$c_0 f_0(x) + \dots + c_m f_m(x) = g'(x),$$

then we can deduce the relation

$$c_0 \int_a^b f_0(x)\,dx + \dots + c_m \int_a^b f_m(x)\,dx = g(b) - g(a)$$

among the definite integrals $\int_a^b f_i(x)\,dx$ provided they exist. Both the functions $f_i(x)$ and the constants c_i may depend on additional parameters, which are not shown here. In order that this works the important point is that the c_i do not depend on the variable of integration. In general, the functions $f_i(x)$ are chosen to be derivatives or shifts in the parameter(s) of the integrand $f(x)$ if we are interested in differential equations or recurrences for the definite integral.

The main task for finding such relations of definite integrals of given functions $f_i(x)$ consists in finding suitable choices for the constants c_i which allow a closed form of the antiderivative $g(x)$ to be computed. We will call this *parametric integration* as it can be viewed as making suitable choices for the parameters c_i occurring in the combined integrand $c_0 f_0(x) + \dots + c_m f_m(x)$. Sometimes the wording *creative telescoping* is used even in the integration context as well since the two concepts are completely analogous.

The approach above also addresses the issue of verifiability. When given such a linear relation of integrals

$$c_0 \int_a^b f_0(x)\,dx + \cdots + c_m \int_a^b f_m(x)\,dx = r$$

the function $g(x)$ may act as a proof certificate of it as we just need to verify

$$c_0 f_0(x) + \cdots + c_m f_m(x) = g'(x) \qquad \text{and} \qquad r = g(b) - g(a),$$

where the left hand sides are directly read off from the integral relation we want to verify.

Symbolic integration

Algorithms to compute indefinite integrals of rational integrands are known for a long time already and many other integrals were computed analytically by hand as mentioned above. Especially in the last century algorithms have been developed capable of dealing with more general classes of integrands in a completely systematic way. In the following we want to give an overview of three different approaches that were taken. We also mention some relevant cornerstones but do not aim at a fully comprehensive survey of the corresponding literature, many other contributions were made. Note that all of those approaches extend to definite integration in one way or the other.

The differential algebra approach represents functions as elements of differential fields and differential rings. These are algebraic structures not only capturing the arithmetic properties of functions but also their differential properties by including derivation as an additional unary operation. In general terms, starting with a prescribed differential field one is interested in indefinite integrals in the same field or in extensions of that field constructed in a certain way. Based on a book by Joseph F. Ritt [Rit] using differential fields Robert H. Risch gave a decision procedure [Ris69, Bro90b] for computing elementary integrals of elementary functions by closely investigating the structure of the derivatives of such functions. Since then this result has been extended in various directions. A parametric version was discussed in [Mac76]. Michael F. Singer et al. generalized this to a parametric algorithm computing elementary integrals over regular Liouvillian fields in the appendix of [SSC85] and Manuel Bronstein gave partial results for more general differential fields constructed by monomials [Bro90a, Bro]. This thesis can be seen as a continuation of this line of research. In [NM77] Arthur C. Norman initiated a variant of Risch's algorithm avoiding its recursive structure, which therefore is sometimes also called the parallel Risch algorithm. The Risch-Norman algorithm can be used in even more general differential fields and has proven to be a rather powerful heuristic in practice, see [Bro, Bro07, Boe10] and references therein. Most results mentioned so far restrict to the case where the generators of the differential fields are algebraically independent. The presence of algebraic relations causes new situations and requires more involved algebraic tools, see [Bro90b, Bro98, Kau08, Boe10] and references therein. Another type of generalization is to search also for certain types of non-elementary integrals over certain differential fields. Some results for this problem have been achieved in [SSC85], see also [Bad06] and the references to the work of Cherry and Knowles in [Bro].

Indefinite integrals of products of special functions that satisfy homogeneous second-order differential equations were considered by Jean C. Piquette. His ansatz for the integral in terms of linear combinations of such products led to a differential system, which after uncoupling he solved by heuristic methods, see [Piq91] and references therein. The holonomic systems approach was initiated by Doron Zeilberger in [Zei90] and puts this on more general and more algorithmic grounds. Functions are represented by the differential and difference operators that annihilate them. The notion of D-finite functions is closely related and refers to functions satisfying homogeneous linear differential equations with rational functions as coefficients. Hence, the derivatives of a D-finite function generate a finite-dimensional vector space over the rational functions. Frédéric Chyzak [Chy00] presented an efficient algorithm for computing indefinite integrals of such functions in the same vector space. The algorithm handles also parametric integration and summation and utilizes Ore algebras to represent the operators corresponding to functions. For extensions and improvements see [CKS09, Kou09].

The rule-based approach operates on the syntactic presentation of the integral by a table of transformation rules. This comes close to what is done when integrating by hand based on integral tables such as [GH, GR]. Also most computer algebra systems make at least partial use of transformations and table look-up. These tables may contain rules for virtually any special function, which makes such algorithms easily extensible in principle. This approach is recently being investigated systematically by Albert D. Rich and David J. Jeffrey [RJ09], who point out several subtle issues related to efficiency.

Structure of the thesis

In Chapter 2 we introduce the reader to differential fields and order functions on them, which are the essential tools for the algorithms later. Some of the basic computations that need to be performed during the execution of the integration algorithm are briefly discussed in Section 2.4. Section 2.6 presents the relevant classes of functions we consider for our algorithm and provides a very general framework of representing a multitude of special functions in terms of differential fields, which has not been considered that way before to our knowledge. Auxiliary definitions and results are provided in Section 2.5. Liouville's theorem on the structure of elementary integrals is discussed in Section 2.7 where we also give new refinements of it, which are corrected versions of some statements from [Bro]. After that, we collect some basic results on Gröbner bases, which will be needed at the end of Section 3.2.

In Chapter 3 we state the problem of parametric elementary integration and provide an algorithm for solving it, which is summarized in Theorem 3.4 and follows the basic ideas and the recursive structure of Risch's algorithm. For this purpose we also identify a large class of differential fields for which the algorithm can be proven to be complete, considerably extending the class of fields for which corresponding results are available in current literature. The description and proof of the algorithm is spread across Sections 3.1 through 3.4 and contains many new contributions, most importantly Theorems 3.9 and 3.15. The detailed steps of the algorithm can be readily extracted from the first half of the proofs of the theorems in these sections. During the algorithm several subproblems need to be solved, some of them are deferred to Chapter 4. In Section 3.5 we discuss extensions of the algorithm to other differential fields. The chapter concludes with a section on examples.

In Chapter 4 we discuss several algorithms needed for solving the differential equations that arise in Chapter 3. Most importantly we need to solve the parametric Risch differential equation discussed in Section 4.1 where we mostly refer to the results presented in [Bro]. In order to make the algorithm as complete as possible for the differential fields considered we also need to incorporate the algorithms discussed in Sections 4.2 and 4.3 where we make some contributions to extend the existing algorithms. In Section 4.2 we outline an algorithm for solving linear ODEs in their coefficient field, which is mainly based on the results presented in [Sin91, Bro92]. We consider the parametric logarithmic derivative problem in Section 4.3.1 and hyperexponential solutions of ODEs in Section 4.3.2. Section 4.3.3 mentions several possibilities to reduce systems of differential equations to scalar ODEs and Section 4.4 presents results which avoid this uncoupling and allow to solve the systems directly in some cases. Most of the results in Section 4.4 are joint work with Moulay A. Barkatou, see also [BR12].

In Chapter 5 we discuss how parametric elementary integration can be applied to compute definite integrals which depend on parameters. Among others many examples from standard integral tables like [GH, GR] were verified this way, but also some non-obvious typos could be detected there. In addition, the implementation was used to evaluate interesting examples of integrals that do not appear in these tables or for which current standard computer algebra software like *Mathematica* or *Maple* do not succeed. Some examples are collected in Section 5.1 to highlight several aspects of the algorithm.

In Appendix A for the convenience of the reader we collect a majority of the common special functions and state their properties needed for representing them in terms of differential fields as explained in Sections 2.6 and 3.5. This also illustrates the broad applicability of the algorithm presented in the thesis.

Chapter 2

Prerequisites

For the convenience of the reader, in this chapter we summarize and discuss most of the notions used later together with some basic facts related to them. Most of the notions and results discussed here are well known in algebra or differential algebra and for further details we refer to [Kap, Bro, CLO] for example. The reader not so familiar with the topic will find Sections 2.2, 2.3, and 2.6 useful at the first reading.

One main contribution to this chapter is the identification of a rich class of special functions that can be represented by Bronstein's notion of monomial extensions in a somewhat flexible way. This class is made more explicit in Appendix A.2. In Section 2.6.2 we provide the general framework and analyze some theoretical aspects of the differential fields obtained. We observe that the properties of the differential fields can be checked by a simplified variant of an algorithm given by Ulmer and Weil [UW96]. At the end of Section 2.5 we observe that Theorem 2.44 can be proven by a result of Singer and Ulmer [SU93] instead of using a result of Hendriks and van der Put [HP95] as was done in [UW96].

Another contribution, which is essential for the integration algorithm, are some of the refined versions of Liouville's theorem given in Section 2.7 correcting statements from [Bro]. Another new refinement is given later by Theorem 3.25.

Note that all fields considered in this thesis are implicitly understood to be of characteristic zero.

2.1 Notation

The natural numbers are considered to satisfy $0 \in \mathbb{N}$ and we denote the positive natural numbers by $\mathbb{N}^+$. For a ring R we denote its group of units by R^* and the ideal generated by a set $S \subseteq R$ is denoted by $\langle S \rangle$. Let K be a field and let S be a set of K-vectors, then by $\operatorname{span}_K S$ we denote the K-vector space generated by S. The field generated by K and another field F is simply denoted by KF and $\overline{K}$ denotes the algebraic closure of K.

Generally we will denote vectors by boldface symbols and matrices by uppercase letters. Also every boldface symbol is a vector, but not every uppercase letter denotes a matrix.

A tuple of vectors is understood as a matrix constructed by those vectors as its columns. We also write linear combinations as products of vectors $(f_0, \dots, f_m) \cdot \mathbf{c} = \sum_{i=0}^{m} c_i f_i$.

Let K be a field and let z be transcendental over K. We will denote the coefficients of polynomials and Laurent polynomials $p \in K[z, \frac{1}{z}]$ by $\operatorname{coeff}(p, z^i)$, which is zero for $i > \deg_z(p)$ and for $i < -\nu_z(p)$. Furthermore, for polynomial division in $K[z]$ we write $a \div b$ for the quotient and $a \bmod b$ for the remainder. For a rational function $f \in K(z)$ we define the denominator of f w.r.t. z as the monic polynomial $\operatorname{den}_z(f) \in K[z]$ of smallest degree such that $\operatorname{den}_z(f)f \in K[z]$, where the numerator of f w.r.t. z is given by $\operatorname{num}_z(f) := \operatorname{den}_z(f)f$. This implies that $\operatorname{den}(0) = 1$ and $\gcd(\operatorname{num}(f), \operatorname{den}(f)) = 1$ as expected. For a polynomial $p \in K[z]$ we use the short notation

$$\sum_{p(\alpha)=0} := \sum_{\substack{\alpha \in \overline{K} \\ p(\alpha)=0}}$$

to write sums over all roots $\alpha \in \overline{K}$ of the polynomial.

All the following notions will be defined precisely in Section 2.3, for now we just give a summary of the notation. We will denote orders of rational functions f at p by $\nu_p(f)$. Intuitively, it can be thought of as the multiplicity of the zero of f at the roots of p with negative values corresponding to poles. For vectors $\mathbf{f}$ we define the order $\nu_p(\mathbf{f})$ as the minimum of the orders of the entries, likewise we define $\nu_p(A)$ for matrices. An order function ν_p has a canonical projection π_p associated to it, which can be thought of as the evaluation of $f \in K(z)$ at the roots of p. Furthermore,

$$\operatorname{llc}_p(f) := \pi_p(f p^{-\nu_p(f)})$$

denotes the local leading coefficient and it can intuitively be understood as the first nonzero coefficient of the p-adic expansion $f = \sum_{i=\nu_p(f)}^{\infty} f_i p^i$.

If D is a derivation on $K(z)$, then ω_p denotes the degree of D at p and $\operatorname{res}_p(f)$ is used to denote the residue of f at p. This should not be confused with the resultant $\operatorname{res}_z(a, b)$ of two polynomials $a, b \in K[z]$. For $b \in K^*$ we define the resultant $\operatorname{res}_z(0, b) := 1$ in order to simplify the statements, since then a vanishing resultant corresponds to common roots of polynomials.

2.2 Differential fields

Definition 2.1. *Let F a field and D a unary operation on it, which is additive and satisfies the product rule, i.e.,*

$$D(f + g) = Df + Dg \qquad \text{and} \qquad D(fg) = fDg + (Df)g.$$

Then D is called a derivation *on F and (F, D) is called a* differential field. *The set of* constants *is denoted by* $\operatorname{Const}_D(F) := \{f \in F \mid Df = 0\}$.

It easily follows from the definition that $\operatorname{Const}_D(F)$ is a differential subfield of F and that D is $\operatorname{Const}_D(F)$-linear. Furthermore, any derivation D also obeys the quotient rule and the logarithmic derivative identity

$$\frac{D(f^n g^m)}{f^n g^m} = n\frac{Df}{f} + m\frac{Dg}{g} \tag{2.1}$$

as well as $Df^n = nf^{n-1}Df$ for $n, m \in \mathbb{Z}$.

Definition 2.2. *Let (F, D) be a differential field and $f \in F$. Then we say that*

1. *f is the* logarithmic derivative *of an element of F if there exists $g \in F^*$ such that $\frac{Dg}{g} = f$, or*
2. *f is the* logarithmic derivative of an F-radical *if there are $g \in F^*$ and $k \in \mathbb{Z} \setminus \{0\}$ such that $\frac{Dg}{kg} = f$.*

For the polynomials over a differential field (F, D) we define the coefficient lifting κ_D of the derivation, which gives a derivation on the rational functions $F(z)$ as well.

Definition 2.3. *Let (F, D) be a differential field and let z be an indeterminate over F. On $F[z]$ we define $\kappa_D : F[z] \to F[z]$ by*

$$\kappa_D \sum_{i=0}^{d} a_i z^i := \sum_{i=0}^{d} (Da_i) z^i.$$

With this definition we can formulate the following variant of the chain rule.

Lemma 2.4. *([Bro, Lemma 3.2.2]) Let (F, D) be a differential field and let z be an indeterminate over F. Let $p \in F[z]$ and $f \in F$ then*

$$D(p(f)) = (\kappa_D p)(f) + \frac{dp}{dz}(f)Df.$$

Consider a differential subfield (K, D) of (F, D) and an element $t \in F$. In general the field extension $K(t)$ of K need not be a differential field again as Dt need not be expressible as an element of $K(t)$. The differential field generated by K and t is given by $K(t, Dt, D^2t, \dots)$ instead, which as a field extension may or may not be finitely generated. We will work a lot with differential field extensions that are generated by adjoining one element in a way such that the field $K(t)$ is a differential field extension of K, i.e., $K(t)$ is closed under D. The following theorem makes the choice explicit which we have when extending the derivation from (K, D) to a differential field extension $(K(t), D)$.

Theorem 2.5. *([Bro, Theorems 3.2.2, 3.2.3]) Let (K, D) be a differential field and let $K(t)$ be the field generated by a new element t.*

1. *If t is transcendental over K, then, for any $w \in K(t)$, D can be uniquely extended to a derivation on $K(t)$ such that $Dt = w$.*
2. *If t is algebraic over K, then D can be uniquely extended to a derivation on $K(t)$. Moreover, if $p \in K[z]$ is such that $p(t) = 0$, then $Dt = -\frac{(\kappa_D p)(t)}{\frac{dp}{dz}(t)}$.*

Theorem 2.6. *([Bro, Theorem 3.2.4]) Let (K, D) be a differential field and let F be an algebraic extension of K. Then, any field automorphism of F over K, i.e., leaving all elements of K fixed, commutes with D. In particular, if F is finitely generated over K, then the trace $\mathrm{Tr} : F \to K$ commutes with D and $\mathrm{Tr}(\frac{Dg}{g}) = \frac{DN(g)}{N(g)}$, where $N : F \to K$ is the norm.*

In particular, the last formula of the theorem implies that if $f \in K$ is the logarithmic derivative of an F-radical, then it is also the logarithmic derivative of a K-radical. In Lemma 3.4.8 of [Bro] a different proof of this fact is provided.

In our considerations and algorithms we mostly will focus on finitely generated differential fields $F = C(t_1, \ldots, t_n)$ with a tower structure $Dt_i \in C(t_1, \ldots, t_{i-1})[t_i]$ in the spirit of [Bro90a]. So the notion of a (differential) monomial is very important to us.

Definition 2.7. *Let (F, D) be a differential field, K a differential subfield, and $t \in F$. Then t is called a* monomial *over (K, D) if*

1. *t is transcendental over K and*
2. *$Dt \in K[t]$.*

If $\deg_t(Dt) \geq 2$ we call t nonlinear.

Later we will work with monomial extensions $(K(t), D)$ of (K, D) such that the field of constants is not extended, i.e., $\mathrm{Const}_D(K(t)) = \mathrm{Const}_D(K)$. A sufficient condition for this will be given in Lemma 2.18. We also need to discuss the notions of normal and special polynomials in this context.

Definition 2.8. *Let t be a monomial over (K, D). Then we call a polynomial $p \in K[t]$ with $\gcd(p, Dp) = 1$* normal *w.r.t. D and we call $f \in K(t)$* simple *w.r.t. D if $\mathrm{den}(f)$ is normal w.r.t. D.*

It can be shown that any factor of a normal polynomial is normal again and that the product of two relatively prime normal polynomials is normal as well. Moreover, normal polynomials are squarefree but the converse need not hold.

Definition 2.9. *Let t be a monomial over (K, D). Then we define the set of polynomials which are* special *w.r.t. D by*

$$S_{K[t]:K} := \{p \in K[t] \mid p|Dp\}$$

and we define the set of special monic irreducible polynomials as

$$S^{irr}_{K[t]:K} := \left\{p \in S_{K[t]:K} \;\middle|\; p \text{ irred. in } K[t], \mathrm{lc}_t(p) = 1\right\}.$$

We say that a special polynomial $p \in S_{K[t]:K}$ (or $p \in S^{irr}_{K[t]:K}$) is of the first kind *if for each $\alpha \in \overline{K}$ with $p(\alpha) = 0$ there are no $g \in K(\alpha)^*$ and $k \in \mathbb{Z} \setminus \{0\}$ such that $\left(\frac{Dt - D\alpha}{t - \alpha}\right)(\alpha) = \frac{Dg}{kg}$. Analogously, these polynomials are collected in the sets $S^1_{K[t]:K}$ and $S^{irr,1}_{K[t]:K}$ respectively. Furthermore, we define the set of elements of $K(t)$ which are* reduced *w.r.t. D by*

$$K(t)_{red} := \left\{\tfrac{a}{b} \;\middle|\; a, b \in K[t], b|Db\right\}.$$

Note that $K^* \subseteq S^1_{K[t]:K} \subseteq S_{K[t]:K}$. Furthermore, $S_{K[t]:K}$ as a multiplicative monoid is generated by K^* and $S^{irr}_{K[t]:K}$ and analogously $S^1_{K[t]:K}$ is generated by K^* and $S^{irr,1}_{K[t]:K}$. The significance of special polynomials being of the first kind consists in their property stated later in Theorem 2.16. We have the following characterization of special polynomials, from which can be seen that any factor of a special polynomial is special again and likewise for special polynomials of the first kind.

Theorem 2.10. *([Bro, Theorem 3.4.3]) Let t be a monomial over (K, D) and define the polynomial $q := Dt \in K[t]$. Let $p \in K[t] \setminus \{0\}$, then $p \in S_{K[t]:K}$ if and only if*

$$D\alpha = q(\alpha)$$

for all roots $\alpha \in \overline{K}$ of p.

From this theorem and Theorem 2.6 we obtain the following result characterizing the special polynomials over algebraic extensions of the coefficient field, cf. Corollary 3.4.1 and Theorem 3.4.4.iii in [Bro].

Corollary 2.11. *Let (K, D) be a differential field, let t be a monomial over (K, D), and let E be an algebraic extension of K. Then t is a monomial over (E, D). Furthermore,*

$$S^{irr}_{E[t]:E} = \{q \in E[t] \mid q \textit{ irred. in } E[t], \mathrm{lc}_t(q) = 1, \exists p \in S^{irr}_{K[t]:K} : q|p\}$$

and analogously

$$S^{irr,1}_{E[t]:E} = \{q \in E[t] \mid q \textit{ irred. in } E[t], \mathrm{lc}_t(q) = 1, \exists p \in S^{irr,1}_{K[t]:K} : q|p\}.$$

2.3 Orders and residues

Definition 2.12. *Let K be a field and let t be transcendental over K. For $p \in K[t] \setminus K$ squarefree and $f \in K(t)$ we define the* order of f at p *by*

$$\nu_p(f) := \sup\left\{\nu \in \mathbb{Z} \;\middle|\; \gcd(\mathrm{den}_t(fp^{-\nu}), p) = 1\right\}.$$

as well as the local ring at p *as $O_p := \{f \in K(t) \mid \nu_p(f) \geq 0\}$. Furthermore, we define the residue ring $K_p := K[t]/\langle p\rangle$ and the* canonical projection $\pi_p : O_p \to K_p$ *by*

$$\pi_p(f) := \mathrm{num}_t(f)b \bmod p$$

where $b \in K[t]$ is a modular inverse of $\mathrm{den}_t(f)$, i.e., $\mathrm{den}_t(f)b \equiv 1 \pmod p$. In addition, we also define the local leading coefficient of f at p *by $\mathrm{llc}_p(0) := 0$ and, for $f \in K(t)^*$,*

$$\mathrm{llc}_p(f) := \pi_p(f^{-\nu_p(f)}).$$

Remark We will use the canonical representative $\pi_p(f) \in \{q \in K[t] \mid \deg_t(q) < \deg_t(p)\}$ instead of equivalence classes from K_p. Also note the following facts.

1. By definition we have $\nu_p(0) = \infty$ and also $O_p = \left\{\frac{a}{b} \;\middle|\; a, b \in K[t], \gcd(b,p) = 1\right\}$ since $\nu_p(f) \geq 0$ is equivalent to $\gcd(\mathrm{den}(f), p) = 1$. This also implies that π_p is well defined since the modular inverse of such a $\mathrm{den}(f)$ always exists.

2. The definition of O_p and π_p agrees with their definition in [Bro], but ν_p does not agree with the definition there unless p is irreducible. The definition given here satisfies $\nu_p(fp^n) = \nu_p(f) + n$ for all squarefree p. We will consider irreducible p most of the time anyway.

3. If p is irreducible, then ν_p is a valuation, O_p is its valuation ring, K_p is its residue field, and $\mathrm{llc}_p : K(t)^* \to K_p^*$ is a group homomorphism. Otherwise, K_p is a ring with zero divisors and π_p is still a ring homomorphism but $\mathrm{llc}_p : K(t)^* \to K_p \setminus \{0\}$ is not even a homomorphism of monoids.

4. If we apply above definition to K and $\tilde{t} := \frac{1}{t}$, then we see that it also covers the case $p = \frac{1}{t}$ on $K(t)$ since $K(\tilde{t}) = K(t)$ and $p \in K[\tilde{t}] \setminus K$ is squarefree (even irreducible). In this case we explicitly have $\nu_p(f) = \deg_t(\mathrm{den}_t(f)) - \deg_t(\mathrm{num}_t(f))$, $O_p = \left\{\frac{a}{b} \;\middle|\; a, b \in K[t], \deg_t(a) \leq \deg_t(b)\right\}$, and $K_p = K$.

5. Note that, if $f \in K[t]$ is a polynomial, then its degree is given by $\deg_t(f) = -\nu_{\frac{1}{t}}(f)$ and its leading coefficient is $\mathrm{lc}_t(f) = \mathrm{llc}_{\frac{1}{t}}(f)$ whereas $\mathrm{llc}_t(f)$ would be its trailing coefficient.

□

It is also true that $\nu_p(f) = \min_{q|p} \nu_q(f)$ where the minimum ranges over all irreducible factors of p. An important situation occurs if $\nu_p(f) = \nu_q(f)$ for all irreducible factors q of p. Following [Abr89, Bro92] we include the following definition.

Definition 2.13. *Let K be a field and let t be transcendental over K. For $p \in K[t] \setminus K$ squarefree and $f \in K(t)$ we say that p is* balanced w.r.t. f *if $\nu_{p_1}(f) = \nu_{p_2}(f)$ for any irreducible factors p_1, p_2 of p. If $S \subseteq K(t)$, then we say that p is balanced w.r.t. S if it is balanced w.r.t. each $f \in S$.*

Remark It is important to note that obviously any irreducible p is balanced w.r.t. any f by definition. This definition also implies that any p in particular is balanced w.r.t. 0 and, for fixed $f \in K(t)$, every factor of p is balanced w.r.t. f if p is. □

We summarize some very important properties of the order function in the following lemma. These will be exploited in our calculations later without further reference.

Lemma 2.14. *Let K be a field and let t be transcendental over K. For $p \in K[t] \setminus K$ squarefree (or $p = \frac{1}{t}$) and $f, g \in K(t)$ we have*

1. $\nu_p(f + g) \geq \min(\nu_p(f), \nu_p(g))$ with equality if $\nu_p(f) \neq \nu_p(g)$ and

2. $\nu_p(fg) \geq \nu_p(f) + \nu_p(g)$ *with equality if* p *is balanced w.r.t.* $\{f, g\}$ *(or* $p = \frac{1}{t}$*).*

Moreover, $f \in K$ *implies* $\nu_p(f) = 0$ *and* $\nu_p(fg) = \nu_p(g)$*.*

Equality holds in the second statement in particular if p is irreducible. Note that the second statement implies $\nu_p(\frac{1}{f}) \leq -\nu_p(f)$ for $f \in K(t)^*$ with equality if p is balanced w.r.t. f (or $p = \frac{1}{t}$). However, $\nu_p(f^n) = n\nu_p(f)$ holds for all $n \in \mathbb{N}^+$ and any squarefree p. An important property of the local leading coefficient is that

$$\nu_p(f - \mathrm{llc}_t(f)p^{\nu_p(f)}) \geq \nu_p(f) + 1. \tag{2.2}$$

One may also consider the p-adic expansion $f = \sum_{i=\nu_p(f)}^{\infty} f_i p^i$ in p with coefficients in K_p. This expansion can be made precise by noting that $K(t)$ can be turned into a metric space with the metric $d(f,g) := 2^{-\nu_p(f-g)}$ and its completion is isomorphic to $K_p((p))$ if we define multiplication in $K_p((p))$ accordingly. The local leading coefficient $\mathrm{llc}_p(f)$ being the first nonzero coefficient in the expansion will be important to us, but we will never make direct use of higher terms in the p-adic expansion nor of the topology induced by ν_p.

Definition 2.15. *Let* $(K(t), D)$ *be a differential field and let* $p \in K[t]$ *be squarefree. Then we define the* degree of D at p *by*

$$\omega_p := \inf_{f \in K(z)^*} \nu_p(Df) - \nu_p(f).$$

The derivation is continuous in the metric mentioned above if and only if $\omega_p > -\infty$. The following theorems describe the relations of the orders of f and Df, which are crucial for our considerations later.

Theorem 2.16. *([Bro, Thm 4.4.2]) Let* (K, D) *be a differential field, let* t *be a monomial over* (K, D)*, and let* $p \in K[t]$ *be irreducible. Then for all* $f \in K(t)^*$ *we have* $\nu_p(Df) \geq 0$ *if* $\nu_p(f) = 0$*; however, if* $\nu_p(f) \neq 0$ *then*

1. $\gcd(p, Dp) = 1$ *implies* $\nu_p(Df) = \nu_p(f) - 1$*,*
2. $p \in S_{K[t]:K}$ *implies* $\nu_p(Df) \geq \nu_p(f)$*, and*
3. $p \in S^1_{K[t]:K}$ *implies* $\nu_p(Df) = \nu_p(f)$*.*

In particular, $\nu_p\left(\frac{Df}{f}\right) \geq -1$ *with equality if and only if* $\gcd(p, Dp) = 1 \wedge \nu_p(f) \neq 0$*.*

Theorem 2.17. *([Bro, Thm 4.4.4]) Let* (K, D) *be a differential field and let* t *be a monomial over* (K, D)*. Let* $f \in K(t)^*$ *then*

$$\nu_{\frac{1}{t}}(Df) \geq \nu_{\frac{1}{t}}(f) - \max(0, \deg_t(Dt) - 1).$$

Moreover, if t *is nonlinear then equality holds if and only if* $\nu_{\frac{1}{t}}(f) \neq 0$*.*

The first theorem implies $\omega_p = -1$ for normal p and $\omega_p \geq 0$ for special p with equality if p is special of the first kind. The second theorem implies $\omega_{\frac{1}{t}} \geq -\max(0, \deg_t(Dt) - 1)$ with equality if t is nonlinear. In particular, for polynomials $f \in K[t]$ we have that

$$\deg_t(Df) \leq \deg_t(f) + \max(0, \deg_t(Dt) - 1). \tag{2.3}$$

The following lemma provides a sufficient condition that in a monomial extension the subfield of constants stays the same. It can be viewed as a refined version of Lemma 3.4.5 in [Bro].

Lemma 2.18. *Let (K, D) be a differential field and let t be a monomial over (K, D). Then*

$$S_{K[t]:K} = S^1_{K[t]:K} \quad \Longrightarrow \quad \mathrm{Const}_D(K(t)) = \mathrm{Const}_D(K).$$

Proof. Assume there exists $c \in \mathrm{Const}_D(K(t)) \setminus \mathrm{Const}_D(K)$. Then from $c \in K(t) \setminus K$ it follows that there is an irreducible $p \in K[t]$ such that $\nu_p(c) \in \mathbb{Z} \setminus \{0\}$. In addition, from $Dc = 0$ we obtain $\nu_p(Dc) = \infty > \nu_p(c)$. Hence $p \in S_{K[t]:K} \setminus S^1_{K[t]:K}$ by Theorem 2.16. $\square$

An algebraic version of the notion of a residue, known from complex analysis, can be found in the following definition, which extends Definition 4.4.1 of [Bro]. It will be used later in Sections 3.2 and 4.3.1.

Definition 2.19. *Let (K, D) be a differential field, let t be a monomial over (K, D), and let $p \in K[t] \setminus K$ be squarefree or $p = \frac{1}{t}$. For $f \in K(t)$ with $\nu_p(f) \geq -\nu_p(\frac{1}{Dp}) - 1$ we define the* residue of f at p *by*

$$\mathrm{res}_p(f) := \pi_p\left(f \frac{p}{Dp}\right).$$

Lemma 2.20. *([Bro, Thm 4.4.1]) Let (K, D) be a differential field, let t be a monomial over (K, D), and let $p \in K[t] \setminus K$ with $\gcd(p, Dp) = 1$. For $f \in K(t)$ with $pf \in O_p$ we have $\mathrm{res}_p(f) = 0$ if and only if $f \in O_p$.*

Lemma 2.21. *([Bro, Corollary 4.4.2.iii]) Let (K, D) be a differential field, let t be a monomial over (K, D), and let $p \in K[t]$ be irreducible with $\gcd(p, Dp) = 1$. Let $f \in K(t)^*$ then $\nu_p(Df) \neq -1$ and $\mathrm{res}_p\left(\frac{Df}{f}\right) = \nu_p(f)$.*

Lemma 2.22. *([Bro, Lemma 4.4.2]) Let (K, D) be a differential field and let t be a monomial over (K, D). Let $p \in K[t]$ irreducible with $\gcd(p, Dp) = 1$, $a \in O_p$, and $b \in K[t]$ such that $\nu_p(b) = 1$, then*

$$\mathrm{res}_p\left(\frac{a}{b}\right) = \pi_p\left(\frac{a}{Db}\right).$$

Lemma 2.23. *([Bro, Lemma 5.6.1]) Let (K, D) be a differential field, let t be a monomial over (K, D), and let $f \in K(t)$ be simple. If there are $h \in K(t)_{red}$, an algebraic extension E of $\mathrm{Const}_D(K)$, $v \in K(t)$, $c_1, \ldots, c_n \in E$, and $u_1, \ldots, u_n \in EK(t)$ such that*

$$f + h = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i},$$

then for any normal irreducible $p \in EK(t)$ we have

$$\mathrm{res}_p(f) = \sum_{i=1}^{n} c_i \nu_p(u_i).$$

2.4 Basic computational tasks

Various kinds of subproblems arise in the course of computing integrals as presented in Chapter 3. Most of them consist in solving linear differential equations in (K, D) and will be dealt with in Chapter 4. Some more basic problems remain and are discussed in this section. In the following chapters we will implicitly use the common computability requirements on differential fields: the basic arithmetic operations in K as well as zero-testing and derivation are computable. In particular, we will assume that we can solve linear systems in K and in $\mathrm{Const}_D(K)$ and compute squarefree factorizations and GCDs of univariate polynomials with coefficients in K as well as the half-extended GCD (i.e. modular inverses) and the extended GCD of two such polynomials as well as orders ν_p. In addition, we also assume that the problems discussed below in this section can be solved algorithmically in finitely many steps. For Problems 2.27 and 2.28 below we will need to represent the constant field as a $\mathbb{Q}$-vector space.

2.4.1 Splitting factorization

Let (K, D) be a differential field and let t be a monomial over (K, D). As can be seen from the definition, an irreducible polynomial from $K[t]$ is either normal or special. So for arbitrary polynomials $p \in K[t]$ we can separate the normal from the special factors, see also [Bro].

Definition 2.24. *Let (K, D) be a differential field and let t be a monomial over (K, D). A factorization $p = p_n p_s$ of $p \in K[t]$ is called a* splitting factorization *w.r.t. D if every irreducible factor of p_n is normal and p_s is special. In that case p_n and p_s are called the normal and special part of p respectively.*

An algorithm for computing splitting factorizations based on computing GCDs of polynomials in $K[t]$ is given in Section 3.5 of [Bro]. It relies on the fact that $p_s = \gcd(p, Dp)$ and $p_n = \frac{p}{p_s}$ is a splitting factorization if p is squarefree.

2.4.2 Balanced factorization

Let (K, D) be a differential field and let t be a monomial over (K, D). As not every polynomial $p \in K$ is balanced w.r.t. a given $f \in K(t)$ we could factor p into irreducibles and each of them would be balanced w.r.t. f. But we can relax this condition, see [Abr89, Bro92], to the following refinement of a squarefree factorization. This is used by some of the algorithms solving linear differential equations discussed in Chapter 4.

Definition 2.25. *Let (K, D) be a differential field and let t be a monomial over (K, D). Let $p \in K[t]$ and let $f \in K(t)$. We say that $p = p_1^{e_1} \dots p_n^{e_n}$ is a* balanced factorization of p w.r.t. f *if each p_i is squarefree and balanced w.r.t. f and $\gcd(p_i, p_j) = 1$ for $i \neq j$. If $S \subseteq K(t)$, then we say that $p = p_1^{e_1} \dots p_n^{e_n}$ is a balanced factorization of p w.r.t. S if it is a balanced factorization w.r.t. each $f \in S$.*

As indicated above a factorization into irreducibles automatically is a balanced factorization w.r.t. any f. An algorithm avoiding complete factorization but relying on GCDs of polynomials in $K[t]$ only can be found in the above references as well.

2.4.3 Constant solutions of linear systems

Let (F, D) be a differential field and $C := \mathrm{Const}(F)$. We need to consider the following problem in (F, D), which arises, for example, in Theorems 3.9 and 3.14 in the integration algorithm presented later but also at several places in the algorithms discussed in Chapter 4.

Problem 2.26. *Given: a matrix $A \in F^{m \times n}$ and a vector $\mathbf{b} \in F^m$.*

Find: a basis $\mathbf{c}_1, \dots, \mathbf{c}_k \in C^n$ of the C-vector space of all solutions $\mathbf{c} \in C^n$ of $A \cdot \mathbf{c} = 0$ as well as a particular solution $\mathbf{c}_0 \in C^n$ of $A \cdot \mathbf{c} = \mathbf{b}$ if it exists.

Solving this problem we always obtain a basis for $\ker(A) \cap C^n$ and, in particular, if $\mathbf{b} = 0$ we can always set $\mathbf{c}_0 = 0$. In conjunction with Gaussian elimination we can exploit the differential structure of (F, D) by noting the following. If $a_1, \dots, a_n, b \in F$ and $\mathbf{c} \in C^n$ satisfy

$$a_1 c_1 + \dots + a_n c_n = b$$

with $a_1, \dots, a_{j-1} \in C$ and $a_j \notin C$ for some j, then by dividing the derivative of both sides by $Da_j \neq 0$ we obtain

$$c_j + \frac{Da_{j+1}}{Da_j} c_{j+1} + \dots + \frac{Da_n}{Da_j} c_n = \frac{Db}{Da_j}.$$

This new equation can be added to the linear system and used for eliminating the entries in the j-th column. Bronstein turned this into an algorithm [Bro, Lemma 7.1.2] for converting the system $A \cdot \mathbf{c} = b$ into an equivalent system $\tilde{A} \cdot \mathbf{c} = \tilde{\mathbf{b}}$ having the same solutions $\mathbf{c} \in C^n$, but with $\tilde{A} \in C^{\tilde{m} \times n}$ and $\tilde{\mathbf{b}} \in F^{\tilde{m}}$. If $\tilde{\mathbf{b}}$ has a non-constant entry then there is no solution $\mathbf{c} \in C^n$, otherwise we compute the solutions of $\tilde{A} \cdot \mathbf{c} = \tilde{\mathbf{b}}$ by plain linear algebra in C. We observe that the size of the system cannot grow arbitrarily and we have $\tilde{m} \in \{m, \dots, m+n\}$ as we add at most one row for each column.

Alternatively, if F is finitely generated over C as a field, we can also exploit the explicit representation of the entries of the system in terms of these generators and compute an equivalent system $\tilde{A} \cdot \mathbf{c} = \tilde{\mathbf{b}}$ having the same solutions $\mathbf{c} \in C^n$ with $\tilde{A} \in C^{\tilde{m} \times n}$ and $\tilde{\mathbf{b}} \in C^{\tilde{m}}$ by comparing coefficients after clearing denominators or performing partial fraction decompositions. More generally, we may as well think of having some C-vector space basis of F and compare coefficients with respect to this basis. Again, from this we compute the constant solutions simply by linear algebra in C. Note that in either case the size of the system can grow arbitrarily large.

2.4.4 Rational solutions of linear systems

Let (F, D) be a differential field. For solving the parametric logarithmic derivative problem in (F, D), see Section 4.3.1, we need to solve the following problem in (F, D).

Problem 2.27. *Given: a matrix* $A \in F^{m \times n}$ *and a vector* $\mathbf{b} \in F^m$.

Find: a basis $\mathbf{c}_1, \ldots, \mathbf{c}_k \in \mathbb{Q}^n$ *of the* $\mathbb{Q}$*-vector space of all solutions* $\mathbf{c} \in \mathbb{Q}^n$ *of* $A \cdot \mathbf{c} = 0$ *as well as a particular solution* $\mathbf{c}_0 \in \mathbb{Q}^n$ *of* $A \cdot \mathbf{c} = \mathbf{b}$ *if it exists.*

For this problem we can apply similar methods to those for computing the constant solutions of a linear system described earlier. Either we directly go from a system in F to a system in $\mathbb{Q}$ based on coefficient comparison, if F is finitely generated over $\mathbb{Q}$ as a field, or we may also first use Bronstein's algorithm to convert to an equivalent system with coefficients in $C := \mathrm{Const}(F)$ and then focus on the representation of C over $\mathbb{Q}$ to reduce it further to a system in $\mathbb{Q}$ by comparing coefficients.

2.4.5 Integer roots of polynomials

Let (F, D) be a differential field. Especially when determining exponents of solutions of differential equations by indicial equations, see Chapter 4, we need to deal with the set of all $n \in \mathbb{Z}$, or $n \in \mathbb{N}$, or $n \in \mathbb{N}^+$ satisfying $p(n) = 0$ for some given polynomial $p \in F[z]$. Often it is enough to just obtain a lower or upper bound for these n.

As a first step towards determining these solutions we also may look for more general roots $c \in C$ or $c \in \overline{C}$ of the polynomial p where $C := \mathrm{Const}_D(F)$. To this end we can compute the special part p_s of p w.r.t. κ_D by a splitting factorization or just by iteratively computing $\gcd(p, \kappa_D p)$. Then the roots of $p_s \in C[z]$ will be exactly the constant roots of p since for any $q \in F[z]$ we have that $q(c) = 0$ and $Dc = 0$ imply $(\kappa_D q)(c) = 0$ by Lemma 2.4. This may be used as a preprocessing step to the following problem.

Problem 2.28. *Given: a differential field* (F, D) *and a polynomial* $p \in F[z]$.

Find: the set $\{n \in \mathbb{Z} \mid p(n) = 0\}$.

Also for this problem we can apply a similar strategy based on comparing coefficients as in the previous section if F or at least C is finitely generated over $\mathbb{Q}$. Either directly or after the preprocessing described above we can decompose the coefficients of

$$p = \sum_{k=0}^{\deg(p)} \Big(\sum_i a_{k,i} b_i\Big) z^k,$$

with $a_{k,i} \in \mathbb{Q}$ and $\{b_i\}_i$ linearly independent over $\mathbb{Q}$, to obtain polynomials

$$q_i := \sum_{k=0}^{\deg(p)} a_{k,i} z^k$$

with coefficients in $\mathbb{Q}$ and compute their GCD $q \in \mathbb{Q}[z]$. Then, we are left with computing the integer roots of q, which is a special case of the problem considered in [Loo83]. In practice, however, it is possible and often preferable to just factor the polynomial q or even p into irreducibles and then inspect the linear factors.

2.5 Liouvillian solutions of differential equations

Apart from introducing the notion of Liouvillian fields and Riccati equations the main purpose of the material summarized in this section is the proof of Theorem 2.53 in the next section. Basically that theorem will be a consequence of work done by Ulmer and Weil [UW96] summarized in Theorem 2.45. Before we can state and prove the necessary results we also need to briefly introduce some notions from differential Galois theory and state some auxiliary results.

Definition 2.29. *Let (F, D) be a differential field, K a differential subfield, and $t \in F$. Then t is called*

1. primitive *over (K, D) if $Dt \in K$,*
2. hyperexponential *over (K, D) if $\frac{Dt}{t} \in K$, or*
3. Liouvillian *over (K, D) if t is algebraic, or primitive, or hyperexponential over (K, D).*

Primitives of some $a \in F$ correspond to indefinite integrals, so we may write $\int a$ for any t from F or from any differential extension of (F, D) with $Dt = a$. Hyperexponential elements behave like exponentials of indefinite integrals and, analogously, we write $e^{\int a}$ for any nonzero t from F or from any differential extension of (F, D) with $\frac{Dt}{t} = a$. Note that neither $\int a$ nor $e^{\int a}$ are unique and may vary by an additive or multiplicative constant, not necessarily from $\mathrm{Const}(F)$, respectively. Using this notation we will always implicitly refer to an arbitrary but fixed representant in order to have different occurrences in one formula denote the same object.

Based on the definition above we define Liouvillian extensions of a differential field in the following way. We will discuss and refine this definition in Section 2.6.1. Liouvillian solutions of some differential equation are solutions in Liouvillian extensions.

Definition 2.30. *Let (K, D) be a differential field and $(F, D) = (K(t_1, \ldots, t_n), D)$ a differential field extension. Then (F, D) is called a* Liouvillian *extension of (K, D), if each t_i is Liouvillian over $(K(t_1, \ldots, t_{i-1}), D)$.*

Definition 2.31. *Let (K, D) be a differential field and let $P(y) = f$ be a (possibly non-linear) differential equation with coefficients in K. Then we say that $P(y) = f$ has a Liouvillian solution over (K, D) if there exist a Liouvillian extension (F, D) of (K, D) and $y \in F$ such that $P(y) = f$.*

Remark It is important to note that the notion of a Liouvillian solution is relative to a given differential field. □

Rosenlicht has proven the following theorem on Liouvillian solutions of a certain class of differential equations by investigating orders on the differential fields only. A special case of it, proven by similar methods, is already contained in Chapter 6 of [Rit].

Theorem 2.32. *([Ros73]) Let (K, D) be a differential field, $m, n \in \mathbb{N}^+$, and let $P \in K[z_0, \ldots, z_k]$ be a polynomial of total degree strictly less than n. If the differential equation*

$$y^n = P(y, Dy, \ldots, D^k y)$$

has a Liouvillian solution over (K, D), then it also has a solution $y \in \overline{K}$.

Later our primary interest, however, will be in linear equations as they arise as subproblems during the integration algorithm. For computational aspects we refer to Chapter 4, where we mainly are interested to compute the solutions in the coefficients field. Here we will review some well-known general facts about the structure of solutions of linear differential solutions. It turns out that there is a type of nonlinear equations intrinsically related to linear ones. To any linear differential we can associate a Riccati equation, which plays an important role in the context of hyperexponential solutions. Note that for $y = e^{\int u}$ the derivatives $D^i y$ are multiples of y, e.g., $Dy = uy$ and $D^2 y = (Du + u^2)y$. This motivates the following definition.

Definition 2.33. *For $i \in \mathbb{N}$ we recursively define the nonlinear differential operators P_i in the following way.*

$$P_0(u) := 1, \qquad P_{i+1}(u) := D(P_i(u)) + uP_i(u)$$

For any $i \in \mathbb{N}$ we have that $P_i(u)$ is a polynomial in $u, Du, \ldots, D^{i-1}u$ with coefficients in $\mathbb{N}$ and total degree i with u^i being the only term of this degree. For example, we have $P_1(u) = u$, $P_2(u) = Du + u^2$, and $P_3(u) = D^2u + 3uDu + u^3$. Note that P_0 and P_1 are the only linear operators among the P_i. With these operators we have

$$D^i e^{\int u} = P_i(u) e^{\int u}$$

and can formulate the following lemma relating equations satisfied by y to equations satisfied by u. More precisely, the correspondence is given by the fact that any nonzero y is a solution of the linear homogeneous differential equation $L(y) = 0$ if and only if $u = \frac{Dy}{y}$ is a solution of the *associated Riccati equation*, which is defined in the satement of the lemma.

Lemma 2.34. *Let (K, D) be a differential field and (F, D) a differential field extension of it. Let $a_0, \ldots, a_n \in K$ and let $L(y) := \sum_{i=0}^{n} a_i D^i y$ and $R(y) := \sum_{i=0}^{n} a_i P_i(y)$. Then for any $u \in F$*

$$L(e^{\int u}) = 0 \iff R(u) = 0.$$

As a consequence we obtain the following well-known result linking Liouvillian solutions of linear differential equations to algebraic solutions of nonlinear differential equations.

Theorem 2.35. *Let (K, D) be a differential field and let $L \in K[D]$ be a linear differential operator of arbitrary order. If $L(y) = 0$ has a Liouvillian solution over (K, D), then it has a solution $y = e^{\int u}$ with $u \in \overline{K}$, i.e., there is a $u \in \overline{K}$ such that $R(u) = 0$.*

Alternatively, the previous theorem has also been proven using Picard-Vessiot theory. The notions defined in the remaining part of this section do not play an important role in the other chapters but are primarily needed here only.

Definition 2.36. *Let (K, D) be a differential field and let $L \in K[D]$ be a linear differential operator of arbitrary order n. The differential field (F, D) generated by $K(y_1, \ldots, y_n)$ is a* Picard-Vessiot extension (PVE) *of (K, D) for $L(y) = 0$ if*

1. *$\{y_1, \ldots, y_n\}$ is a fundamental system for $L(y) = 0$ and*
2. $\mathrm{Const}(F) = \mathrm{Const}(K)$.

Remark If $\mathrm{Const}(K)$ is algebraically closed, then a PVE for $L(y) = 0$ exists and is unique up to differential isomorphism. □

Definition 2.37. *Let (K, D) be a differential field with $C := \mathrm{Const}(K)$ algebraically closed and let $L \in K[D]$ with PVE (F, D). The group $G(L)$ of all differential automorphisms of (F, D) which leave all elements of K fixed is called the* differential Galois group *of $L(y) = 0$.*

Later we will sometimes consider the differential Galois group of operators with coefficients from a differential field (K, D) whose field of constants C is (potentially) not algebraically closed. In this case we will implicitly consider the differential Galois group to be defined in terms of a PVE of $(\overline{C}K, D)$.

Remark Any $g \in G(L)$ is an invertible C-linear map $g : V(L) \to V(L)$ acting on the space of solutions $V(L) = \{y \in F \mid L(y) = 0\}$, which by definition of (F, D) has $\dim_C(V(L)) = n$ where n is the order of the differential operator L. So fixing a basis of $V(L)$ it can be represented as an element of $GL(n, C)$, in other words $G(L)$ is isomorphic to a subgroup of $GL(n, C)$. Since a classification of the subgroups of $SL(n, C)$ is known many results of differential Galois theory rely on the following property. □

Definition 2.38. *Let (K, D) be a differential field with $C := \mathrm{Const}(K)$ algebraically closed and let $L \in K[D]$. The differential Galois group $G(L)$ is called* unimodular *if it has a matrix representation as $G(L) \subseteq SL(n, C)$.*

Remark If for some $L = D^n + a_{n-1}D^{n-1} + \cdots + a_0 \in K[D]$ we define a new operator $\tilde{L}$ by $\tilde{L}(y) := L(ye^{\int u})e^{-\int u}$ with $u \in K$, then it has the form $\tilde{L} = D^n + (a_{n-1} + nu)D^{n-1} + \ldots$ with all coefficients in K as well. This change of variable is particularly useful in view of the following theorem, which provides a criterion for checking unimodularity of $G(L)$ based on the coefficient of order $n - 1$ only. □

Theorem 2.39. *Let (K, D) be a differential field with $C := \mathrm{Const}(K)$ algebraically closed and let $L = D^n + a_{n-1}D^{n-1} + \cdots + a_0 \in K[D]$. Then $G(L)$ is unimodular if and only if a_{n-1} is the logarithmic derivative of an element of K.*

Theorem 2.40. *Let (K, D) be a differential field and let $L \in K[D]$ be a linear differential operator of arbitrary order. Then $L(y) = 0$ has a fundamental system in $\overline{K}$ if and only if its differential Galois group is finite.*

Definition 2.41. *Let (K, D) be a differential field. An operator $L \in K[D]$ is called* reducible *over K if there exist $L_1, L_2 \in K[D] \setminus K$ such that $L = L_1 \circ L_2$, otherwise L is called* irreducible *over K. Also the corresponding differential equation $L(y) = 0$ is called reducible or irreducible, respectively.*

Note that a first-order right factor $L = \tilde{L} \circ (D - u)$ is equivalent to a hyperexponential solution of the homogeneous equation $L(e^{\int u}) = 0$. We need to introduce one more definition before we can state the criteria for Liouvillian solutions of second-order equations we are interested in.

Definition 2.42. *Let (K, D) be a differential field with $C := \mathrm{Const}(K)$ algebraically closed and let $L \in K[D]$ monic with PVE (F, D). For $n \in \mathbb{N}^+$ we define the n-*th symmetric power *of L as the smallest order monic linear operator $L^{\circledS n} \in K[D]$ such that for all $y_1, \ldots, y_n \in V(L) = \{y \in F \mid L(y) = 0\}$ we have $L^{\circledS n}(y_1 \cdot \ldots \cdot y_n) = 0$.*

For computing symmetric powers see [SU93, BMW97] for example, where it is also proven that for second-order operators the n-th symmetric power has order $n + 1$.

We will focus on second-order equations now, for which Kovacic gave the first complete algorithm to compute Liouvillian solutions in case of rational function coefficients [Kov86]. Singer and Ulmer [SU93] relate the existence of Liouvillian solutions of linear differential equations to factorization properties of their symmetric powers. In particular, they prove in their Proposition 4.4 that a second-order equation $L(y) = 0$ has a Liouvillian solution if and only if $L^{\circledS 6}(y)$ is reducible. While in theory it also would be possible to use this criterion, in practice we prefer computation of in-field solutions over reducibility checks in view of the results in Chapter 4. Ulmer and Weil [UW96] managed to give an algorithm for computing Liouvillian solutions of second-order equations $L(y) = 0$ based on in-field solutions of the associated Riccati equation and some symmetric powers of L. Fakler [Fak97] gave a variant of this algorithm which gives nicer formulas in many cases. We will rely on the ideas of Ulmer and Weil, but modify their algorithm to a simpler version which suffices for our purpose. Before we proceed to the two main ingredients for Theorem 2.53 we need to state a trivial lemma, which is essential for what follows.

Lemma 2.43. *Let (K, D) be a differential field, $C := \mathrm{Const}(K)$, and let $L \in K[D]$ be a linear differential operator of arbitrary order. If $L(y) = 0$ has a solution $y \in \overline{C}K^*$, then it also has a solution $y \in K^*$.*

Proof. Write $y = \sum_{i=0}^{d-1} y_i \alpha^i$ for some $\alpha \in \overline{C}$, d its algebraic degree over C, and $y_i \in K$. As $D\alpha = 0$, we have $L(y) = \sum_{i=0}^{d-1} L(y_i)\alpha^i$. So $L(y) = 0$ implies $L(y_i) = 0$ for all i, form which the result trivially follows since not all y_i are zero. □

Next, we prove a criterion to check, by solving some auxiliary equations in K, when a second-order linear operator $L = D^2 + a_1 D + a_0$ with coefficients in K is irreducible over $\overline{C}K$, i.e., cannot be written as $L = (D + \alpha_1) \circ (D + \alpha_2)$ for some $\alpha_1, \alpha_2 \in \overline{C}K$. Note that for second-order operators reducibility over $\overline{C}K$ is equivalent to a solution $u \in \overline{C}K$ of the associated Riccati equation, but we want to avoid computations in $\overline{C}K$.

Theorem 2.44. *Let (K, D) be a differential field, $C := \mathrm{Const}(K)$, and let $L(y) := D^2y + a_1Dy + a_0y$ with $a_0, a_1 \in K$. If $Du = -u^2 - a_1u - a_0$ does not have a solution $u \in K$ and $L^{\circledS 2}(y) = 0$ does not have a solution $y \in K^*$, then L is irreducible over $\overline{C}K$.*

Proof. Assume that L is reducible over $\overline{C}K$ and $Du = -u^2 - a_1u - a_0$ does not have a solution $u \in K$. Then $Du = -u^2 - a_1u - a_0$ has a solution $u_1 \in \overline{C}K \setminus K$ and any conjugate of u_1 is also a solution of $Du = -u^2 - a_1u - a_0$. This means that $Du = -u^2 - a_1u - a_0$ has (at least) two distinct solutions $u_1, u_2 \in \overline{C}K$ and the corresponding solutions $y_i = e^{\int u_i}$, $i = 1, 2$, of $L(y) = 0$ are linearly independent over $\overline{C}$ since $W(y_1, y_2) = (u_2 - u_1)y_1y_2 \neq 0$. By Proposition 4.2.ii from [SU93] we have $y_1y_2 \in \overline{C}K^*$, which is a solution of $L^{\circledS 2}(y) = 0$, and hence $L^{\circledS 2}(y) = 0$ has a solution $y \in K^*$ by Lemma 2.43. □

The following theorem is the key to an algorithmic check for Liouvillian solutions of second-order linear equations by linking their existence to in-field solutions of higher-order linear differential equations. It is a specialized version of the results of Ulmer and Weil [UW96], for the case where one is merely interested in the existence of Liouvillian solutions and not in their actual computation. A version of this result can also be found as Proposition 8 in [Fak97], so we only sketch the proof here.

Theorem 2.45. *Let (K, D) be a differential field, $C := \mathrm{Const}(K)$, and let $L(y) := D^2y + a_1Dy + a_0y$ with $a_0, a_1 \in K$. Assume L is irreducible over $\overline{C}K$ and has unimodular differential Galois group $G(L)$, then the following hold.*

1. *$L(y) = 0$ does not have a Liouvillian solution over (K, D) if and only if $L^{\circledS 12}(y) = 0$ does not have a solution $y \in K^*$.*
2. *If $L(y) = 0$ does not have a fundamental system $\{y_1, y_2\} \subseteq \overline{K}$, then $L(y) = 0$ does not have a Liouvillian solution over K if and only if $L^{\circledS 4}(y) = 0$ does not have a solution $y \in K^*$.*

Proof. Consider $(\overline{C}K, D)$ and apply Lemmas 3.2 and 3.3 from [UW96]. Then the result follows by Lemma 2.43 above. For the second statement consider Theorem 2.40 in addition. □

2.6 Algebraic representation of functions

Generally speaking, we aim at representing given functions by elements of differential fields $(F, D) = (C(t_1, \dots, t_n), D)$ which are generated in such a way that each t_i is a monomial over $(F_{i-1}, D) = (C(t_1, \dots, t_{i-1}), D)$ with additional properties. One important technical property will be $\mathrm{Const}(F_{i-1}(t_i)) = \mathrm{Const}(F_{i-1})$, such that in total we have $\mathrm{Const}(F) = C$. Also the presence of special polynomials can cause difficulties later. A very general criterion addressing both issues is given by the following theorem.

Theorem 2.46. *Let (K, D) be a differential field and let $q \in K[z]$ be a polynomial with coefficients in K. Let t from some differential field extension of (K, D) such that $Dt = q(t)$. Then $Dy = q(y)$ does not have a solution $y \in \overline{K}$ if and only if t is transcendental over K, $\operatorname{Const}(K(t)) = \operatorname{Const}(K)$, and $S^{irr}_{K[t]:K} = \emptyset$.*

Proof. First, assume that $Dy = q(y)$ does not have a solution $y \in \overline{K}$. Since $y = t$ is a solution of $Dy = q(y)$ it then is transcendental over K. Furthermore, Theorem 2.10 implies $S^{irr}_{K[t]:K} = \emptyset$, which in turn implies $\operatorname{Const}(K(t)) = \operatorname{Const}(K)$ by Lemma 2.18. Conversely, assume that t is transcendental over K and $\alpha \in \overline{K}$ is such that $D\alpha = q(\alpha)$. Let $p \in K[t]$ be the minimal polynomial of α then $p \in S^{irr}_{K[t]:K}$ by Theorem 2.10. □

We will give specialized variants of this theorem adapted to classes of functions that frequently arise in practice. Each of the Theorems 2.50, 2.51, and 2.53 will provide a criterion that can be checked algorithmically for a large class of differential fields (K, D) using algorithms discussed in Chapter 4.

2.6.1 Liouvillian functions

Before we discuss Liouvillian functions we start by describing a very basic class of functions. The *elementary functions* are those which can be constructed from rational functions by the following operations in addition to the basic arithmetic operations: taking the logarithm, applying the exponential function, and solving algebraic equations with elementary functions as coefficients. In particular this means that the composition of elementary functions is an elementary function again and so are powers $f(x)^{g(x)}$ of elementary functions.

Examples: rational and algebraic functions, logarithms, c^x and x^c, trigonometric functions and their inverses, hyperbolic functions and their inverses, etc.

Recall that trigonometric and hyperbolic functions can be expressed in terms of exponentials and their inverses can be expressed in terms of logarithms of algebraic functions. When representing functions in differential fields we are mainly interested in their differential properties. To this end we recall that the derivatives of logarithms and exponentials are given by

$$\frac{d}{dx}\ln(a(x)) = \frac{a'(x)}{a(x)} \tag{2.4}$$

$$\frac{d}{dx}\exp(a(x)) = \exp(a(x))a'(x), \tag{2.5}$$

where the latter also may be written in the form of a logarithmic derivative

$$\frac{\frac{d}{dx}\exp(a(x))}{\exp(a(x))} = a'(x). \tag{2.6}$$

By forgetting about the special structure of the right hand sides in (2.4) and (2.6) we are led to define functions satisfying one of the equations

$$\frac{d}{dx}y(x) = a(x) \tag{2.7}$$

$$\frac{\frac{d}{dx}y(x)}{y(x)} = a(x) \tag{2.8}$$

for some given $a(x)$. This way the notion of elementary functions is generalized naturally to give *Liouvillian functions*. In other words, Liouvillian functions are the functions obtained from rational functions by the basic arithmetic operations, by taking primitive functions $\int a(x)\,dx$, by taking hyperexponential functions $e^{\int a(x)\,dx}$, and by solving algebraic equations with Liouvillian functions as coefficients. Again, the composition of Liouvillian functions as well as powers $f(x)^{g(x)}$ of Liouvillian functions are Liouvillian. Several special Functions can be found in the class of Liouvillian functions as illustrated by the following list of examples.

Examples: logarithmic and exponential integrals, error functions, Fresnel integrals, polylogarithms, incomplete Beta and Gamma functions, etc.

Note that there are a few equivalent definitions of the class of Liouvillian functions. For instance, we need not start the construction from the rational functions but it suffices to start from the set of constants because the rational functions are obtained by the basic arithmetic operations from constants and the identity function, which in turn is a primitive function of the constant 1. Similarly, we may also choose to keep the operation of applying the exponential function instead of replacing it by taking hyperexponential functions as the latter operation can obviously be decomposed into applying the exponential function to a primitive function. Alternatively, we may also summarize taking primitive and hyperexponential functions into taking solutions of linear first-order differential equations. More precisely, the class of Liouvillian functions may also be constructed from the set of constants by the basic arithmetic operations and taking particular solutions of

$$y'(x) = a(x)y(x) + b(x) \tag{2.9}$$

and of algebraic equations with Liouvillian coefficients each. Note that the more flexible formulation does not yield a bigger class of functions since the solutions of (2.9) may be expressed in terms of the previous operations by $y(x) = e^{\int a(x)\,dx} \int \frac{b(x)}{e^{\int a(x)\,dx}}\,dx$ exploiting the hidden undetermined constants in the notation.

Now we turn to the corresponding definitions of differential fields. Starting from some field of constants C Liouvillian functions are represented by the elements of Liouvillian extensions of $(C, 0)$ as defined by Definitions 2.29 and 2.30. As special case of Liouvillian extensions we now define regular Liouvillian extensions and elementary extensions, which will play a prominent role later.

Definition 2.47. *Let (F, D) be a differential field, K a differential subfield, and $t \in F$. Then we call t*

1. *a* logarithm *over (K, D) if there exists $a \in K$ such that $Dt = \frac{Da}{a}$,*
2. *an* exponential *over (K, D) if there exists $a \in K$ such that $\frac{Dt}{t} = Da$, or*

3. elementary *over* (K, D) *if* t *is algebraic, or a logarithm, or an exponential over* (K, D).

Definition 2.48. *Let* (K, D) *be a differential field and* $(F, D) = (K(t_1, \dots, t_n), D)$ *a differential field extension. Then* (F, D) *is called*

1. regular Liouvillian *extension of* (K, D), *if*
   (a) *all* t_i *are algebraically independent over* K,
   (b) $\mathrm{Const}(F) = \mathrm{Const}(K)$, *and*
   (c) *each* t_i *is Liouvillian over* $(K(t_1, \dots, t_{i-1}), D)$, *or*
2. elementary *extension of* (K, D), *if each* t_i *is elementary over* $(K(t_1, \dots, t_{i-1}), D)$.

Elementary functions are represented by the elements of elementary extensions of $(C(x), \frac{d}{dx})$. Comparing this to the case of Liouvillian functions above we observe that an elementary extension of $(C, 0)$ would not contain any non-constant elements. Furthermore, we note that an elementary extension is also a Liouvillian extension but need not be a regular Liouvillian extension. Also keep in mind that an elementary or Liouvillian extension of some differential field (K, D) does not only contain elementary or Liouvillian functions unless K does.

The importance of the notion of elementary extensions for our work comes from specifying how we want to express the integrals we are looking for. In general, for finding an antiderivative of a given $f \in F$ we can always define a Liouvillian extension of (F, D) such that it contains a g with $Dg = f$, e.g., by simply adjoining a g defined that way. But for practical applications this is not very useful unless we can find a way to express this g in terms of functions that can be handled. Typically the differential field F represents a specific set of functions that can be handled, so if we restrict to find an antiderivative g in F for example, then such a g automatically is meaningful to us. However, we can also be more general and apply elementary functions to those functions in order to construct an antiderivative, which is made precise by the following definition.

Definition 2.49. *Let* (F, D) *be a differential field and* $f \in F$. *Then we say that* f *has an* elementary integral over (F, D) *if there exist an elementary extension* (E, D) *of* (F, D) *and* $g \in E$ *such that* $Dg = f$.

The structure of elementary integrals will be investigated in more detail in Section 2.7, which will be a key to the integration algorithm later in Chapter 3. Liouville's theorem (see Theorem 2.58) proves the well-known fact that it suffices to introduce in a certain way logarithms of functions represented by elements in F, which again should provide functions that can be handled.

The following two theorems characterize Liouvillian monomials. In particular, they provide criteria to check whether a given Liouvillian extension $(K(t), D)$ of (K, D) is regular. In addition, $S^{irr}_{K[t]:K}$ is determined in these cases.

Theorem 2.50. *([Bro, Thm 5.1.1]) Let* (K, D) *be a differential field and let* t *be primitive over* (K, D). *If* Dt *is not the derivative of an element of* K, *then* t *is transcendental over* K, $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$, *and* $S^{irr}_{K[t]:K} = \emptyset$. *Conversely, if* t *is transcendental over* K *and* $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$, *then* Dt *is not the derivative of an element of* K.

Remark If the condition of the previous theorem is not satisfied, i.e., if there exists $g \in K$ such that $Dt = Dg$, then for $c := t - g$ we have $K(t) = K(c)$ and $Dc = 0$. So, if $t \notin K$ and hence $c \notin K$, then this means that we find t in a differential field that is generated from K and a new constant c. □

Theorem 2.51. *([Bro, Thm 5.1.2]) Let (K, D) be a differential field and let t be hyperexponential over (K, D). If $\frac{Dt}{t}$ is not the logarithmic derivative of a K-radical then t is transcendental over K, $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$, and $S^{irr}_{K[t]:K} = S^{irr,1}_{K[t]:K} = \{t\}$. Conversely, if t is transcendental over K and $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$, then $\frac{Dt}{t}$ is not the logarithmic derivative of a K-radical.*

Remark If the condition of the previous theorem is not satisfied, i.e., if there exist $g \in K^*$ and $k \in \mathbb{Z} \setminus \{0\}$ such that $\frac{Dt}{t} = \frac{Dg}{kg}$, then for $c := \frac{t^k}{g}$ we have $Dc = 0$. This means that t is algebraic over (or, if k can be chosen ± 1, even contained in) a differential field that is generated from K and a (possibly) new constant c. □

2.6.2 A class of non-Liouvillian functions

In the previous section we saw that the class of Liouvillian functions contains quite a number of special functions in addition to elementary functions. But by far not all special functions are Liouvillian. However, a vast majority of non-Liouvillian special functions appearing in applications satisfy second-order differential equations, many examples of which will be mentioned shortly. Hence it seems desirable to generalize our considerations beyond Liouvillian functions and include solutions of linear second-order equations as well. We consider the more flexible formulation of two first order ODEs and start with a homogeneous version.

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \tag{2.10}$$

The following list contains many of the common special functions which satisfy a system of the form (2.10) where the coefficients of the matrix are elementary functions (mostly rational functions even).

Examples: orthogonal polynomials, associated Legendre functions, Bessel functions, Airy functions, complete elliptic integrals, Whittaker functions, Mathieu functions, hypergeometric functions, Heun functions, etc.

Let $\Phi(x) := \begin{pmatrix} \varphi_1(x) & \tilde{\varphi}_1(x) \\ \varphi_2(x) & \tilde{\varphi}_2(x) \end{pmatrix}$ be a fundamental matrix of (2.10). We do not represent the solutions of (2.10) as such, but apart from $\varphi_1(x)$ we consider the transformed functions

$$v(x) := \frac{\varphi_2(x)}{\varphi_1(x)}, \qquad \tilde{v}(x) := \frac{\tilde{\varphi}_1(x)}{\varphi_1(x)}, \qquad w(x) := \det \Phi(x) \tag{2.11}$$

instead. Then $v(x)$ satisfies a Riccati differential equation (2.12) and $w(x)$ satisfies Liouville's formula (2.14). More precisely $\varphi_1(x), v(x), \tilde{v}(x), w(x)$ satisfy the following system.

$$\begin{aligned}
v'(x) &= -a_{12}(x)v(x)^2 + (a_{22}(x) - a_{11}(x))v(x) + a_{21}(x) && (2.12)\\
\varphi_1'(x) &= (a_{12}(x)v(x) + a_{11}(x))\varphi_1(x) && (2.13)\\
w'(x) &= (a_{11}(x) + a_{22}(x))w(x) && (2.14)\\
\tilde{v}'(x) &= \tfrac{a_{12}(x)w(x)}{\varphi_1(x)^2} && (2.15)
\end{aligned}$$

The important feature of this system is that it is uncoupled in the sense of a cascading system, which makes it fit into our tower framework. The equations (2.13)-(2.15) are of a type that is covered by Definition 2.48.1 already. To deal with solutions of (2.10) we just need to incorporate functions defined by equations of type (2.12), which are covered by Definition 2.7. Hence we represent the solutions of (2.10) in terms of the functions $v(x), \varphi_1(x), w(x), \tilde{v}(x)$:

$$\varphi_2(x) = v(x)\varphi_1(x), \qquad \tilde{\varphi}_1(x) = \tilde{v}(x)\varphi_1(x), \qquad \tilde{\varphi}_2(x) = \frac{w(x)}{\varphi_1(x)} + v(x)\tilde{v}(x)\varphi_1(x).$$

We can extend this to solutions of inhomogeneous equations as well, which can be viewed as a natural generalization of (2.9).

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix} \tag{2.16}$$

Below we will give explicit formulas of a version of variation of the constants that fit to the definitions above. The special functions in the following list of examples are solutions of (2.16) with rational function coefficients but not of (2.10) with rational function coefficients.

Examples: Struve functions, Anger functions, Weber functions, Lommel functions, Scorer functions, etc.

Let $\Phi(x) := \begin{pmatrix} \varphi_1(x) & \tilde{\varphi}_1(x) \\ \varphi_2(x) & \tilde{\varphi}_2(x) \end{pmatrix}$ again be a fundamental matrix of the homogeneous system and fix a particular solution $(y_1(x), y_2(x))^T$ of the inhomogeneous system. In addition to $\varphi_1(x), v(x), \tilde{v}(x), w(x)$, as defined above, we consider

$$\lambda(x) := \frac{y_1(x)\varphi_2(x) - y_2(x)\varphi_1(x)}{\det \Phi(x)} \qquad \text{and} \qquad \tilde{\lambda}(x) := \frac{y_1(x)\tilde{\varphi}_2(x) - y_2(x)\tilde{\varphi}_1(x)}{\det \Phi(x)}. \tag{2.17}$$

The derivatives of these functions can be expressed in terms of $\varphi_1(x), v(x), \tilde{v}(x), w(x)$:

$$\begin{aligned}
\lambda'(x) &= \tfrac{v(x)b_1(x) - b_2(x)}{w(x)}\varphi_1(x) && (2.18)\\
\tilde{\lambda}'(x) &= \tfrac{v(x)b_1(x) - b_2(x)}{w(x)}\tilde{v}(x)\varphi_1(x) + \tfrac{b_1(x)}{\varphi_1(x)} && (2.19)
\end{aligned}$$

Note that we would have obtained these equations also from solving (2.16) by variation of the constants with the ansatz $(y_1(x), y_2(x))^T = \Phi(x) \cdot (\tilde{\lambda}(x), -\lambda(x))^T$. Obviously (2.18) and (2.19) are covered by Definition 2.48.1 as well. Hence we represent the solutions of (2.16) in terms of the functions $v(x), \varphi_1(x), w(x), \tilde{v}(x), \lambda(x), \tilde{\lambda}(x)$ as follows:

$$y_1(x) = (\tilde{\lambda}(x) - \lambda(x)\tilde{v}(x))\varphi_1(x), \qquad y_2(x) = (\tilde{\lambda}(x) - \lambda(x)\tilde{v}(x))v(x)\varphi_1(x) - \frac{\lambda(x)w(x)}{\varphi_1(x)}.$$

The differential fields resulting from the discussion in this section are closely related to the class of 2-*solvable* differential fields considered in [Ngu09] and references therein. We want to investigate the objects which satisfy (2.12)-(2.15) in more detail in terms of differential algebra. The most important part will be to characterize properties of monomials t with $\deg_t(Dt) = 2$, which we do in Theorem 2.53 below in analogy to Theorems 2.50 and 2.51.

To begin with, let (K, D) be a differential field. For $a_{11}, a_{12}, a_{21}, a_{22} \in K$ with $a_{12} \neq 0$, we consider the following first order system modeling (2.11).

$$\begin{pmatrix} Dy_1 \\ Dy_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.20}$$

Let φ_1, φ_2 from some differential extension of (K, D) such that $\varphi_1 \neq 0$ and (φ_1, φ_2) is a solution of (2.20). Then $v = \frac{\varphi_2}{\varphi_1}$ will satisfy the following first order nonlinear equation, cf. (2.12).

$$Dv = -a_{12}v^2 + (a_{22} - a_{11})v + a_{21}. \tag{2.21}$$

Observe that a Riccati equation $Dv = av^2 + bv + c$ is the associated Riccati equation of some second-order differential equation if and only if $a = -1$. The following trivial lemma states that any Riccati equation $Dv = av^2 + bv + c$ can be brought to the standard form $Du = -u^2 + r$ by a linear transform, the proof is a straightforward calculation so we omit it.

Lemma 2.52. *Let (K, D) be a differential field, let $a, b, c \in K$ with $a \neq 0$, and define $r := -\frac{D^2a}{2a} + \frac{3}{4}\left(\frac{Da}{a}\right)^2 + \frac{b}{2}\frac{Da}{a} - \frac{Db}{2} + \frac{b^2}{4} - ac \in K$. Let u, v from some differential extension of (K, D) such that $u = -av - \frac{1}{2}\left(\frac{Da}{a} + b\right)$, then*

$$Dv = av^2 + bv + c \qquad \Longleftrightarrow \qquad Du = -u^2 + r.$$

Note that $y = \varphi_1$ is a solution of the following second-order homogeneous linear equation

$$D^2y - \left(\frac{Da_{12}}{a_{12}} + \mathrm{Tr}(A)\right) Dy + \left(a_{11}\frac{Da_{12}}{a_{12}} - Da_{11} + \det(A)\right) y = 0, \tag{2.22}$$

where $\mathrm{Tr}(A) = a_{11} + a_{22}$ and $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. In view of Theorem 2.39 the corresponding equation with unimodular Galois group regardless of the values of the coefficients a_{ij} reads

$$D^2y - \frac{Da_{12}}{a_{12}}Dy + \left(\frac{a_{11} - a_{22}}{2}\frac{Da_{12}}{a_{12}} - D\frac{a_{11} - a_{22}}{2} - \frac{\mathrm{Tr}(A)^2}{4} + \det(A)\right) y = 0. \tag{2.23}$$

In particular, for a second-order equation $D^2y + a_1Dy + a_0 = 0$ it means that the existence of a solution $u \in \overline{K}$ of the associated Riccati equation $Du = -u^2 - a_1u - a_0$ is equivalent to the existence of a Liouvillian solution y over (K, D). Hence, in view of Theorem 2.10, special polynomials are linked to Liouvillian solutions.

Theorem 2.53. *Let (K,D) be a differential field and let $a,b,c \in K$ with $a \neq 0$. Let t be from some differential field extension of (K,D) such that $Dt = at^2 + bt + c$. Then*

1. *$Dv = av^2 + bv + c$ has no solution $v \in K$ and*
2. *$L^{\circledS 12}(y) = 0$ has no solution $y \in K$ where*
$$L(y) := D^2y - \tfrac{Da}{a}Dy + (ac - \tfrac{b^2}{4} + \tfrac{Db}{2} - \tfrac{Da}{a}\tfrac{b}{2})y$$

if and only if t is transcendental over K, $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$, and $S^{irr}_{K[t]:K} = \emptyset$. Furthermore, if these conditions are satisfied and $(\tilde{K},D)$ is a Liouvillian extension of (K,D), then the same conditions are satisfied with K replaced by $\tilde{K}$.

Proof. Assume $Dv = av^2 + bv + c$ has no solution $v \in K$, then by the correspondence $u = -av - \frac{b}{2}$ also the associated Riccati equation of L
$$Du = -u^2 + \tfrac{Da}{a}u - (ac - \tfrac{b^2}{4} + \tfrac{Db}{2} - \tfrac{Da}{a}\tfrac{b}{2})$$
has no solution $u \in K$. Assume further that $L^{\circledS 12}(y) = 0$ has no solution $y \in K$. Then in particular also $L^{\circledS 2}(\tilde{y}) = 0$ has no solution $\tilde{y} \in K$ as otherwise we would have $L^{\circledS 12}(\tilde{y}^6) = 0$ by definition of symmetric powers. Altogether, Theorems 2.39, 2.44, and 2.45 imply that $L(y) = 0$ does not have a Liouvillian solution over (K,D). Consequently, by Theorem 2.35 there is no solution $u \in \overline{K}$ of the associated Riccati equation of L above. Equivalently, by $u = -av - \frac{b}{2}$ there is no $v \in \overline{K}$ such that $Dv = av^2 + bv + c$. To complete this part of the proof we apply Theorem 2.46.

For the converse we assume that t is transcendental over K as well as $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$ and we prove the existence of a nontrivial special polynomial in case condition 1 or 2 is violated. On the one hand, if $Dv = av^2 + bv + c$ has a solution $v \in K$, then $t - v \in S^{irr}_{K[t]:K}$ since it divides its derivative $D(t-v) = at^2 + bt + c - (av^2 + bv + c) = (a\cdot(t+v) + b)(t-v)$. On the other hand, if $L^{\circledS 12}(y) = 0$ has a solution $y \in K$, then there is a special polynomial $t^{12} - \frac{Dy}{y}t^{11} + b_{10}t^{10} \cdots + b_0$ with coefficients $b_0, \ldots, b_{10} \in K$ given by Theorem 2.1 from [UW96].

Moreover, for a Liouvillian extension $(\tilde{K},D)$ of (K,D) recall that, if the conditions are satisfied in K, then in the first part of the proof we showed that there is no $v \in \overline{K}$ such that $Dv = av^2 + bv + c$. Hence by Theorem 2.32 it follows that $Dv = av^2 + bv + c$ does not have a solution v in the algebraic closure of $\tilde{K}$ either, so we can apply Theorem 2.46 in $\tilde{K}$ as well. □

Note that if we apply this theorem to (2.21), then the L obtained coincides with (2.23).

Remark Under some additional assumptions the previous theorem admits some modifications, which may make the algorithmic check faster.

1. If b is the logarithmic derivative of an element from K, then we can replace the linear operator above by
$$L(y) := D^2y - (\tfrac{Da}{a} + b)Dy + acy,$$
which still has unimodular Galois group in this case.

2. Independent of the previous modification, if for $L(y) = 0$ there is no fundamental system $\{y_1, y_2\} \subseteq \overline{K}$, then we can replace $L^{\circledS 12}(y) = 0$ by $L^{\circledS 4}(y) = 0$ in condition 2 in view of the second statement in Theorem 2.45.

3. Even without additional assumptions the last part of the theorem allows us to restrict the algorithmic check of the conditions to a differential subfield of K which still contains the coefficients a, b, c and of which (K, D) is a Liouvillian extension.

$\square$

Now we proceed with the construction of an appropriate differential field for modeling solutions of (2.10). We assume in the following that extending the field K by a v defined by (2.21) satisfies the conditions of Theorem 2.53. For modeling φ_1 and φ_2 we still need to extend $(K(v), D)$ further. The main point is that they are hyperexponential over $(K(v), D)$, or more precisely

$$\frac{D\varphi_1}{\varphi_1} = a_{12}v + a_{11} \tag{2.24}$$

and $\frac{D\varphi_2}{\varphi_2} = a_{22} + \frac{a_{21}}{v}$. Since by assumption in (2.20) we have $a_{12} \neq 0$, we can apply the following corollary of Theorem 2.51 to this situation.

Corollary 2.54. *Let (K, D) be a differential field and let v be a monomial over (K, D) such that $S^{irr}_{K[v]:K} = \emptyset$. Let t be from some differential extension of $(K(v), D)$ such that $\frac{Dt}{t} \in K[v] \setminus K$. Then t is transcendental over $K(v)$, $\mathrm{Const}(K(v,t)) = \mathrm{Const}(K)$, and $S^{irr}_{K(v)[t]:K(v)} = S^{irr,1}_{K(v)[t]:K(v)} = \{t\}$.*

Proof. Theorem 2.16 implies that for any $g \in K(v)^*$ and $k \in \mathbb{Z} \setminus \{0\}$ with $\frac{Dg}{kg} \notin K$ we even have $\frac{Dg}{kg} \notin K[v]$ by definition of v. Hence $\frac{Dt}{t}$ is not the logarithmic derivative of a $K(v)$-radical and Theorem 2.51 implies that t is transcendental over $K(v)$, $\mathrm{Const}(K(v,t)) = \mathrm{Const}(K(v))$, and $S^{irr}_{K(v)[t]:K(v)} = S^{irr,1}_{K(v)[t]:K(v)} = \{t\}$. Lastly, Lemma 2.18 implies $\mathrm{Const}(K(v)) = \mathrm{Const}(K)$. $\square$

So far by Theorem 2.53 and the corollary above we proved that for a solution $(y_1, y_2) = (\varphi_1, \varphi_2)$ of (2.20) which is not Liouvillian over (K, D), first, φ_1 and φ_2 are algebraically independent over K and, second, $\mathrm{Const}(K(\varphi_1, \varphi_2)) = \mathrm{Const}(K)$.

Now we want to complete (φ_1, φ_2) to a fundamental matrix $\Phi := \begin{pmatrix} \varphi_1 & \tilde{\varphi}_1 \\ \varphi_2 & \tilde{\varphi}_2 \end{pmatrix}$ of (2.20). It is straightforward to verify that the Wronskian $w := \det(\Phi)$ and the quotient $\tilde{v} = \frac{\tilde{\varphi}_1}{\varphi_1}$ satisfy the following equations.

$$\frac{Dw}{w} = \mathrm{Tr}(A) \tag{2.25}$$

$$D\tilde{v} = \frac{a_{12}w}{\varphi_1^2} \tag{2.26}$$

So for adjoining such a w to our differential field we can rely on Theorem 2.51 and for adjoining such a $\tilde{v}$ afterwards we can use Theorem 2.50. However we can also exploit the special structure of the defining equations and apply the following corollary of Theorem 2.50.

Corollary 2.55. *Let* (K, D) *be a differential field, let* v *be a nonlinear monomial over* (K, D) *such that* $S^{irr}_{K[v]:K} = \emptyset$, *and let* φ_1 *be hyperexponential over* $(K(v), D)$ *such that* $\frac{D\varphi_1}{\varphi_1} \in K[v]$ *and* $\deg_v(\frac{D\varphi_1}{\varphi_1}) = \deg_v(Dv) - 1 > 0$. *Let* t *be from some differential extension of* $(K(v, \varphi_1), D)$ *such that* $Dt = a\varphi_1^n$ *with* $a \in K^*$ *and* $n \in \mathbb{Z} \setminus \{0\}$. *If there is no* $m \in \mathbb{N}^+$ *such that* $m \operatorname{lc}_v(Dv) + n \operatorname{lc}_v(\frac{D\varphi_1}{\varphi_1}) = 0$, *then* t *is transcendental over* $K(v, \varphi_1)$, $\operatorname{Const}(K(v, \varphi_1, t)) = \operatorname{Const}(K)$, *and* $S^{irr}_{K(v,\varphi_1)[t]:K(v,\varphi_1)} = \emptyset$.

Proof. In order to apply Theorem 2.50 we need to show that $a\varphi_1^n$ is not the derivative of an element from $K(v, \varphi_1)$. Assume the opposite, then there exists $g \in K(v)$ such that $D(g\varphi_1^n) = a\varphi_1^n$. In other words $Dg + n\frac{D\varphi_1}{\varphi_1}g = a$, from which we obtain $g \in K[v]$ by virtue of Theorem 2.16. But $Dg + n\frac{D\varphi_1}{\varphi_1}g$ has degree $\deg_v(g) + \deg_v(Dv) - 1 > 0$ in v with $\operatorname{lc}_v(Dg + n\frac{D\varphi_1}{\varphi_1}g) = \deg_v(g)\operatorname{lc}_v(Dv) + n\operatorname{lc}_v(\frac{D\varphi_1}{\varphi_1})\operatorname{lc}_v(g) \neq 0$. Altogether, it follows that $a\varphi_1^n$ is not the derivative of an element from $K(v, \varphi_1)$. Hence the claim follows by Theorem 2.50. □

We summarize the situation in the following theorem. The proof is an immediate application of the previous results.

Theorem 2.56. *Let* (K, D) *be a differential field and let* $a_{11}, a_{12}, a_{21}, a_{22} \in K$ *such that* $a_{12} \neq 0$, (2.21) *has no solution* $v \in K$, *and* $L^{ⓢ12}(y) = 0$ *has no solution* $y \in K$ *where* L *is given by* (2.23). *Let* $w, v, \varphi_1, \tilde{v}$ *from some differential extension of* (K, D) *with* (2.25), (2.21), (2.24), *and* (2.26), *then with* $\tilde{K} := K(w)$

1. $v, \varphi_1, \tilde{v}$ *are algebraically independent over* $\tilde{K}$,
2. $\operatorname{Const}_D(\tilde{K}(v, \varphi_1, \tilde{v})) = \operatorname{Const}_D(\tilde{K})$,
3. $S^{irr}_{\tilde{K}[v]:\tilde{K}} = S^{irr}_{\tilde{K}(v,\varphi_1)[\tilde{v}]:\tilde{K}(v,\varphi_1)} = \emptyset$ *and* $S^{irr}_{\tilde{K}(v)[\varphi_1]:\tilde{K}(v)} = S^{irr,1}_{\tilde{K}(v)[\varphi_1]:\tilde{K}(v)} = \{\varphi_1\}$, *and*
4. *a fundamental matrix for* (2.20) *is given by*

$$\begin{pmatrix} \varphi_1 & \tilde{v}\varphi_1 \\ v\varphi_1 & \frac{w}{\varphi_1} + v\tilde{v}\varphi_1 \end{pmatrix}.$$

Proof. First, since by definition w is Liouvillian over (K, D), Theorem 2.53 implies that v is transcendental over $\tilde{K}$ and $S^{irr}_{\tilde{K}[v]:\tilde{K}} = \emptyset$. Second, we note $a_{12} \neq 0$ and so we can apply Corollary 2.54 over $\tilde{K}$ to show that φ_1 is transcendental over $\tilde{K}(v)$ and $S^{irr}_{\tilde{K}(v)[\varphi_1]:\tilde{K}(v)} = S^{irr,1}_{\tilde{K}(v)[\varphi_1]:\tilde{K}(v)} = \{\varphi_1\}$. Next, note that there is no $m \in \mathbb{N}^+$ such that $m\operatorname{lc}_v(Dv) - 2\operatorname{lc}_v(\frac{D\varphi_1}{\varphi_1}) = -(m+2)a_{12} = 0$. Hence $\tilde{v}$ is transcendental over $\tilde{K}(v, \varphi_1)$, $\operatorname{Const}(\tilde{K}(v, \varphi_1, \tilde{v})) = \operatorname{Const}(\tilde{K})$, and $S^{irr}_{\tilde{K}(v,\varphi_1)[\tilde{v}]:\tilde{K}(v,\varphi_1)} = \emptyset$ by Corollary 2.55. Last, it is straightforward to verify that the matrix given above has determinant w and is indeed a fundamental matrix for (2.20). □

As a byproduct we obtain the following corollary on the algebraic independence of non-Liouvillian functions.

Corollary 2.57. *Let (K, D) be a differential field and let $a_{11}, a_{12}, a_{21}, a_{22} \in K$ such that (2.20) does not have a nonzero Liouvillian solution over (K, D). Let $\Phi := \begin{pmatrix} \varphi_1 & \tilde{\varphi}_1 \\ \varphi_2 & \tilde{\varphi}_2 \end{pmatrix}$ be a fundamental matrix of (2.20) and let $w := \det(\Phi)$. Then $\varphi_1, \varphi_2, \tilde{\varphi}_1$ are algebraically independent over $K(w)$ and $\mathrm{Const}_D(K(\varphi_1, \varphi_2, \tilde{\varphi}_1, \tilde{\varphi}_2)) = \mathrm{Const}_D(K(w))$.*

In order to determine all algebraic relations of $\varphi_1, \varphi_2, \tilde{\varphi}_1, \tilde{\varphi}_2$ over K we also invoke Theorem 2.51 and the remark following it. This shows that non-Liouvillian solutions of (2.20) at most satisfy algebraic relations of the form $\det(\Phi)^k - g = 0$ where $g \in K$ and $k \in \mathbb{N}^+$.

2.6.3 Inverse functions

It may happen that although we cannot represent a given function in our tower framework we a least can represent its inverse function. We briefly respond to this situation in this section. Assume the function g is preferable over the function g^{-1} and we encounter an integrand $f(x, g^{-1}(x))$ involving g^{-1}. This generalizes the case $f(x, g^{-1}(x)) = g^{-1}(x)^n$ considered in a note by Parker [Par55]. Then, by the change of variable $x = g(u)$ we can transform the integral into a form that is easier to deal with algorithmically.

$$\int f(x, g^{-1}(x))\, dx = \int f(g(u), u) g'(u)\, du \tag{2.27}$$

For example, it can be proven that the Lambert W function is not a Liouvillian function [BCDJ08], but it is the inverse function of the elementary function $g(u) = ue^u$. Already in [CGHJK96] it was proposed to use Risch's algorithm in combination with the change of variables $x = g(u) = ue^u$ in order to integrate expressions involving $g^{-1}(x) = W(x)$. This can be extended to expressions like $W(x^c)$ and the Wright function $\omega(x) = W(e^x)$, which are not Liouvillian either [BCDJ08], by choosing $g(u) = (ue^u)^{1/c}$ and $g(u) = \ln(u) + u$, respectively.

Other functions which can be represented by inverses of Liouvillian functions include the elliptic functions of Jacobi and Weierstraß, for details we refer to Section A.3 in the appendix.

2.7 Liouville's theorem and some refinements

Before we discuss refinements of Liouville's theorem recall that the notion of an elementary integral is defined relative to an underlying differential field and does not require the integral to represent an elementary function. For example, the integral $\int \frac{1}{\ln(x)}\, dx$ is not elementary over the differential field $(\mathbb{Q}(x, \ln(x)), \frac{d}{dx})$, but it is elementary over the differential field $(\mathbb{Q}(x, \ln(x), \mathrm{li}(x)), \frac{d}{dx})$ as it equals $\mathrm{li}(x)$, even though $\mathrm{li}(x)$ is not an elementary function. This also motivates some of the refinements later.

Liouville's theorem provides a very crucial theoretical foundation that facilitates algorithmic computation of elementary integrals over some differential field as it predicts a rather restrictive structure of a possible integral. We recall a modern version of it, which will be used in Section 3.2.

Theorem 2.58. *(Liouville's Theorem [Bro, Thm 5.5.3]) Let (F, D) be a differential field and $C := \mathrm{Const}(F)$. If $f \in F$ has an elementary integral over (F, D), then there are $v \in F$, $c_1, \ldots, c_n \in \overline{C}$, and $u_1, \ldots, u_n \in F(c_1, \ldots, c_n)^*$ such that*

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}. \tag{2.28}$$

In view of this theorem we always can express an elementary integral $\int f$ as the sum of two parts: a $v \in F$, which then is called the *rational part*, and a sum of logarithms $\sum c_i \log(u_i)$, which is called the *logarithmic part* of the integral.

However, we also need refined versions of Liouville's theorem in order to justify several of the results in Chapter 3. As a start, we recall a refinement for reduced elements in $F = K(t)$.

Theorem 2.59. *([Bro, Thm 5.7.1]) Let (K, D) be a differential field and let t be a monomial over (K, D) such that $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. If $f \in K(t)_{red}$ has an elementary integral over $(K(t), D)$, then there are $v \in K(t)_{red}$, $c_1, \ldots, c_n \in \overline{C}$, and $u_1, \ldots, u_n \in S_{K(c_1,\ldots,c_n)[t]:K(c_1,\ldots,c_n)}$ such that (2.28).*

Based on this it is not difficult to prove the following corollary for nonlinear t without special polynomials, which we will utilize in Section 3.3.

Corollary 2.60. *([Bro, Corollary 5.11.1]) Let t be a nonlinear monomial over (K, D) with $S^{irr} = \emptyset$ and let $f \in K[t]$ with $\deg_t(f) < \deg_t(Dt)$. If f has an elementary integral over $(K(t), D)$, then $f \in K$.*

Now we turn to further refinements, which will be important in Section 3.4. These are inspired by Exercise 5.5 in Bronstein's book [Bro]. The following theorem will be used as the main ingredient of Theorem 3.15. Another refinement will be presented in Section 3.5 as Theorem 3.25.

Theorem 2.61. *Let (K, D) be a differential field and let t be a monomial over (K, D) such that $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Let $f \in K$ be such that f has an elementary integral over $(K(t), D)$. Furthermore, let $v \in K(t)_{red}$, $c_1, \ldots, c_n \in \overline{C}$, and $u_1, \ldots, u_n \in S_{K(c_1,\ldots,c_n)[t]:K(c_1,\ldots,c_n)}$ as in Theorem 2.59. Then, in addition, the following properties are satisfied:*

1. *If t is a nonlinear monomial over (K, D), then $\nu_{\frac{1}{t}}(v) \geq 0$.*
2. *If $S^{irr}_{K[t]:K} = S^{irr,1}_{K[t]:K}$, then $v \in K[t]$.*
3. *If $S^{irr}_{K[t]:K} = \emptyset$, then $v \in K[t]$ and $u_1, \ldots, u_n \in K(c_1, \ldots, c_n)$.*

Proof. Note that $\frac{Du_i}{u_i} \in K(c_1, \dots, c_n)[t]$, $Dv \in K(t)_{red}$ and $f \in K$ imply $Dv \in K[t]$. First, assume that t is a nonlinear monomial over (K, D) and $\nu_{\frac{1}{t}}(v) \neq 0$. By Theorem 2.17 we obtain $\nu_{\frac{1}{t}}(v) = d - 1 - \deg_t(Dv)$, where $d := \deg_t(Dt) \geq 2$. Likewise we have $\deg_t\left(\frac{Du_i}{u_i}\right) \leq d - 1$, which yields $\deg_t(Dv) = \deg_t\left(f - \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}\right) \leq d - 1$. Altogether this implies $\nu_{\frac{1}{t}}(v) \geq 0$.

Next, assume that $S^{irr}_{K[t]:K} = S^{irr,1}_{K[t]:K}$ and let $p \in S^{irr}_{K[t]:K}$. Then we have $\nu_p(Dv) \geq 0$ because $Dv \in K[t]$. If $\nu_p(v) < 0$, then Theorem 2.16 would imply $\nu_p(Dv) < 0$. Hence $\nu_p(v) \geq 0$ for all $p \in S^{irr}_{K[t]:K}$, which implies $v \in K[t]$.

Finally, assume that $S^{irr}_{K[t]:K} = \emptyset$. From this it trivially follows that $v \in K[t]$ and using Corollary 2.11 it follows that $u_1, \dots, u_n \in K(c_1, \dots, c_n)$. □

To conclude this section we state a related result on the structure of integrals in a certain type of Liouvillian extensions analogous to Liouville's theorem for elementary integrals.

Theorem 2.62. *Let (K, D) be a differential field and define $C := \mathrm{Const}_D(K)$. Let $w_1, \dots, w_n \in K$ such that no non-trivial C-linear combination has an integral in K. Then for any $f \in K$ that has an integral in $K(t_1, \dots, t_n)$, where $Dt_i = w_i$, there are $v \in K$ and $c_1, \dots, c_n \in C$ such that*

$$f = Dv + \sum_{i=1}^{n} c_i w_i.$$

2.8 Gröbner bases

As we will not deal with Gröbner bases very much, we restrict our presentation to very superficially recalling some notions that we will need in Section 3.2. For a more extensive treatment of the theory of Gröbner bases we refer to standard textbooks on this topic such as [CLO].

Let K be a field and let $K[x_1, \dots, x_n]$ be the (commutative) ring of polynomials in n indeterminates. Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal and let $<$ be an admissible term order, then a set $G \subseteq K[x_1, \dots, x_n]$ is called a *Gröbner basis of I w.r.t. $<$* if $I = \langle G \rangle$ and for every $p \in I \setminus \{0\}$ there is a $g \in G$ such that $\mathrm{lt}(g) | \mathrm{lt}(p)$.

Recall that a Gröbner basis G is called *minimal* if no leading term of an element from G divides the leading term of any other polynomial in G and it is called *reduced* if no leading term of an element from G divides any term of any other polynomial in G. For a given ordering a reduced Gröbner basis consisting of monic polynomials is unique. A proper ideal $I \subseteq K[x_1, \dots, x_n]$ is called *zero-dimensional* if for every $i \in \{1, \dots, n\}$ it contains a nonzero polynomial from $K[x_i]$. Equivalently, if G is a Gröbner basis of I, then I is zero-dimensional if for every $i \in \{1, \dots, n\}$ there is a $g \in G$ such that $\mathrm{lt}(g)$ is of the form x_i^k. The *radical* of an ideal $I \subseteq K[x_1, \dots, x_n]$ is defined as

$$\mathrm{Rad}(I) := \left\{p \in K[x_1, \dots, x_n] \;\middle|\; \exists m \in \mathbb{N}^+ : p^m \in I\right\}$$

and an ideal is called radical if $I = \mathrm{Rad}(I)$.

Later the Gröbner basis of interest can be computed easily by the FGLM algorithm in our setting. This algorithm (see [FGLM93] for a detailed description) converts a given Gröbner basis w.r.t. one ordering, which typically is cheaper to compute, into a Gröbner basis w.r.t. another given ordering, which would typically be computationally expensive to obtain otherwise. The idea of the FGLM algorithm is explained best by considering a natural generalization of the problem, which was treated in [MMM93]. We briefly hint the main idea of the algorithm along these lines, for more details see the two references just given.

Assume we are given a K-linear map $\varphi : K[x_1, \ldots, x_n] \to V$ such that its kernel is a zero-dimensional ideal and we want to compute a Gröbner basis of $\ker(\varphi)$ w.r.t. a given ordering $<$. We start with $G := \emptyset$ and $B := \emptyset$ and proceed through the power products one by one according to the ordering $<$ and at the end G will be the Gröbner basis we are looking for and B will be a basis of $K[x_1, \ldots, x_n]/\ker(\varphi)$. For each power product p, if it is not in $\langle G\rangle$, we check whether there is a linear dependence $\varphi(p) + \sum_i c_i\varphi(b_i) = 0$ with $c_i \in K$ and $b_i \in B$. If so, then by linearity of φ we know that $g := p + \sum_i c_ib_i$ is in our ideal and we include g into G, otherwise we include p into B. Then we proceed with the next bigger power product which is not in $\langle G\rangle$. Since by assumption the ideal $\ker(\varphi)$ is zero-dimensional after finitely many steps there are no power products to consider anymore and the algorithm terminates with G being the reduced Gröbner basis of $\ker(\varphi)$ w.r.t. $<$.

For applying the FGLM algorithm to the Gröbner basis G_1 we would choose $V := K[x_1, \ldots, x_n]/\langle G_1\rangle$ and φ computes the normal form modulo the ideal w.r.t. the first ordering. Also the Buchberger-Möller algorithm, which computes the vanishing ideal of a finite set of given points in $\overline{K}^n$, can be understood this way by choosing $V := \overline{K}^m$ and defining φ to be the evaluation functional at the m given points.

We also will exploit the following structure theorem for lexicographic Gröbner bases of bivariate polynomials.

Theorem 2.63. *(Structure Theorem [Laz85, Thm 1]) Let K be a field, consider the (commutative) polynomial ring $K[x, y]$ with lexicographic ordering $x < y$.*

1. *Let $\{P_0, \ldots, P_m\} \in K[x, y]$ be a minimal Gröbner basis of an ideal in $K[x, y]$ such that $\mathrm{lt}(P_{i-1}) < \mathrm{lt}(P_i)$ for all $i \in \{1, \ldots, m\}$. Then*

$$\forall i \in \{0, \ldots, m\} : P_i = Q_{i+1}\cdot \ldots \cdot Q_{m+1}\cdot R\cdot S_i,$$

where $Q_1, \ldots, Q_{m+1} \in K[x]$, $Q_{m+1} = \mathrm{cont}_y(P_m)$, $R = \mathrm{pp}_y(P_0) \in K[x, y]$, $S_0 = 1$, and $S_1, \ldots, S_m \in K[x, y]$ such that for all $i \in \{1, \ldots, m\}$:

   (a) *S_i is monic w.r.t. y,*

   (b) *$\deg_y(S_{i-1}) < \deg_y(S_i)$, and*

   (c) *$S_i \in \langle Q_{j+1}\cdot \ldots \cdot Q_{i-1}\cdot S_j \mid j \in \{0, \ldots, i-1\}\rangle$.*

2. *Every set of polynomials which satisfies the preceding conditions is a Gröbner basis; it is minimal if and only if $\forall i \in \{1, \ldots, m\} : Q_i \notin K$.*

In Section 3.2 we will also need the following three lemmas which are generalizations of the Lemmas 2.1, 2.2.iii, and 2.3 from [Czi95] with essentially the same proofs. The proofs given here are more detailed and make explicit use of the structure theorem above instead of reproving the relevant parts.

Lemma 2.64. *Let $a, b, c \in K[t]$ with $b \neq 0$ squarefree and $\gcd(b, c) = 1$, let z be an indeterminate over $K[t]$. Then the ideal $I := \langle a - zc, b\rangle \subseteq K[t, z]$ is zero-dimensional and radical. Moreover, $\{b, z - pa\}$ is a minimal Gröbner basis of I w.r.t. lexicographic ordering $t < z$ for $p \in K[t]$ such that $pc \equiv 1 \pmod b$.*

Proof. First, we show that $\{b, z-pa\}$ is a minimal Gröbner basis of I w.r.t. lexicographic ordering $t < z$. Since $\gcd(b, c) = 1$ such a $p \in K[t]$ always exists and let $q \in K[t]$ such that $pc + qb = 1$. Hence we have $(-p)\cdot(a - zc) + (zq)\cdot b = -pa + zpc + zqb = z - pa$, i.e., $z - pa \in I$. On the other hand $(qa)\cdot b + (-c)\cdot(z - pa) = a - zc$. Thus $\{b, z - pa\}$ is a minimal Gröbner basis of I w.r.t. the lexicographic ordering $t < z$.

Now, for proving zero-dimensionality we show that the corresponding algebraic variety of the ideal I is a finite set (alternative proof: read it off from leading terms of Gröbner basis above). To this end, let $\beta_1, \dots \beta_d \in \overline{K}$ be the roots of $b \in K[t]$. From $\gcd(b, c) = 1$ it follows that $c(\beta_i) \neq 0$ for all $i \in \{1, \dots, d\}$. Hence for each β_i there is exactly one $\alpha_i \in \overline{K}$ such that $a(\beta_i) - \alpha_i c(\beta_i) = 0$. So the system of equations $a(t) - z\cdot c(t) = 0, b(t) = 0$ has only finitely many solutions $(t, z) \in \overline{K}^2$.

Next, we show that the radical ideal $\mathrm{Rad}(I)$ is contained in I. Let $r \in \mathrm{Rad}(I)$ and reduce it by $\{b, z - pa\}$ as follows: $r(t, z)$ is reduced by $z - pa$ to $r(t, p(t)a(t))$, which in turn is reduced by b to some $\tilde{r} \in K[t]$ with $\deg(\tilde{r}) < \deg(b)$. In addition, $\tilde{r}$ vanishes on the $\deg(b)$ distinct roots (in $\overline{K}$) of b because of $\tilde{r} \in \mathrm{Rad}(I)$. Altogether this implies $\tilde{r} = 0$, i.e., $r \in I$. □

Lemma 2.65. *Let $a, b, c \in K[t]$ with $b \neq 0$ squarefree and $\gcd(b, c) = 1$, let z be an indeterminate over $K[t]$, and let $\{P_0, \dots, P_m\} \subseteq K[z, t]$ be a minimal Gröbner basis of the ideal $I := \langle a - zc, b\rangle \subseteq K[z, t]$ w.r.t. lexicographic ordering $z < t$ such that $\mathrm{lt}(P_0) < \mathrm{lt}(P_i)$ for all $i \in \{1, \dots, m\}$.*
Then $P_0 \in K[z]$ is the squarefree part of $r(z) := \mathrm{res}_t(a - zc, b) \in K[z]$.

Proof. By the elimination property $\{P_0, \dots, P_m\} \cap K[z]$ is a Gröbner basis of $I \cap K[z]$. Since by Lemma 2.64 the ideal I is zero-dimensional $\{P_0, \dots, P_m\} \cap K[z]$ is not empty. Since P_0 is the basis element with smallest leading term we obtain $P_0 \in K[z]$. From the minimality of the Gröbner basis we conclude $\{P_0, \dots, P_m\} \cap K[z] = \{P_0\}$. So the roots of $P_0 \in K[z]$ are those $\alpha \in \overline{K}$ such that the polynomials $\{P_0(\alpha, t), \dots, P_m(\alpha, t)\} \subseteq \overline{K}[t]$ have a common root in $\overline{K}$. In addition, by Lemma 2.64 the ideal I is radical, hence also $I \cap K[z] = \langle P_0\rangle$ is radical. This implies that P_0 is squarefree.
The roots of $r \in K[z]$ are those $\alpha \in \overline{K}$ such that $a - \alpha c \in \overline{K}[t]$ and b have a common root in $\overline{K}$. Now, $\{a - zc, b\}$ and $\{P_0, \dots, P_m\}$ generate the same ideal (in $K[z, t]$) so by the evaluation homomorphism $z \mapsto \alpha$ also $\{a - \alpha c, b\}$ and $\{P_0(\alpha, t), \dots, P_m(\alpha, t)\}$ generate the same ideal (in $\overline{K}[t]$). Hence the roots of r and P_0 are the same. □

Lemma 2.66. *Let $a, b, c \in K[t]$ with $b \neq 0$ squarefree and $\gcd(b, c) = 1$, let z be an indeterminate over $K[t]$, and let $\{P_0, \dots, P_m\} \subseteq K[z, t]$ be a minimal Gröbner basis of*

the ideal $I := \langle a - zc, b\rangle \subseteq K[z,t]$ *w.r.t. lexicographic ordering* $z < t$ *with* $\mathrm{lt}(P_{i-1}) < \mathrm{lt}(P_i)$ *for all* $i \in \{1,\dots,m\}$. *Furthermore, let* $Q_1,\dots,Q_{m+1} \in K[z]$ *and* $R, S_0,\dots,S_m \in K[z,t]$ *be as in Theorem 2.63.*
Then for any $\alpha \in \overline{K}$ *root of* $r(z) := \mathrm{res}_t(a - zc, b) \in K[z]$ *there is a unique* $i \in \{1,\dots,m\}$ *such that* $Q_i(\alpha) = 0$. *With this* i *we have*

$$S_i(\alpha,t) = \gcd(a - \alpha c, b) \in K(\alpha)[t].$$

Proof. From Lemma 2.65 we know that $R = 1$ and $P_0 = Q_1 \cdot \ldots \cdot Q_{m+1}$ is squarefree and has the same roots as r. So there is a unique $i \in \{1,\dots,m{+}1\}$ such that $Q_i(\alpha) = 0$. Since by Lemma 2.64 I is zero-dimensional we have $\deg(Q_{m+1}) = 0$, otherwise for the roots $\tilde{\alpha} \in \overline{K}$ of Q_{m+1} all $P_j(\tilde{\alpha},t)$ would vanish on all $t \in \overline{K}$ (alternative proof: otherwise we would have $z \mid \mathrm{lt}(P_j)$ for all j). So $i \neq m+1$.

Next, using this i we prove $\forall k \in \{0,\dots,m\} : P_i(\alpha,t) | P_k(\alpha,t)$ by induction on k. For $k < i$ we have $Q_i | P_k$ and hence $P_k(\alpha,t) = 0$; for $k = i$ we have $P_k(\alpha,t) \neq 0$ by the uniqueness of i and the statement is trivial. For $k \in \{i{+}1,\dots,m\}$ Theorem 2.63 implies $S_k \in \langle Q_{j+1} \cdot \ldots \cdot Q_{k-1} \cdot S_j \mid j \in \{0,\dots,k{-}1\}\rangle$. Multiplication by $Q_k \cdot \ldots \cdot Q_{m+1}$ yields $Q_k P_k \in \langle P_j \mid j \in \{0,\dots,k{-}1\}\rangle$. Hence we obtain $Q_k P_k = \sum_{j=0}^{k-1} T_j P_j$ for some $T_j \in K[z,t]$. Evaluation at $z = \alpha$ yields

$$Q_k(\alpha) P_k(\alpha,t) = \sum_{j=0}^{k-1} T_j(\alpha,t) P_j(\alpha,t) \in K(\alpha)[t].$$

By the induction hypothesis each summand of the right hand side is divisible by $P_i(\alpha,t)$. Dividing by $Q_k(\alpha) \in K(\alpha)^*$ concludes the induction step.

Now, from this it follows that $\gcd(P_k(\alpha,t) \mid k {\in} \{0,\dots,m\}) = S_i(\alpha,t)$, note that $S_i(\alpha,t)$ is monic by Theorem 2.63. But we also have $\gcd(P_k(\alpha,t) \mid k {\in} \{0,\dots,m\}) = \gcd(a - \alpha c, b)$, since by the evaluation homomorphism $z \mapsto \alpha$ we know that $\{P_k(\alpha,t) \mid k {\in} \{0,\dots,m\}\}$ and $\{a - \alpha c, b\}$ generate the same ideal in $K(\alpha)[t]$. $\square$

Chapter 3

Indefinite integration

In order to compute antiderivatives or, in other words, indefinite integrals of given functions we need to specify which kind of objects we accept as a result. In principle one could always define a new function as an antiderivative of a given function, but for practical purposes this is not useful unless we can identify an antiderivative which can be expressed in terms of known functions in a suitable manner, which often is called a *closed form* of the integral. There are various notions of what may be considered suitable, for a discussion of some of them we refer to the introduction in Chapter 1. Moreover, we will always consider parametric integration in the sense that there is not only one integrand but several integrands and we are interested in finding all their constant-linear combinations which have an integral of the type specified. This will be of particular importance in Chapter 5. Alternatively, we could think of this situation as having one integrand only which depends linearly on some parameters and we want to determine the possible constant values of those parameters such that there is a corresponding antiderivative of the type specified. In this chapter let (K, D) be a differential field of characteristic 0 and, unless specified otherwise, let t be a monomial over (K, D).

The main contributions to the algorithm for solving Problem 3.2 below, apart from identifying suitable sufficient conditions for Theorem 3.4, consist in the algorithm presented in Theorem 3.9 and in the correct statement of Theorem 3.15. In Theorem 3.10 we also show how Czichowski's algorithm can be extended to this general setting to compute the elementary extensions needed, which is published in [Raa12]. Theorem 3.8 and Section 3.3 contain small generalizations of results in [Bro]. The main algorithm is summarized in Theorem 3.4, which also relies on some contributions made in Chapter 4. The results presented here will almost all be algorithmic in nature as we discussed the underlying theory and most of the results needed in the previous chapter already. In Section 3.5.1 we present a generalization of our algorithm to another class of differential fields, see Theorem 3.30, which is based on a translation of ideas of Campbell [Cam88] into the language of differential fields and on an adapted version of our algorithm given in Theorem 3.9.

One natural algebraic formulation of the problem of indefinite integration is the following, where the integrals sought for are limited to lie in the same differential field as the integrands. Typically this differential field represents a specific set of functions that can be handled, so such an integral automatically is useful to us.

Problem 3.1 (limited integration)**.** *Given: a differential field* (F, D) *and* $f_0, \dots, f_m \in F$.

Find: $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ *and* $g_1, \dots, g_n \in F$ *such that* $\{(g_j, \mathbf{c}_j) \mid j \in \{1, \dots, n\}\}$ *is a basis of the* C*-vector space of all solutions* $(g, \mathbf{c}) = (g, c_0, \dots, c_m) \in F \times C^{m+1}$ *of*

$$Dg = \sum_{i=0}^{m} c_i f_i.$$

This is a special case of the parametric Risch differential equation problem and will be discussed in Section 4.1. It is easy to see that for each $\mathbf{c} \in C^{m+1}$ the corresponding $g \in F$ is unique up to an additive constant from C.

In order to be able to compute more integrals we want to go beyond limited integration and relax the restriction on the form of the integrals. So we will focus on elementary integrals defined in Definition 2.49 instead. In addition to arising as a subproblem in our algorithm for finding elementary integrals, the limited integration problem can also be used to find integrals in a predefined extension of (F, D) of certain type, see Theorem 2.62, and to check the assumption of that theorem.

Problem 3.2 (parametric elementary integration)**.** *Given: a differential field* (F, D) *and* $f_0, \dots, f_m \in F$.

Find: $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$, *where* $C := \mathrm{Const}(F)$, *and corresponding* $g_1, \dots, g_n$ *from some elementary extensions of* (F, D) *such that*

$$Dg_j = (f_0, \dots, f_m) \cdot \mathbf{c}_j$$

and $\{\mathbf{c}_j \mid j \in \{1, \dots, n\}\}$ *is a basis of the* C*-vector space of all* $\mathbf{c} = (c_0, \dots, c_m) \in C^{m+1}$ *for which* $\sum\limits_{i=0}^{m} c_i f_i$ *has an elementary integral over* (F, D).

Liouville's theorem (see Theorem 2.58) states that it suffices to introduce logarithms of functions represented by elements in F. In particular, the g_j will be given in the following form

$$g = v + \sum_{k=1}^{l} \sum_{Q_k(\alpha)=0} \alpha \log(S_k(\alpha)),$$

where $v \in F$, $Q_1, \dots, Q_l \in C[z]$ are squarefree and $S_1, \dots, S_l \in F[z]$, which we consider suitable here. For clarification it should be mentioned that $(f_0, \dots, f_m) \cdot \mathbf{c}$ refers to the scalar product of vectors and will often be used to denote the linear combination $\sum_{i=0}^{m} c_i f_i$ determined by the entries of the vector $\mathbf{c}$. Our aim is to solve the problem of parametric elementary integration over $(F, D) := (K(t), D)$. It turns out that we are able give a decision procedure under some additional assumptions based on the following definition.

Definition 3.3. *We call a differential field* $(F, D) = (C(t_1, \dots, t_n), D)$ admissible, *if*

1. *all* t_i *are algebraically independent over* C,
2. $\mathrm{Const}(F) = C$, *and*

3. for each t_i and $F_{i-1} := C(t_1, \dots, t_{i-1})$ either

(a) t_i is a Liouvillian monomial over (F_{i-1}, D), or

(b) there is a $q \in F_{i-1}[t_i]$ with $\deg(q) \geq 2$ such that

i. $Dt_i = q(t_i)$ and

ii. $Dy = q(y)$ does not have a solution $y \in \overline{F_{i-1}}$.

In addition we require in an admissible differential field that we can solve all problems discussed in Section 2.4 over each F_{i-1}, including factorization of polynomials.

Before stating the main result let us comment on the definition of admissible differential fields. Note that the very last condition on q is used to ensure that there are no special polynomials, i.e., $S^{irr}_{F_{i-1}[t_i]:F_{i-1}} = \emptyset$, in a nonlinear monomial t_i by Theorem 2.46. Additionally, Theorem 2.32 ensures that such t_i cannot be found in any Liouvillian extension of (F_{i-1}, D). For a given differential field with $\deg_{t_i}(Dt_i) \leq 2$ for all i, as it arises in modeling the Liouvillian and non-Liouvillian functions discussed in Section 2.6, the results in that section provide criteria for checking inductively that the field is admissible. The main result of this thesis can now be stated as follows.

Theorem 3.4. *Let $(F, D) = (C(t_1, \dots, t_n), D)$ be an admissible differential field with the restriction that for any $i, j, k \in \{1, \dots, n\}$, $i < j < k$, such that t_k is a Liouvillian monomial and t_i, t_j are non-Liouvillian monomials none of the monomials $t_{i+1}, \dots, t_{j-1}$ is allowed to be hyperexponential.*
Then we can solve the parametric elementary integration problem over (F, D).

The description of the algorithm, which in particular gives a proof of this theorem, extends over this chapter and Chapter 4. Even though the theorem is stated only for towers of monomial extensions which satisfy certain conditions the algorithm to be presented can still be applied heuristically to fields where some of the conditions do not hold. In that case it may happen that not all solutions are found, but in practice it still may happen that at least some integrals are found. Any result returned, consisting of a g from some elementary extension of (F, D) and a $\mathbf{c} \in C^{m+1}$, will at least be a correct solution of $Dg = (f_0, \dots, f_m) \cdot \mathbf{c}$ as long as all t_i are algebraically independent over C. If there is an algebraic relation among the generators, then the algorithm might lead to a division by zero as well. We will discuss heuristic generalizations to more generals fields in Section 3.5.

Before we give a proof of the theorem above by combining all the results that will follow, we briefly present an overview of the structure of the algorithm. The algorithm follows the general recursive structure of its precursors—Risch's algorithm [Ris69] and generalizations like those in [SSC85, Bro90a]—proceeding through the tower of transcendental extensions one by one. Starting with the integrands in $F = C(t_1, \dots, t_n)$ at each step the focus lies entirely on the generator $t_i =: t$ of the current level in the tower, integrands from $C(t_1, \dots, t_i) =: K(t)$ are reduced to integrands from the differential subfield $C(t_1, \dots, t_{i-1}) = K$, portions of the integral can be computed during this reduction, and the algorithm proceeds recursively. This means that intuitively one should think of K and t in this chapter to be some F_{i-1} and t_i of some admissible differential field, in particular both (K, D) and $(K(t), D)$ are admissible themselves. However, we do not make this a formal requirement as many results are more general. At each level in the tower the computation goes through the following phases.

For computing elementary integrals over $(K(t), D)$ we start by applying Hermite reduction for reducing the denominator of the integrands, which initially are arbitrary elements of $K(t)$. Secondly, we apply the residue criterion, which tells us how to construct some of the logarithms needed in the integral and also produces conditions on the linear combinations of the integrands. The remaining integrands are reduced elements of $K(t)$ and the next phase determines bounds on the denominator and degree of the integral and computes its coefficients by comparing coefficients, which leads to auxiliary problems in K. This gives conditions on the linear combinations of the integrands as well and the remaining integrands are from K. Still, as a final phase we have to reduce the problem of elementary integration over $(K(t), D)$ to elementary integration over (K, D) for these integrands, which may add additional integrands from K to consider in the linear combinations.

Altogether, this reduces the problem of parametric elementary integration over $(K(t), D)$ to the problem of parametric elementary integration over (K, D) and we proceed recursively. The following sections reflect the four phases of the algorithm as outlined above and present the computations in detail; solving the auxiliary problems is deferred to Chapter 4. Utilizing the results discussed in all those sections and taking care of the conditions they impose on the differential field we now prove Theorem 3.4. Note that the base case of the recursion is parametric elementary integration over the trivial differential field $(C, 0)$, which is trivial indeed as any elementary extension (E, D) of $(C, 0)$ only contains constants, i.e., $\mathrm{Const}(E) = E$.

Proof of Theorem 3.4. The proof proceeds by induction on $n \in \mathbb{N}$.
$n = 0$: This case means $F = C$. We compute a basis $\mathbf{c}_1, \dots, \mathbf{c}_k \in C^{m+1}$ of the C-vector space $\{\mathbf{c} \in C^{m+1} \mid (f_0, \dots, f_m) \cdot \mathbf{c} = 0\}$ and set $g_j := 0$ for all $j \in \{1, \dots, k\}$. This trivially solves the problem since for any $f \in C$ which has an elementary integral over $(F, D) = (C, 0)$ we obtain $f = 0$ by (2.28).
$n > 0$: Let $K := C(t_1, \dots, t_{n-1})$, so $F = K(t_n)$ and $\mathrm{Const}(K(t_n)) = \mathrm{Const}(K)$. The admissible field (K, D) also satisfies the restriction on the ordering of monomials, so by induction hypothesis we can solve the parametric elementary integration problem over (K, D). Next, by the computability assumptions made on the field (F, D) the assumptions of Theorem 3.16 are satisfied. Now, we distinguish two cases following Definition 3.3.
Case 1: If t_n is a Liouvillian monomial, then (K, D) also satisfies the assumptions of Theorem 4.2 by the restriction on the ordering of the monomials we imposed. Hence we can solve Problems 3.1 and 4.1 in (K, D). Finally, Theorem 3.17 or 3.18 implies that we can solve the parametric elementary integration problem also over (F, D), depending on whether t_n is a primitive or a hyperexponential monomial respectively.
Case 2: If t_n is a nonlinear monomial, then the assumptions of Theorem 3.19 are satisfied, which implies that we can solve the parametric elementary integration problem over (F, D) as well. □

Remark Since this aspect will not be stressed in what follows, we want to emphasize here that, if desired, at any intermediate step of the algorithm one can make the

$\mathbf{c}_1, \ldots, \mathbf{c}_n$ linearly independent over C by simply computing a basis and computing the corresponding combinations of the g_j. $\square$

3.1 Reducing the denominator

As a first step for finding integrals of elements from $K(t)$ we can apply a procedure that is called Hermite reduction. It extracts part of the integral and leaves integrands with simpler denominators. Since Hermite reduction can always be done this step does not restrict the possible linear combinations $\sum_{i=0}^{m} c_i f_i$ in any way. Hence it can be applied to each f_i independently after splitting off the reduced part of f_i.

Theorem 3.5. *([Bro, Thm 5.3.1]) Let $a, b \in K[t]$ such that no $p \in S^{irr}_{K[t]:K}$ divides b. Then using the extended Euclidean algorithm (EEA) in $K[t]$ we can compute $g \in K(t)$ such that $\frac{a}{b} - Dg$ is simple.*

3.2 Finding field extensions

For determining which field extensions are necessary in order to represent an elementary integral over $(K(t), D)$ we need a constructive version of Liouville's theorem. The main theoretical tool for achieving this is the Rothstein-Trager resultant given by (3.1) below. Theorems 3.9 and 3.10 contain new relevant algorithms. Before we can prove the correctness of these algorithms we need some properties of the Rothstein-Trager resultant. The following lemmas are slightly generalized versions of Lemma 4.4.3 and Theorem 4.4.3 from [Bro]. They exhibit the fundamental relation of residues $\operatorname{res}_q(\frac{a}{b})$, the Rothstein-Trager resultant $\operatorname{res}_t(a - zDb, b)$, and gcd's of the form $\gcd(a - \alpha Db, b)$.

Lemma 3.6. *Let $q \in K[t]$ be irreducible with $\gcd(q, Dq) = 1$ and let $a, b \in K[t]$ with $\nu_q(b) = 1$. Then for any $\alpha \in K$*

$$q | \gcd(a - \alpha Db, b) \quad \Longleftrightarrow \quad \operatorname{res}_q\left(\frac{a}{b}\right) = \alpha.$$

Proof. Since $\nu_q(b) = 1$ implies $q|b$, we have that $q|\gcd(a - \alpha Db, b)$ is obviously equivalent to $q|(a - \alpha Db)$. From $\nu_q(b) = 1$ by Theorem 2.16 it also follows that $\nu_q(Db) = 0$, hence $q|(a - \alpha Db)$ is equivalent to $\alpha = \pi_q\left(\frac{a}{Db}\right)$. To complete the proof, we note that $\pi_q\left(\frac{a}{Db}\right) = \operatorname{res}_q(\frac{a}{b})$ by Lemma 2.22. $\square$

Lemma 3.7. *Let $a, b \in K[t]$ with $b \neq 0$ and $\gcd(b, Db) = 1$ and let z be an indeterminate over $K[t]$. Define $r := \operatorname{res}_t(a - zDb, b) \in K[z]$ as in (3.1) below. Then for any $\alpha \in K$*

$$r(\alpha) = 0 \quad \Longleftrightarrow \quad \exists q \in K[t] \text{ irred.} : q|b \wedge \operatorname{res}_q\left(\frac{a}{b}\right) = \alpha.$$

Proof. We make use of the fact that $\forall \beta \in K : r(\beta) = 0 \Leftrightarrow \deg(\gcd(a - \beta Db, b)) > 0$. First assume that $\deg(\gcd(a - \alpha Db, b)) > 0$, i.e., there exists $q \in K[t]$ irreducible such

that $q|\gcd(a-\alpha Db,b)$. Since $\gcd(b,Db)=1$ we infer $\nu_q(b)=1$ and $\gcd(q,Dq)=1$. Now we obtain $\operatorname{res}_q(\frac{a}{b})=\alpha$ by Lemma 3.6.
Conversely, assume that there exists $q\in K[t]$ irreducible with $q|b$ such that $\operatorname{res}_q(\frac{a}{b})=\alpha$. From $\gcd(b,Db)=1$ it follows that $\nu_q(b)=1$ and $\gcd(q,Dq)=1$. Lemma 3.6 now implies $q|\gcd(a-\alpha Db,b)$, i.e., $\deg(\gcd(a-\alpha Db,b))>0$. $\square$

The following important theorem is a corrected and stronger version of Theorem 5.6.1 from [Bro] and can be considered the main theorem on the Rothstein-Trager resultant. Although all necessary proof ingredients can be adapted in a straightforward way, we give the proof explicitly.

Theorem 3.8. *Assume $C:=\operatorname{Const}(K(t))=\operatorname{Const}(K)$. Let $a,b\in K[t]$ with $b\neq 0$ and $\gcd(b,Db)=1$ and let z be an indeterminate over $K[t]$. Define*

$$r:=\operatorname{res}_t(a-zDb,b)\in K[z] \tag{3.1}$$

and

$$g:=\sum_{r(\alpha)=0}\alpha\frac{Dg_\alpha}{g_\alpha}\in\overline{K}(t), \tag{3.2}$$

where $g_\alpha:=\gcd(a-\alpha Db,b)\in K(\alpha)[t]$ for each root $\alpha\in\overline{K}$ of r.

1. *Then $g\in K(t)$ and $\frac{a}{b}-g\in K[t]$.*
2. *If there exists $h\in K(t)_{red}$ such that $h+\frac{a}{b}$ has an elementary integral over $(K(t),D)$, then all roots $\alpha\in\overline{K}$ of r are in $\overline{C}$.*
3. *If E is an algebraic extension of C such that there are $h\in K(t)_{red}$, $v\in K(t)$, $c_1,\dots,c_n\in E$, and $u_1,\dots,u_n\in EK(t)$ with $h+\frac{a}{b}=Dv+\sum_{i=1}^{n}c_i\frac{Du_i}{u_i}$, then E contains all roots $\alpha\in\overline{K}$ of r.*

Proof. Let $r=r_1^{k_1}\cdot\ldots\cdot r_n^{k_n}$ be a factorization of r into irreducibles in $K[z]$. Then for each $i\in\{1,\dots,n\}$ we have that $\sum_{r_i(\alpha)=0}\alpha\frac{Dg_\alpha}{g_\alpha}\in K(t)$, hence $g=\sum_{i=1}^{n}\sum_{r_i(\alpha)=0}\alpha\frac{Dg_\alpha}{g_\alpha}\in K(t)$. Since for all roots $\alpha\in\overline{K}$ of r by definition $g_\alpha|b$ in $K(\alpha)[t]$ we obtain $\operatorname{den}(g)|b$ in $K[t]$. Now let $q\in\overline{K}[t]$ be an irreducible factor of b in $\overline{K}[t]$ then $\gcd(q,Dq)=1$ and $\beta:=\operatorname{res}_q(\frac{a}{b})\in\overline{K}$. Then $\nu_q(g_\alpha)=0$ for all $\alpha\in\overline{K}\setminus\{\beta\}$ by Lemma 3.6 and by Lemma 3.7 we have $r(\beta)=0$. Altogether, this implies $\sum_{r(\alpha)=0}\alpha\nu_q(g_\alpha)=\beta\nu_q(g_\beta)=\beta$. Hence by Lemma 2.21 we obtain $\operatorname{res}_q(\frac{a}{b}-g)=\operatorname{res}_q(\frac{a}{b})-\sum_{r(\alpha)=0}\alpha\nu_q(g_\alpha)=0$ independent of β and q. From this it follows that $\frac{a}{b}-g\in K[t]$ by Lemma 2.20.

Next, we prove statement 2. Let $\alpha\in\overline{K}$ be a root of r, then by Theorem 2.58 there are $v\in K(t)$, $c_1,\dots,c_n\in\overline{C}$, and $u_1,\dots,u_n\in K(c_1,\dots,c_n,t)^*$ such that $h+\frac{a}{b}=Dv+\sum_{i=1}^{n}c_i\frac{Du_i}{u_i}$. Then by Lemma 3.7 applied in $K(c_1,\dots,c_n)[t]$ there exists an irreducible

$q \in K(c_1, \ldots, c_n)[t]$ with $\gcd(q, Dq) = 1$ and $\operatorname{res}_q(\frac{a}{b}) = \alpha$. Hence by Lemma 2.23 we obtain $\alpha = \operatorname{res}_q(\frac{a}{b}) = \sum_{i=1}^{n} c_i \nu_q(u_i) \in \overline{C}$.

For proving the last statement let $\alpha \in \overline{K}$ be a root of r again. By Lemma 3.7 applied in $\overline{K}[t]$ there exists an irreducible $q \in \overline{K}[t]$ with $\gcd(q, Dq) = 1$ and $\alpha = \operatorname{res}_q(\frac{a}{b})$. Consequently, we have $\alpha = \sum_{i=1}^{n} c_i \nu_q(u_i) \in E$ by Lemma 2.23. □

Remark Note that the condition $\operatorname{Const}(K(t)) = \operatorname{Const}(K)$, which was omitted in [Bro], is essential in the theorem above, for otherwise statement 2 need not be true. E.g., let $(K, D) := (\mathbb{Q}(x), \frac{d}{dx})$ and let t be transcendental over K with $Dt = 1$, then $t - x \in \operatorname{Const}(K(t)) \setminus \operatorname{Const}(K)$. With $a = t - x$ and $b = t$ we have $r = z + x$. So the only root $-x$ is not in $\overline{\operatorname{Const}(K)}$, not even in $\overline{\operatorname{Const}(K(t))}$, nevertheless $\frac{a}{b} = a\frac{Db}{b} = D(a \log(b))$ has an elementary integral over $(K(t), D)$. □

The (proof of the) following theorem provides an effective way to exploit Theorem 3.8 for parametric elementary integration and has been one of the missing building blocks for a full parametric algorithm in [Bro]. In [SSC85] a different approach has been taken instead, essentially relying on irreducible factorization of b in $\overline{C}K[t]$ with subsequent partial fraction decomposition of all $\frac{a_j}{b}$ (in the notation of the following theorem).

Theorem 3.9. *Assume $C := \operatorname{Const}(K(t)) = \operatorname{Const}(K)$ and that we can find a basis for the constant solutions of linear systems with coefficients from K (see Section 2.4.3). Let $a_0, \ldots, a_m, b \in K[t]$ with $b \neq 0$ and $\gcd(b, Db) = 1$ and let z be an indeterminate. Then using half-extended GCDs (i.e. modular inverses) in $K[t]$ we can compute linear independent $\mathbf{c}_1, \ldots, \mathbf{c}_n \in C^{m+1}$ such that*

1. *If $h + \frac{(a_0,\ldots,a_m)\cdot\mathbf{c}}{b} \in K(t)$ has an elementary integral over $(K(t), D)$ for some $\mathbf{c} \in C^{m+1}$ and $h \in K(t)_{red}$, then $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \ldots, n\} \exists r \in C[z] : \frac{(a_0,\ldots,a_m)\cdot\mathbf{c}_j}{b} - \sum_{r(\alpha)=0} \alpha \frac{Dg_\alpha}{g_\alpha} \in K[t]$,*
   where $g_\alpha := \gcd((a_0, \ldots, a_m) \cdot \mathbf{c}_j - \alpha Db, b) \in K(\alpha)[t]$ for all roots $\alpha \in \overline{C}$ of r.

Proof. First, we want to construct $q_0, \ldots, q_m \in K[t]$ such that $\deg(q_i) < \deg(b)$ and

$$\forall \beta \in \overline{K}, b(\beta) = 0 : q_i(\beta) = D\left(\frac{a_i(\beta)}{(Db)(\beta)}\right) \tag{3.3}$$

for all $i \in \{0, \ldots, m\}$. To this end, by the half-extended Euclidean algorithm in $K[t]$ we compute for each $i \in \{0, \ldots, m\}$ polynomials $p_i, \tilde{p}_i \in K[t]$ such that $\deg(p_i) < \deg(b)$, $\deg(\tilde{p}_i) < \deg(b)$, and

$$a_i \equiv p_i Db \pmod{b} \tag{3.4}$$

$$\frac{dp_i}{dt} \cdot \kappa_D b \equiv \tilde{p}_i \frac{db}{dt} \pmod{b}. \tag{3.5}$$

Since $\gcd(b, Db) = 1$ implies $\gcd(b, \frac{db}{dt}) = 1$ such p_i and $\tilde{p}_i$ exist. Now we compute

$$q_i := \kappa_D p_i - \tilde{p}_i$$

having $\deg(q_i) < \deg(b)$ and arrange the coefficients in a matrix $A \in K^{\deg(b)\times(m+1)}$ by

$$A := (\mathrm{coeff}(q_i, t^j))_{j,i},$$

where $j \in \{0, \dots, \deg(b)-1\}$ and $i \in \{0, \dots, m\}$. Finally, we compute a C-vector space basis $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ of $\ker(A) \cap C^{m+1}$.

For verifying (3.3) we take $\beta \in \overline{K}$ such that $b(\beta) = 0$ and calculate $p_i(\beta) = \frac{a_i(\beta)}{(Db)(\beta)}$ by (3.4) as well as $\tilde{p}_i(\beta) = \frac{\frac{dp_i}{dt}(\beta)\cdot(\kappa_D)(\beta)}{\frac{db}{dt}(\beta)} = \frac{dp_i}{dt}(\beta)\frac{D(b(\beta)) - \frac{db}{dt}(\beta)\cdot D\beta}{\frac{db}{dt}(\beta)} = -\frac{dp_i}{dt}(\beta)\cdot D\beta$ by (3.5) and Lemma 2.4. Form this we obtain $q_i(\beta) = (\kappa_D p_i)(\beta) - \tilde{p}_i(\beta) = (\kappa_D p_i)(\beta) + \frac{dp_i}{dt}(\beta)\cdot D\beta = D(p_i(\beta)) = D\left(\frac{a_i(\beta)}{(Db)(\beta)}\right)$ using Lemma 2.4 again. Now let $\mathbf{c} \in C^{m+1}$ be fixed and we define $q := (q_0, \dots, q_m) \cdot \mathbf{c} \in K[t]$. Then, by construction $\deg(q) < \deg(b)$. The roots of $r := \mathrm{res}_t((a_0, \dots, a_m) \cdot \mathbf{c} - zDb, b) \in K[z]$ are those $\alpha \in \overline{K}$ such that there exists a $\beta \in \overline{K}$ with $b(\beta) = 0$ and $(a_0(\beta), \dots, a_m(\beta)) \cdot \mathbf{c} - \alpha\cdot(Db)(\beta) = 0$. Hence if $\beta \in \overline{K}$ ranges over the roots of b then $\alpha = \frac{(a_0(\beta),\dots,a_m(\beta))\cdot\mathbf{c}}{(Db)(\beta)}$ ranges over the roots of r. By (3.3) this implies

$$\{q(\beta) \mid \beta \in \overline{K}, b(\beta) = 0\} = \{D\alpha \mid \alpha \in \overline{K}, r(\alpha) = 0\}. \tag{3.6}$$

For verifying the first part of the statement assume that there exists $h \in K(t)_{red}$ such that $h + \frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b} \in K(t)$ has an elementary integral over $(K(t), D)$. By Theorem 3.8.2 we then have that $\alpha \in \overline{C}$ for all roots of r, i.e., $q(\beta) = 0$ for all roots $\beta \in \overline{K}$ of b by (3.6). Since b is squarefree it has $\deg(b)$ distinct roots in $\overline{K}$ and it follows that $q = 0$. Consequently, by definition we have $A \cdot \mathbf{c} = 0$, i.e., $\mathbf{c} \in \mathrm{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ as required. For verifying the second part of the statement we fix some $j \in \{1, \dots, n\}$ and assume $\mathbf{c} = \mathbf{c}_j$ above. Then $q = (q_0, \dots, q_m) \cdot \mathbf{c}_j = (1, t, \dots, t^{\deg(b)-1}) \cdot A \cdot \mathbf{c}_j = 0$. So by (3.6) all roots $\alpha \in \overline{K}$ of r lie in $\overline{C}$. Therefore $\frac{r}{\mathrm{lc}(r)} \in C[z]$ and it fulfils the statement by Theorem 3.8.1. □

Still, for computing elementary integrals using Theorem 3.9 we would need to compute all the g_α as gcd's in various $K(\alpha)[t]$. There are two methods for avoiding gcd computation in algebraic extensions at this point. In [Bro] it is shown how the idea of Lazard, Rioboo and Trager of using the subresultant PRS for computing the Rothstein-Trager resultant (3.1) to obtain the g_α can be carried over from rational functions to this general setting of monomials t. We do not discuss this here. Instead we show how Czichowski's idea of using a bivariate Gröbner basis to obtain the g_α carries over from rational functions to this general setting as well. This is also published in [Raa12].

Theorem 3.10. *Assume $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$ and that we can find a solution of linear systems with coefficients from K if there is one. Let $a, b \in K[t]$ with $b \neq 0$ and $\gcd(b, Db) = 1$ and let z be an indeterminate. Then using the half-extended Euclidean algorithm in $K[t]$ we can compute $Q_1, \dots, Q_m \in K[z]$ and $S_1, \dots, S_m \in K[z, t]$ such that*

1. *$r \in K[z]$ as defined in (3.1) has all its roots $\alpha \in \overline{K}$ lying in $\overline{C}$ if and only if $Q_1, \dots, Q_m \in C[z]$ and*

2. *$Q_1, \ldots, Q_m$ are squarefree, $S_1, \ldots, S_m$ are monic w.r.t. t and*

$$\frac{a}{b} - \sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \frac{DS_i(\alpha, t)}{S_i(\alpha, t)} \in K[t].$$

Proof. First, using the half-extended Euclidean algorithm in $K[t]$ we compute $p \in K[t]$ such that

$$pDb \equiv 1 \pmod{b}.$$

Then $\{b, z - pa\} \subseteq K[z, t]$ is a Gröbner basis of $\langle a - zDb, b\rangle$ w.r.t. lexicographic ordering $t < z$ by Lemma 2.64. From this, by the FGLM algorithm (see Section 2.8), we compute a monic minimal Gröbner basis $\{P_0, \ldots, P_m\} \subseteq K[z, t]$ for the same ideal but w.r.t. lexicographic ordering $z < t$, with $\operatorname{lt}(P_{i-1}) < \operatorname{lt}(P_i)$ for all $i \in \{1, \ldots, m\}$. By the assumption on finding solutions of linear systems over K and Lemma 2.64 we can do this. Next, for $i \in \{0, \ldots, m\}$ we extract

$$R_i := \operatorname{lc}_t(P_i) \in K[z]$$

and, finally, we compute for $i \in \{1, \ldots, m\}$

$$Q_i := \frac{R_{i-1}}{R_i} \in K(z) \quad \text{and} \quad S_i := \frac{P_i}{R_i} \in K(z)[t].$$

Now we verify the desired properties. By construction $S_1, \ldots, S_m$ are monic w.r.t. t. Additionally, since the ideal is zero-dimensional we have $\operatorname{lc}_t(P_m) = \operatorname{lc}(P_m) = 1$ and $\deg_t(P_0) = 0$, hence $\operatorname{cont}_t(P_m) = 1$ and $\operatorname{pp}_t(P_0) = 1$. So by Theorem 2.63 we get $Q_1, \ldots, Q_m \in K[z]$, $S_1, \ldots, S_m \in K[z, t]$ and $P_0 = Q_1 \cdot \ldots \cdot Q_m$. Now Lemma 2.65 implies that $\{\alpha \in \overline{K} \mid r(\alpha) = 0\}$ is the disjoint union of $\{\alpha \in \overline{K} \mid Q_i(\alpha) = 0\}$ for $i \in \{1, \ldots, m\}$ and $Q_1, \ldots, Q_m$ are squarefree. From this assertion 1 follows trivially since by construction $\operatorname{lc}(Q_i) = \frac{\operatorname{lc}(P_{i-1})}{\operatorname{lc}(P_i)} = 1$. Also assertion 2 follows immediately using Theorem 3.8.1 and Lemma 2.66. □

Remark Regarding the algorithmic efficiency in the proof of Theorem 3.10 note the following:

1. The Gröbner basis $\{b, z - pa\}$ of I is minimal. Computing $p \in K[t]$ with $\deg(p) < \deg(b)$ such that $pDb \equiv a \pmod{b}$ instead, we would obtain $\{b, z - p\}$ as a reduced Gröbner basis for I, which shortens computation of normal forms in the FGLM algorithm.

2. During execution of the FGLM algorithm $P_0 \in K[z]$ is the first element of the Gröbner basis that is computed. In view of Theorems 3.8.2 and 3.10.1 this can be used as a necessary criterion whether $h + \frac{a}{b}$ can have an elementary integral over $(K(t), D)$ without computing the full Gröbner basis $\{P_0, \ldots, P_m\}$.

3. It can be shown that $\deg(b) = \dim_K(K[z, t]/I) = \sum_{i=1}^{m} \deg(Q_i) \deg_t(S_i)$. This can be exploited during the FGLM algorithm in the following way. When computing P_k

we consider all partitions of $\deg(b) - \sum_{i=1}^{k-1} \deg(Q_i) \deg_t(S_i)$ into $m_0 := \deg_z(\mathrm{lt}(P_{k-1}))$ parts where each part is greater than $\deg_t(S_{k-1})$. By looking at the size m_1 and multiplicity m_2 of the smallest part in each of those partitions we obtain restrictions on the possible leading terms $z^{m_0-m_2}t^{m_1}$ of P_k. Thereby we can identify some steps in the FGLM algorithm where the linear system will not have a solution. More explicitly, exactly the monomials $1, t, \ldots, t^{\deg_t(P_m)-1}$ can be dropped from the candidates for leading monomials.

4. Defining $S_i := P_i \in K[z, t]$ instead of computing the quotient $\frac{P_i}{R_i}$ we would retain all necessary properties (except monicity) since $\gcd(Q_i, P_i) = 1$. In this case we still have $\sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \frac{DS_i(\alpha,t)}{S_i(\alpha,t)} - g = \sum_{i=1}^{m} \sum_{Q_i(\alpha)=0} \alpha \frac{DR_i(\alpha)}{R_i(\alpha)} \in K$, where g is as in (3.2).

□

3.3 Integration of reduced integrands

With the algorithms presented so far we can reduce integrands from $K(t)$ to integrands from $K(t)_{red}$, we will summarize this in Theorem 3.16 later. In this section we present results for further reducing integrands from $K(t)_{red}$ to integrands from K. These not only depend on $\deg_t(Dt)$ but also upon the knowledge of $S^{irr}_{K[t]:K}$. Therefore, we cannot treat all monomials t exactly the same way, but have to introduce small variations for different cases.

A simplified view on the setting reveals the following principle for integration of polynomials from $K[t]$. For (part of) the integral of a polynomial $f \in K[t]$ we make the ansatz

$$g = \sum_{k=1}^{m} g_i t^i \in K[t],$$

where $m := \deg_t(f) + 1 - d$ and $d := \deg_t(Dt)$, cf. property (2.3). Then we compare the coefficients of the powers of t in $Dg = f$ starting from $t^{\deg(f)+\max(1-d,0)}$ down to $t^{\max(d,1)}$. Thereby we obtain equations for the $g_i \in K$. If $d \geq 2$, then these are trivial to solve, but for $d \leq 1$ this leads to solving certain differential equations in (K, D), which we treat in Chapter 4. The following subsections will fill in the correct details and will turn this into a rigorous algorithm. We treat the two types of Liouvillian monomials as well as the case of nonlinear monomials for which $S^{irr}_{K[t]:K} = \emptyset$ is known.

3.3.1 Primitive extensions

For primitive monomials t we know $K(t)_{red} = K[t]$ and we have to consider the possible drop in the degree that may occur when differentiating a polynomial from $K[t]$ with constant leading coefficient. The following theorem is a straightforward generalization of Theorem 5.8.1 in [Bro].

Theorem 3.11. *Let t be primitive over (K, D) and $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. If we can solve the limited integration problem in (K, D), then for any $a_0, \ldots, a_m \in K(t)_{red}$ we can compute $\mathbf{c}_1, \ldots, \mathbf{c}_n \in C^{m+1}$ and $b_1, \ldots, b_n \in K[t]$ such that:*

1. *If $\sum_{i=0}^{m} c_i a_i = (a_0, \ldots, a_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$ for some $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \mathrm{span}_C\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \ldots, n\} : (a_0, \ldots, a_m) \cdot \mathbf{c}_j - Db_j \in K$.*

Proof. From Theorem 2.50 we know that $S = K$, hence $a_0, \ldots, a_m \in K[t]$. We prove the statement by induction on $N = \max(\max_i \deg_t(a_i), 0)$.
$N = 0$: Let $\mathbf{c}_1, \ldots, \mathbf{c}_{m+1} \in C^{m+1}$ be a basis of C^{m+1}, then with $b_1 = \cdots = b_{m+1} = 0$ the statement is trivially fulfilled.
$N > 0$: By assumption we can solve the limited integration problem in (K, D). In particular we can compute $\mathbf{e}_1, \ldots, \mathbf{e}_{n_N} \in C^{m+1}$, $\bar{c}_1, \ldots, \bar{c}_{n_N} \in C$ and $\bar{b}_1, \ldots, \bar{b}_{n_N} \in K$ such that the set $\{(\bar{b}_j, \mathbf{e}_j, \bar{c}_j) \mid j \in \{1, \ldots, n_N\}\} \subseteq K \times C^{m+2}$ is a basis of the C-vector space of all solutions $(\bar{b}, e_0, \ldots, e_{m+1}) \in K \times C^{m+2}$ of

$$D\bar{b} = e_0 \cdot \mathrm{coeff}(a_0, t^N) + \cdots + e_m \cdot \mathrm{coeff}(a_m, t^N) + e_{m+1} \cdot (N+1)Dt.$$

Now for each $j \in \{1, \ldots, n_N\}$ we compute

$$\tilde{a}_j := (a_0, \ldots, a_m) \cdot \mathbf{e}_j + D(\bar{c}_j t^{N+1} - \bar{b}_j t^N) \in K[t].$$

Obviously we have $\deg_t(\tilde{a}_j) \leq N$, but in addition we also have that $\mathrm{coeff}(\tilde{a}_j, t^N) = (\mathrm{coeff}(a_0, t^N), \ldots, \mathrm{coeff}(a_m, t^N)) \cdot \mathbf{e}_j + \bar{c}_j \cdot (N+1)Dt - D\bar{b}_j = 0$. Altogether this implies that $\max(\max_j \deg_t(\tilde{a}_j), 0) < N$, and by induction hypothesis applied to $\tilde{a}_1, \ldots, \tilde{a}_{n_N}$ we can compute $\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n \in C^{n_N}$ and $\tilde{b}_1, \ldots, \tilde{b}_n \in K[t]$ such that:

1. If $(\tilde{a}_1, \ldots, \tilde{a}_{n_N}) \cdot \tilde{\mathbf{c}}$ has an elementary integral over $(K(t), D)$ for some $\tilde{\mathbf{c}} \in C^{n_N}$, then $\tilde{\mathbf{c}} \in \mathrm{span}_C\{\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n\}$.
2. $\forall j \in \{1, \ldots, n\} : (\tilde{a}_1, \ldots, \tilde{a}_{n_N}) \cdot \tilde{\mathbf{c}}_j - D\tilde{b}_j \in K$.

In a final step we compute $(\mathbf{c}_1, \ldots, \mathbf{c}_n) := (\mathbf{e}_1, \ldots, \mathbf{e}_{n_N}) \cdot (\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n) \in C^{(m+1)\times n}$ (as multiplication of matrices represented by their columns) and

$$b_j := \left(-\bar{c}_1 t^{N+1} + \bar{b}_1 t^N, \ldots, -\bar{c}_{n_N} t^{N+1} + \bar{b}_{n_N} t^N\right) \cdot \tilde{\mathbf{c}}_j + \tilde{b}_j \in K[t].$$

Now we check that these $\mathbf{c}_1, \ldots, \mathbf{c}_n$ and $b_1, \ldots, b_n$ satisfy the statement of the theorem. To this end, let $\mathbf{c} \in C^{m+1}$ be fixed. Assume that $f := (a_0, \ldots, a_m) \cdot \mathbf{c} \in K[t]$ has an elementary integral over $(K(t), D)$, then by Theorem 2.59 there are $v \in K[t]$, $d_1, \ldots, d_l \in \overline{C}$, and $u_1, \ldots, u_l \in K(d_1, \ldots, d_l)$ such that $f = Dv + \sum d_i \frac{Du_i}{u_i}$. Since $\deg_t(Dv) \leq \max(\deg_t(Dv + \sum d_i \frac{Du_i}{u_i}), 0) \leq N$, there are $\bar{c} \in C$, $\bar{b} \in K$, $\tilde{v} \in K[t]$ such that $\deg_t(\tilde{v}) < N$ and $v = \bar{c}t^{N+1} + \bar{b}t^N + \tilde{v}$. Hence we have

$$\sum c_i \,\mathrm{coeff}(a_i, t^N) = \mathrm{coeff}(f, t^N) = \mathrm{coeff}(Dv, t^N) = \bar{c} \cdot (N+1)Dt + D\bar{b}.$$

So by construction of $\mathbf{e}_1, \dots, \mathbf{e}_{n_N}$, $\bar{c}_1, \dots, \bar{c}_{n_N}$, and $\bar{b}_1, \dots, \bar{b}_{n_N}$ there is $\tilde{\mathbf{c}} \in C^{n_N}$ such that $\bar{b} = \sum \tilde{c}_j \bar{b}_j$, $\mathbf{c} = \sum \tilde{c}_j \mathbf{e}_j$, and $-\bar{c} = \sum \tilde{c}_j \bar{c}_j$. From this we obtain

$$f = (a_0, \dots, a_m) \cdot \sum \tilde{c}_j \mathbf{e}_j = \sum \tilde{c}_j \cdot (\tilde{a}_j - D(\bar{c}_j t^{N+1} - \bar{b}_j t^N)) = \sum \tilde{c}_j \tilde{a}_j + D(\bar{c} t^{N+1} + \bar{b} t^N).$$

Consequently, $(\tilde{a}_1, \dots, \tilde{a}_{n_N}) \cdot \tilde{\mathbf{c}} = D\tilde{v} + \sum d_i \frac{Du_i}{u_i}$ has an elementary integral over $(K(t), D)$, i.e., $\tilde{\mathbf{c}} \in \operatorname{span}_C\{\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_n\}$. From this it follows that $\mathbf{c} = (\mathbf{e}_1, \dots, \mathbf{e}_{n_N}) \cdot \tilde{\mathbf{c}} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. In order to verify $(a_0, \dots, a_m) \cdot \mathbf{c}_j - Db_j \in K$ we just need to check, for fixed $j \in \{1, \dots, n\}$, that by construction of $\mathbf{c}_j$, b_j, and $\tilde{a}_1, \dots, \tilde{a}_{n_N}$ we have

$$(a_0, \dots, a_m) \cdot \mathbf{c}_j - Db_j = (a_0, \dots, a_m) \cdot (\mathbf{e}_1, \dots, \mathbf{e}_{n_N}) \cdot \tilde{\mathbf{c}}_j - Db_j = (\tilde{a}_1, \dots, \tilde{a}_{n_N}) \cdot \tilde{\mathbf{c}}_j - D\tilde{b}_j,$$

which is from K by induction hypothesis. □

Remark In the context of the proof above by Theorem 2.50 the equation $D\bar{b} = Dt$ does not have a solution $\bar{b} \in K$. Hence for each $\mathbf{e} \in C^{m+1}$ there is at most one $\bar{c} \in C$ such that the limited integration problem considered above has a solution $(\bar{b}, \mathbf{e}, \bar{c}) \in K \times C^{m+2}$. However, for each $(\mathbf{e}, \bar{c}) \in C^{m+2}$ a solution is not unique, if it exists, since $(1, 0, \dots, 0) \in K \times C^{m+2}$ is a solution of the limited integration problem. □

3.3.2 Hyperexponential extensions

In the case of a hyperexponential monomial t we know $K(t)_{red} = K[t, \frac{1}{t}]$, so we have to consider Laurent polynomials instead of polynomials. The following theorem is a straightforward generalization of Theorem 5.9.1 in [Bro].

Theorem 3.12. *Let t be hyperexponential over (K, D) and $C := \operatorname{Const}(K(t)) = \operatorname{Const}(K)$. If we can solve parametric Risch differential equations in (K, D), then for any $a_0, \dots, a_m \in K(t)_{red}$ we can compute $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ and $b_1, \dots, b_n \in K[t, \frac{1}{t}]$ such that:*

1. *If $\sum_{i=0}^{m} c_i a_i = (a_0, \dots, a_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$ for some $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \dots, n\} : (a_0, \dots, a_m) \cdot \mathbf{c}_j - Db_j \in K$.*

Proof. According to Theorem 2.51 we have $K(t)_{red} = K[t, \frac{1}{t}]$. We prove the statement by induction on $N = \max(\max_i \deg_t(a_i), 0)$ and $M = \min(\min_i \nu_t(a_i), 0)$ ranging over $\mathbb{N}$ and $-\mathbb{N}$ respectively.
$N = M = 0$: Let $\mathbf{c}_1, \dots, \mathbf{c}_{m+1} \in C^{m+1}$ be a basis of C^{m+1} and $b_1 = \dots = b_{m+1} = 0$. Then the statement is trivially fulfilled.
$N > 0 \vee M < 0$: Let $L \in \{M, N\} \setminus \{0\}$ be fixed and assume that the theorem holds for all a_i with $\deg_t(a_i), \nu_t(a_i) \in \{-\infty, M, \dots, N, \infty\} \setminus \{L\}$. Then by assumption we can compute $\mathbf{e}_1, \dots, \mathbf{e}_{n_L} \in C^{m+1}$ and $\bar{b}_1, \dots, \bar{b}_{n_L} \in K$ such that the set $\{(\bar{b}_j, \mathbf{e}_j) \mid j \in \{1, \dots, n_L\}\} \subseteq$

$K \times C^{m+1}$ is a basis of the C-vector space of all solutions $(\bar{b}, e_0, \ldots, e_m) \in K \times C^{m+1}$ of the parametric Risch differential equation

$$D\bar{b} + L\frac{Dt}{t}\cdot\bar{b} = e_0\cdot \operatorname{coeff}(a_0, t^L) + \cdots + e_m\cdot \operatorname{coeff}(a_m, t^L).$$

Now, for each $j \in \{1, \ldots, n_L\}$ compute

$$\tilde{a} := (a_0, \ldots, a_m) \cdot \mathbf{e}_j - (D\bar{b}_j + L\tfrac{Dt}{t}\cdot\bar{b}_j)t^L.$$

Then obviously $\max(\max_j \deg_t(\tilde{a}_j), 0) \leq N$ and $\min(\min_j \nu_t(\tilde{a}_j), 0) \geq M$. In addition, because of $\operatorname{coeff}(\tilde{a}_j, t^L) = (\operatorname{coeff}(a_0, t^L), \ldots, \operatorname{coeff}(a_m, t^L))\cdot\mathbf{e}_j - (D\bar{b}_j + L\frac{Dt}{t}\cdot\bar{b}_j) = 0$ at least one of the inequalities is strict. Hence by induction hypothesis applied to $\tilde{a}_1, \ldots, \tilde{a}_{n_L}$ we can compute $\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n \in C^{n_L}$ and $\tilde{b}_1, \ldots, \tilde{b}_{n_L} \in K[t, \frac{1}{t}]$ such that:

1. If $(\tilde{a}_1, \ldots, \tilde{a}_{n_L})\cdot\tilde{\mathbf{c}}$ has an elementary integral over $(K(t), D)$ for some $\tilde{\mathbf{c}} \in C^{n_L}$, then $\tilde{\mathbf{c}} \in \operatorname{span}_C\{\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n\}$.
2. $\forall j \in \{1, \ldots, n\} : (\tilde{a}_1, \ldots, \tilde{a}_{n_L}) \cdot \tilde{\mathbf{c}}_j - D\tilde{b}_j \in K$.

In a final step we compute $(\mathbf{c}_1, \ldots, \mathbf{c}_n) := (\mathbf{e}_1, \ldots, \mathbf{e}_{n_L}) \cdot (\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n) \in C^{(m+1)\times n}$ (as multiplication of matrices represented by their columns) and

$$b_j := \left(\bar{b}_1 t^L, \ldots, \bar{b}_{n_L} t^L\right) \cdot \tilde{\mathbf{c}}_j + \tilde{b}_j \in K[t, \tfrac{1}{t}].$$

Now we verify that these $\mathbf{c}_1, \ldots, \mathbf{c}_n \in C^{m+1}$ and $b_1, \ldots, b_n \in K[t, \frac{1}{t}]$ satisfy the statements of the theorem. To this end, let $\mathbf{c} \in C^{m+1}$ fixed such that $f := (a_0, \ldots, a_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$. Then by Theorem 2.59 there are $v \in K[t, \frac{1}{t}]$, $d_1, \ldots, d_l \in \overline{C}$, and $u_1, \ldots, u_l \in S_{K(d_1,\ldots,d_l)[t]:K(d_1,\ldots,d_l)}$ such that $f = Dv + \sum d_i \frac{Du_i}{u_i}$. In particular we have $\sum d_i \frac{Du_i}{u_i} \in K$. Next, let $\bar{b} := \operatorname{coeff}(v, t^L) \in K$, then $(\bar{b}, \mathbf{c})$ satisfies

$$D\bar{b} + L\tfrac{Dt}{t}\cdot\bar{b} = \operatorname{coeff}(Dv, t^L) = \operatorname{coeff}(f, t^L) = (\operatorname{coeff}(a_0, t^L), \ldots, \operatorname{coeff}(a_m, t^L)) \cdot \mathbf{c},$$

i.e., by construction of $\bar{b}_1, \ldots, \bar{b}_{n_L}$ and $\mathbf{e}_1, \ldots, \mathbf{e}_{n_L}$ there is $\tilde{\mathbf{c}} \in C^{n_L}$ such that $\bar{b} = \sum \tilde{c}_j \bar{b}_j$ and $\mathbf{c} = \sum \tilde{c}_j \mathbf{e}_j$. This reveals that

$$(\tilde{a}_1, \ldots, \tilde{a}_{n_L}) \cdot \tilde{\mathbf{c}} = \textstyle\sum \tilde{c}_j \tilde{a}_j = (a_0, \ldots, a_m) \cdot \sum \tilde{c}_j \mathbf{e}_j - \sum \tilde{c}_j D(\bar{b}_j t^L) = f - D(\bar{b} t^L)$$

has an elementary integral over $(K(t), D)$. Hence by construction of $\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n$ we obtain $\tilde{\mathbf{c}} \in \operatorname{span}_C\{\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_n\}$. From this by construction of $\mathbf{c}_1, \ldots, \mathbf{c}_n$ it follows that $\mathbf{c} = \sum \tilde{c}_j \mathbf{e}_j \in \operatorname{span}_C\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$. To conclude the proof, for fixed $j \in \{1, \ldots, n\}$, we verify $(a_0, \ldots, a_m) \cdot \mathbf{c}_j - Db_j \in K$. By construction of $\mathbf{c}_j$, b_j, and $\tilde{a}_1, \ldots, \tilde{a}_{n_L}$ we can check

$$(a_0, \ldots, a_m) \cdot \mathbf{c}_j - Db_j = (a_0, \ldots, a_m) \cdot (\mathbf{e}_1, \ldots, \mathbf{e}_{n_L}) \cdot \tilde{\mathbf{c}}_j - Db_j = (\tilde{a}_1, \ldots, \tilde{a}_{n_L}) \cdot \tilde{\mathbf{c}}_j - D\tilde{b}_j,$$

which is from K by construction of $\tilde{b}_j$ and $\tilde{\mathbf{c}}_j$. □

Remark In the context of the proof above and in view of Theorem 2.51 the homogeneous Risch differential equation $D\bar{b} + L\frac{Dt}{t}\cdot\bar{b} = 0$ does not have a non-trivial solution $\bar{b} \in K^*$, for otherwise $\frac{Dt}{t} = -\frac{D\bar{b}}{L\bar{b}}$ would be the logarithmic derivative of a K-radical. Hence for each $\mathbf{e} \in C^{m+1}$ there is at most one $\bar{b} \in K$ such that $(\bar{b}, \mathbf{e})$ solves the parametric Risch differential equation above. $\square$

3.3.3 Nonlinear extensions

If t is a nonlinear monomial, then we can apply what Bronstein called polynomial reduction. It is based on the fact that $\operatorname{lc}(Df) = \deg(f)\operatorname{lc}(f)$ for polynomials in this setting, which enables us to read off the coefficients of f from the coefficients of Df. So by the following theorem we can reduce the polynomial part of the integrands to have degree less than $\deg_t(Dt)$.

Theorem 3.13. *([Bro, Thm 5.4.1]) Let t be a nonlinear monomial over (K, D). Then for any $a \in K[t]$ we can compute $b \in K[t]$ such that $a - Db \in K[t]$ has degree less than $\deg_t(Dt)$.*

If the monomial t is such that there are no special polynomials, then the previous theorem immediately gives rise to the following corollary, in the spirit of Theorems 3.11 and 3.12. Theorem 5.11.1 in [Bro] already contains a non-parametric version of it, i.e., with $m = 0$.

Corollary 3.14. *Let t be a nonlinear monomial over (K, D) with $S^{irr}_{K[t]:K} = \emptyset$ and let $C := \operatorname{Const}(K(t))$. Assume that we can find a basis for the constant solutions of linear systems with coefficients from K (see Section 2.4.3). Then for any $a_0, \dots, a_m \in K(t)_{red}$ we can compute $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ and $b_1, \dots, b_n \in K[t]$ such that:*

1. *If $\sum_{i=0}^{m} c_i a_i = (a_0, \dots, a_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$ for some $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \dots, n\} : (a_0, \dots, a_m) \cdot \mathbf{c}_j - Db_j \in K$.*

Proof. The assumption $S^{irr} = \emptyset$ implies $K(t)_{red} = K[t]$ and $C = \operatorname{Const}(K(t)) = \operatorname{Const}(K)$. For each $i \in \{0, \dots, m\}$ by virtue of Theorem 3.13 we compute $\tilde{b}_i \in K[t]$ such that $\tilde{a}_i := a_i - D\tilde{b}_i \in K[t]$ has $\deg_t(\tilde{a}_i) < \deg_t(Dt)$. Now we compute a basis $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ of the constant solutions of the linear system arising from comparing the coefficients of powers of t in $\sum_{i=0}^{m}(\tilde{a}_i \div t)c_i = 0$. By assumption we can do this. Then for each $j \in \{1, \dots, n\}$ we compute $b_j := (\tilde{b}_0, \dots, \tilde{b}_m) \cdot \mathbf{c}_j$.

Now we verify that these $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ and $b_1, \dots, b_n \in K[t]$ satisfy the required properties. To this end, let $\mathbf{c} \in C^{m+1}$ fixed such that $f := (a_0, \dots, a_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$. Then $\tilde{a} := (\tilde{a}_0, \dots, \tilde{a}_m) \cdot \mathbf{c} = f - (D\tilde{b}_0, \dots, D\tilde{b}_m) \cdot \mathbf{c} \in K[t]$ has an elementary integral over $(K(t), D)$ as well and satisfies $\deg(\tilde{a}) < \deg_t(Dt)$. Hence by Corollary 2.60 we obtain $\tilde{a} \in K$. So by construction of $\mathbf{c}_1, \dots, \mathbf{c}_n$ we find $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. On the other hand, we have that $(a_0, \dots, a_m) \cdot \mathbf{c}_j - Db_j = (\tilde{a}_0, \dots, \tilde{a}_m) \cdot \mathbf{c}_j \in K$ for all $j \in \{1, \dots, n\}$. $\square$

3.4 Main recursive step

Now we reduced the integrands to lie in K, still we want to find integrals which are elementary over $(K(t), D)$. If t is elementary over (K, D), then this obviously is equivalent to finding integrals elementary over (K, D). In order to apply our algorithm recursively we have to reduce this to a problem of finding elementary integrals over (K, D) also in the case where t is non-elementary over (K, D). As a minimalistic example we note that $\int 1$ is not elementary over (C, D), where $C = \mathbb{Q}$, but it is elementary over $(C(t), D)$, where $Dt = 1$. We will see a more elaborate example after the following theorem. Manuel Bronstein commented as follows on the algorithms presented in his book:

> "Note that when t itself is not elementary over (K, D), then the problems of deciding whether an element of K has an elementary integral over $(K(t), D)$ are fundamentally different, so our algorithms will produce proofs of nonintegrability only if the integrand is itself an elementary function." [Bro, p. 157]

> "The only obstruction to a complete algorithm for Liouvillian integrands is the case where t is a nonelementary primitive over (K, D): even though we can reduce the problem to an integrand in K, the problem becomes however to determine whether $f \in K$ has an elementary integral over $(K(t), D)$, and although there are algorithms for special types of primitive monomials, this problem has not been solved for general monomials." [Bro, p. 136]

By the refinements of Liouville's theorem given in Section 2.7 this issue is resolved. A special case is already implicitly contained in [SSC85, Thm A1]. The following theorem summarizes the relevant cases for admissible monomials.

Theorem 3.15. *Assume $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$ and let $f \in K$ such that f has an elementary integral over $(K(t), D)$. Then the following statements hold.*

1. *If t is elementary over (K, D), then f has an elementary integral over (K, D).*
2. *If t is primitive over (K, D), then there exists a $c \in C$ such that $f - cDt$ has an elementary integral over (K, D).*
3. *If t is hyperexponential over (K, D), then there exists a $c \in C$ such that $f - c\frac{Dt}{t}$ has an elementary integral over (K, D).*
4. *If t is a nonlinear monomial over (K, D) and $S^{irr} = \emptyset$, then f has an elementary integral over (K, D).*

Proof. By assumption we have $f \in K(t)_{red}$, so Theorem 2.59 implies that there are $v \in K(t)_{red}$, $c_1, \dots, c_n \in \overline{C}$, and $u_1, \dots, u_n \in S_{K(c_1,\dots,c_n)[t]:K(c_1,\dots,c_n)}$ such that

$$f = Dv + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}.$$

If t is elementary over (K, D), then v as well as all c_i and u_i are from the elementary extension $K(c_1, \ldots, c_n, t)$ of K. So f has an elementary integral over (K, D).

Now let us assume that t is primitive over (K, D). Then by Theorems 2.50 and 2.61.3 we have $v \in K[t]$ and $u_i \in K(c_1, \ldots, c_n)$. Consequently, $Dv = f - \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} \in K$. From this it follows that $v = ct + a$ for some $a \in K$ and $c \in C$, i.e.,

$$f - cDt = Da + \sum_{i=1}^{n} c_i \frac{Du_i}{u_i}$$

has an elementary integral over (K, D).

Now let us assume that t is hyperexponential over (K, D). Then by Theorem 2.51 we have $S^{irr}_{K[t]:K} = S^{irr,1}_{K[t]:K} = \{t\}$. Hence by Corollary 2.11 there are $a_1, \ldots, a_n \in K(c_1, \ldots, c_n)$ and $m_1, \ldots, m_n \in \mathbb{N}$ such that $u_i = a_i t^{m_i}$. Define $\tilde{c} := \sum_{i=1}^{n} c_i m_i \in C(c_1, \ldots, c_n)$ so that $\sum_{i=1}^{n} c_i \frac{Du_i}{u_i} = \tilde{c}\frac{Dt}{t} + \sum_{i=1}^{n} c_i \frac{Da_i}{a_i}$. Theorem 2.61.2 implies that $v \in K[t]$ and we have $Dv = f - \tilde{c}\frac{Dt}{t} - \sum_{i=1}^{n} c_i \frac{Da_i}{a_i} \in K$. From this we obtain $v \in K$. Since $E := K(c_1, \ldots, c_n)$ is a finite algebraic extension of K the set of embeddings $\sigma : E \to \overline{K}$ leaving all elements from K fixed is finite. Denote these by $\sigma_1, \ldots, \sigma_d$ then $T(\alpha) := \frac{1}{d} \sum_{j=1}^{d} \sigma_j(\alpha) = \frac{1}{d} \operatorname{Tr}^E_K(\alpha)$ is a K-linear map from E to K that leaves all elements from K fixed as well. With $c := T(\tilde{c}) \in C$ we have that

$$f - c\frac{Dt}{t} = T\left(f - \tilde{c}\frac{Dt}{t}\right) = T(Dv) + \sum_{i=1}^{n} T\left(c_i \frac{Da_i}{a_i}\right) = Dv + \sum_{i=1}^{n} \sum_{j=1}^{d} \frac{\sigma_j(c_i)}{d} \frac{D\sigma_j(a_i)}{\sigma_j(a_i)}$$

has an elementary integral over (K, D) since $\sigma_j(c_i) \in \overline{C}$ and $\sigma_j(a_i) \in \overline{K}$.

Now let us assume that t is a nonlinear monomial over (K, D) such that $S^{irr} = \emptyset$. Then by Theorem 2.61 it follows immediately that $v \in K$ and $u_i \in K(c_1, \ldots, c_n)$, i.e., f has an elementary integral over (K, D). $\square$

Example Consider the field $(F, D) = (C(t_1, t_2, t_3), D)$ with $C = \mathbb{Q}$, $Dt_1 = 1$, $Dt_2 = \frac{1}{t_1}$, and $Dt_3 = \frac{1}{t_2}$. Then $t_1 \leftrightarrow x$, $t_2 \leftrightarrow \ln(x)$, and $t_3 \leftrightarrow \operatorname{li}(x)$. We want to compute an elementary integral of $\frac{(t_1+1)^2}{t_1 t_2} + t_3$ over (F, D). In a first step we reduce the integrand to an element from $K = C(t_1, t_2)$ by Theorem 3.11

$$\int \frac{(x+1)^2}{x\ln(x)} + \operatorname{li}(x)\, dx = x\operatorname{li}(x) + \int \frac{2x+1}{x\ln(x)}\, dx.$$

The remaining integral is not elementary over (K, D) so we cannot find the integral by applying the integration algorithm over (K, D) directly. However, Theorem 3.15 tells us to consider the rewriting

$$\int \frac{2x+1}{x\ln(x)}\, dx = c\operatorname{li}(x) + \int \frac{2x+1}{x\ln(x)} - c\frac{1}{\ln(x)}\, dx$$

and apply parametric elementary integration over (K, D) to find also a value for $c \in \mathbb{Q}$. Indeed we succeed with $c = 2$ as found by Theorem 3.9 and altogether obtain the integral

$$\int \frac{(x+1)^2}{x\ln(x)} + \mathrm{li}(x)\,dx = (x+2)\mathrm{li}(x) + \ln(\ln(x)).$$

Remark The converse of Theorem 3.15 trivially is true: if one of the four conditions is fulfilled then f has an elementary integral over $(K(t), D)$. □

Now we are in the position to prove that we can reduce the problem of parametric elementary integration over $(K(t), D)$ to parametric elementary integration over (K, D) by gluing together the results presented in the previous sections. Note that the corresponding results in Bronstein's book (Theorems 5.8.2, 5.9.2, and 5.11.1 in [Bro]) are weaker in two main aspects. First, they do not cover the parametric case but work with single integrands only and, second, they merely reduce the integrand to an element from K for which still elementary integrals over $(K(t), D)$ need to be found. We compensate these issues mainly by incorporating our Theorems 3.9 and 3.15.

Let us start by summarizing the results of Sections 3.1 and 3.2 as the computations there are uniform for all monomials t. For the assumptions on computability see Section 2.4.

Theorem 3.16. *Assume $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$ and assume that we can compute splitting factorizations in $K[t]$, find a basis for the constant solutions of linear systems with coefficients from K, and find a solution of linear systems with coefficients from K if there is one. Then, for any $f_0, \ldots, f_m \in K(t)$ we can compute $\mathbf{c}_1, \ldots, \mathbf{c}_n \in C^{m+1}$ and $g_1, \ldots, g_n$ from some elementary extension of (K, D) such that:*

1. *If $(f_0, \ldots, f_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$ for some $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \mathrm{span}_C\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \ldots, n\} : (f_0, \ldots, f_m) \cdot \mathbf{c}_j - Dg_j \in K(t)_{red}$.*

Proof. First, we treat all f_i separately and for each of them we compute a splitting factorization of $\mathrm{den}_t(f_i)$ and based on this we compute $\bar{a}_i, \bar{b}_i \in K[t]$ and $\bar{f}_i \in K(t)_{red}$ such that $f_i = \frac{\bar{a}_i}{\bar{b}_i} + \bar{f}_i$ and all irreducible factors of $\bar{b}_i$ are normal. Next, by Hermite reduction (Theorem 3.5) we compute $\bar{g}_i \in K(t)$ such that $\frac{\bar{a}_i}{\bar{b}_i} - D\bar{g}_i$ is simple for all $i \in \{0, \ldots, m\}$. Now, we write all integrands with a common denominator.

$$b := \mathrm{lcm}\left(\mathrm{den}\left(\tfrac{\bar{a}_i}{\bar{b}_i} - D\bar{g}_i\right), \ldots, \mathrm{den}\left(\tfrac{\bar{a}_i}{\bar{b}_i} - D\bar{g}_i\right)\right)$$

$$\tilde{a}_i := \left(\frac{\bar{a}_i}{\bar{b}_i} - D\bar{g}_i\right) b \in K[t]$$

Then, by applying Theorem 3.9 to $\tilde{a}_0, \ldots, \tilde{a}_m$ and b we compute $\mathbf{c}_1, \ldots, \mathbf{c}_n \in C^{m+1}$ with the properties stated there and use Theorem 3.10 for computing the corresponding $Q_{j,1}, \ldots, Q_{j,m_j} \in C[z]$ and $S_{j,1}, \ldots, S_{j,m_j} \in K[z, t]$ such that

$$\tilde{f}_j := \frac{(\tilde{a}_0, \ldots, \tilde{a}_m)}{b} \cdot \mathbf{c}_j - \sum_{k=1}^{m_j} \sum_{Q_{j,k}(\alpha)=0} \alpha \frac{DS_{j,k}(\alpha, t)}{S_{j,k}(\alpha, t)} \in K[t]$$

for all $j \in \{1, \dots, n\}$. As a final step we set

$$g_j := (\bar{g}_0, \dots, \bar{g}_m) \cdot \mathbf{c}_j + \sum_{k=1}^{m_j} \sum_{Q_{j,k}(\alpha)=0} \alpha \log(S_{j,k}(\alpha, t)).$$

Now we check the properties claimed for $\mathbf{c}_1, \dots, \mathbf{c}_n$ and $g_1, \dots, g_n$. So we fix a $\mathbf{c} \in C^{m+1}$ such that $f := (f_0, \dots, f_m) \cdot \mathbf{c}$ has an elementary integral over $(K(t), D)$. Define $h := (\bar{f}_0, \dots, \bar{f}_m) \cdot \mathbf{c} \in K(t)_{red}$. Then, $h + \frac{(\tilde{a}_0, \dots, \tilde{a}_m)}{b} \cdot \mathbf{c} = f - (D\bar{g}_0, \dots, D\bar{g}_m) \cdot \mathbf{c}$ has an elementary integral as well and by Theorem 3.9 it follows that $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. Finally, it is easy to verify that

$$(f_0, \dots, f_m) \cdot \mathbf{c}_j - Dg_j =$$
$$\left(\frac{\bar{a}_0}{\bar{b}_0} + \bar{f}_0 - D\bar{g}_0, \dots, \frac{\bar{a}_m}{\bar{b}_m} + \bar{f}_m - D\bar{g}_m \right) \cdot \mathbf{c}_j - \sum_{k=1}^{m_j} \sum_{Q_{j,k}(\alpha)=0} \alpha \frac{DS_{j,k}(\alpha, t)}{S_{j,k}(\alpha, t)} =$$
$$(\bar{f}_0, \dots, \bar{f}_m) \cdot \mathbf{c}_j + \tilde{f}_j \in K(t)_{red}.$$

□

Remark If t is such that $S^{irr} = \emptyset$, then the situation is simpler and we can start with Hermite reduction right away since in the proof above we will have $\bar{f}_i = 0$ for all i in this case. □

The following three theorems show for each of the three cases considered in Section 3.3 under which assumptions on the underlying differential field (K, D) we can solve the parametric elementary integration problem over $(K(t), D)$. The proofs are basically all the same and differ by small details only.

Theorem 3.17. *Let t be primitive over (K, D) and let $C := \operatorname{Const}(K(t)) = \operatorname{Const}(K)$. In addition to the assumptions of Theorem 3.16, assume that we can solve the limited integration problem in (K, D) as well as the parametric elementary integration problem over (K, D). Then for any $f_0, \dots, f_m \in K(t)$ we can compute $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ and $g_1, \dots, g_n$ from some elementary extension of (K, D) such that:*

1. *If $(f_0, \dots, f_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$ for some $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \dots, n\} : Dg_j = (f_0, \dots, f_m) \cdot \mathbf{c}_j$.*

Proof. From Theorem 2.50 we know that $S^{irr} = \emptyset$ and so by the assumptions we can compute $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}$ and $g_1, \dots, g_m$ from some elementary extension of $(K(t), D)$ as in Theorem 3.16 following the remark after it. Then, from

$$\tilde{f}_j := (f_0, \dots, f_m) \cdot \bar{\mathbf{c}}_j - D\bar{g}_j \in K(t)_{red},$$

$j \in \{1, \dots, \bar{n}\}$, we compute $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}} \in C^{\bar{n}}$ and $\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}} \in K[t]$ according to Theorem 3.11, which we can do by the assumptions. After that, we set

$$\hat{f}_j := (\tilde{f}_1, \dots, \tilde{f}_{\bar{n}}) \cdot \tilde{\mathbf{c}}_j - D\tilde{g}_j \in K$$

for all $j \in \{1, \dots, \tilde{n}\}$ and solve the parametric elementary integration problem over (K, D) for $\hat{f}_1, \dots, \hat{f}_{\hat{n}}, Dt \in K$ in order to obtain $\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n \in C^{\tilde{n}}$, $\hat{d}_1, \dots, \hat{d}_n \in C$, and $\hat{g}_1, \dots, \hat{g}_n$ from some elementary extension of (K, D) such that for all $j \in \{1, \dots, n\}$

$$D\hat{g}_j = (\hat{f}_1, \dots, \hat{f}_{\hat{n}}) \cdot \hat{\mathbf{c}}_j + \hat{d}_j Dt.$$

Finally, we compute

$$g_j := ((\bar{g}_1, \dots, \bar{g}_{\bar{n}}) \cdot (\tilde{c}_1, \dots, \tilde{c}_{\tilde{n}}) + (\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}})) \cdot \hat{\mathbf{c}}_j + \hat{g}_j - \hat{d}_j t$$

for all $j \in \{1, \dots, n\}$ and

$$(\mathbf{c}_1, \dots, \mathbf{c}_n) := (\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}) \cdot (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot (\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n).$$

Now, for these $\mathbf{c}_1, \dots, \mathbf{c}_n$ and $g_1, \dots, g_n$ we verify the claims made in the statements above. To this end, we fix a $\mathbf{c} \in C^{m+1}$ such that $f := (f_0, \dots, f_m) \cdot \mathbf{c}$ has an elementary integral over $(K(t), D)$. By construction of $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}$ there exists $\tilde{c} \in C^{\bar{n}}$ such that $\mathbf{c} = (\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}) \cdot \tilde{\mathbf{c}}$ and $\tilde{f} := f - (D\bar{g}_1, \dots, D\bar{g}_{\bar{n}}) \cdot \tilde{\mathbf{c}} \in K(t)_{red}$ has an elementary integral over $(K(t), D)$. Hence, by construction there is a $\hat{\mathbf{c}} \in C^{\tilde{n}}$ such that $\tilde{\mathbf{c}} = (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot \hat{\mathbf{c}}$ and $\hat{f} := \tilde{f} - (\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}}) \cdot \hat{\mathbf{c}} \in K$ has an elementary integral over $(K(t), D)$. Now, from Theorem 3.15 it follows that there exists $\hat{d} \in C$ such that $\hat{f} + \hat{d}Dt \in K$ has an elementary integral over (K, D). Consequently, $(\hat{\mathbf{c}}, \hat{d}) \in \operatorname{span}_C\{(\hat{\mathbf{c}}_1, \hat{d}_1), \dots, (\hat{\mathbf{c}}_n, \hat{d}_n)\}$ by construction and in particular also $\hat{\mathbf{c}} \in \operatorname{span}_C\{\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n\}$. Hence, we obtain $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ by construction of $\mathbf{c}_1, \dots, \mathbf{c}_n$. From the construction above it is also straightforward to verify $Dg_j = (f_0, \dots, f_m) \cdot \mathbf{c}_j$ for all $j \in \{1, \dots, n\}$. □

Remark If in the above corollary t is a logarithm and hence elementary over (K, D), then we do not need to include Dt when computing elementary integrals of $\hat{f}_1, \dots, \hat{f}_{\hat{n}}$ over (K, D). In fact, the $\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n$ obtained would certainly be linear dependent otherwise since $(\mathbf{0}, 1) \in \operatorname{span}_C\{(\hat{\mathbf{c}}_1, \hat{d}_1), \dots, (\hat{\mathbf{c}}_n, \hat{d}_n)\}$ would correspond to the integrand Dt, which has an elementary integral over (K, D). □

Theorem 3.18. *Let t be hyperexponential over (K, D) and let $C := \operatorname{Const}(K(t)) = \operatorname{Const}(K)$. In addition to the assumptions of Theorem 3.16, assume that we can solve parametric Risch differential equations in (K, D) as well as the parametric elementary integration problem over (K, D). Then for any $f_0, \dots, f_m \in K(t)$ we can compute $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ and $g_1, \dots, g_n$ from some elementary extension of (K, D) such that:*

1. *If $(f_0, \dots, f_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$ for some $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \dots, n\} : Dg_j = (f_0, \dots, f_m) \cdot \mathbf{c}_j$.*

Proof. By the assumptions we can compute $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}$ and $g_1, \dots, g_m$ from some elementary extension of $(K(t), D)$ as in Theorem 3.16. Then, from

$$\tilde{f}_j := (f_0, \dots, f_m) \cdot \bar{\mathbf{c}}_j - D\bar{g}_j \in K(t)_{red},$$

$j \in \{1, \ldots, \bar{n}\}$, we compute $\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_{\tilde{n}} \in C^{\bar{n}}$ and $\tilde{g}_1, \ldots, \tilde{g}_{\tilde{n}} \in K[t, \frac{1}{t}]$ according to Theorem 3.12, which we can do by the assumptions. After that, we set

$$\hat{f}_j := (\tilde{f}_1, \ldots, \tilde{f}_{\tilde{n}}) \cdot \tilde{\mathbf{c}}_j - D\tilde{g}_j \in K$$

for all $j \in \{1, \ldots, \tilde{n}\}$ and solve the parametric elementary integration problem over (K, D) for $\hat{f}_1, \ldots, \hat{f}_{\hat{n}}, \frac{Dt}{t} \in K$ in order to obtain $\hat{\mathbf{c}}_1, \ldots, \hat{\mathbf{c}}_n \in C^{\tilde{n}}$, $\hat{d}_1, \ldots, \hat{d}_n \in C$, and $\hat{g}_1, \ldots, \hat{g}_n$ from some elementary extension of (K, D) such that for all $j \in \{1, \ldots, n\}$

$$D\hat{g}_j = (\hat{f}_1, \ldots, \hat{f}_{\hat{n}}) \cdot \hat{\mathbf{c}}_j + \hat{d}_j \frac{Dt}{t}.$$

Finally, we compute

$$g_j := ((\bar{g}_1, \ldots, \bar{g}_{\bar{n}}) \cdot (\tilde{c}_1, \ldots, \tilde{c}_{\tilde{n}}) + (\tilde{g}_1, \ldots, \tilde{g}_{\tilde{n}})) \cdot \hat{\mathbf{c}}_j + \hat{g}_j - \hat{d}_j \log(t)$$

for all $j \in \{1, \ldots, n\}$ and

$$(\mathbf{c}_1, \ldots, \mathbf{c}_n) := (\bar{\mathbf{c}}_1, \ldots, \bar{\mathbf{c}}_{\bar{n}}) \cdot (\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot (\hat{\mathbf{c}}_1, \ldots, \hat{\mathbf{c}}_n).$$

Now, for these $\mathbf{c}_1, \ldots, \mathbf{c}_n$ and $g_1, \ldots, g_n$ we verify the claims made in the statements above. To this end, we fix a $\mathbf{c} \in C^{m+1}$ such that $f := (f_0, \ldots, f_m) \cdot \mathbf{c}$ has an elementary integral over $(K(t), D)$. By construction of $\bar{\mathbf{c}}_1, \ldots, \bar{\mathbf{c}}_{\bar{n}}$ there exists $\tilde{c} \in C^{\bar{n}}$ such that $\mathbf{c} = (\bar{\mathbf{c}}_1, \ldots, \bar{\mathbf{c}}_{\bar{n}}) \cdot \tilde{\mathbf{c}}$ and $\tilde{f} := f - (D\bar{g}_1, \ldots, D\bar{g}_{\bar{n}}) \cdot \tilde{\mathbf{c}} \in K(t)_{red}$ has an elementary integral over $(K(t), D)$. Hence, by construction there is a $\hat{\mathbf{c}} \in C^{\tilde{n}}$ such that $\tilde{\mathbf{c}} = (\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot \hat{\mathbf{c}}$ and $\hat{f} := \tilde{f} - (\tilde{g}_1, \ldots, \tilde{g}_{\tilde{n}}) \cdot \hat{\mathbf{c}} \in K$ has an elementary integral over $(K(t), D)$. Now, from Theorem 3.15 it follows that there exists $\hat{d} \in C$ such that $\hat{f} + \hat{d}\frac{Dt}{t} \in K$ has an elementary integral over (K, D). Consequently, $(\hat{\mathbf{c}}, \hat{d}) \in \operatorname{span}_C\{(\hat{\mathbf{c}}_1, \hat{d}_1), \ldots, (\hat{\mathbf{c}}_n, \hat{d}_n)\}$ by construction and in particular also $\hat{\mathbf{c}} \in \operatorname{span}_C\{\hat{\mathbf{c}}_1, \ldots, \hat{\mathbf{c}}_n\}$. Hence, we obtain $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ by construction of $\mathbf{c}_1, \ldots, \mathbf{c}_n$. From the construction above it is also straightforward to verify $Dg_j = (f_0, \ldots, f_m) \cdot \mathbf{c}_j$ for all $j \in \{1, \ldots, n\}$. □

Remark Analogously to the previous remark, if in the above corollary t is an exponential and hence elementary over (K, D), then we do not need to include $\frac{Dt}{t}$ when computing elementary integrals of $\hat{f}_1, \ldots, \hat{f}_{\hat{n}}$ over (K, D). In fact, the $\hat{\mathbf{c}}_1, \ldots, \hat{\mathbf{c}}_n$ obtained would certainly be linear dependent otherwise since $(\mathbf{0}, 1) \in \operatorname{span}_C\{(\hat{\mathbf{c}}_1, \hat{d}_1), \ldots, (\hat{\mathbf{c}}_n, \hat{d}_n)\}$ would correspond to the integrand $\frac{Dt}{t}$, which has an elementary integral over (K, D). □

Theorem 3.19. *Let t be such that $\deg_t(Dt) \geq 2$ and $S^{irr}_{K[t],K} = \emptyset$ and let $C := \operatorname{Const}(K(t))$. In addition to the assumptions of Theorem 3.16, assume that we can solve the parametric elementary integration problem over (K, D). Then for any $f_0, \ldots, f_m \in K(t)$ we can compute $\mathbf{c}_1, \ldots, \mathbf{c}_n \in C^{m+1}$ and $g_1, \ldots, g_n$ from some elementary extension of (K, D) such that:*

1. *If $(f_0, \ldots, f_m) \cdot \mathbf{c} \in K(t)$ has an elementary integral over $(K(t), D)$ for some $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \ldots, n\} : Dg_j = (f_0, \ldots, f_m) \cdot \mathbf{c}_j$.*

Proof. From Lemma 2.18 we know that $C = \mathrm{Const}(K(t)) = \mathrm{Const}(K)$ and so by the assumptions we can compute $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}$ and $g_1, \dots, g_m$ from some elementary extension of $(K(t), D)$ as in Theorem 3.16 following the remark after it. Then, from

$$\tilde{f}_j := (f_0, \dots, f_m) \cdot \bar{\mathbf{c}}_j - D\bar{g}_j \in K(t)_{red},$$

$j \in \{1, \dots, \bar{n}\}$, we compute $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}} \in C^{\bar{n}}$ and $\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}} \in K[t]$ according to Corollary 3.14, which we can do by the assumptions. After that, we set

$$\hat{f}_j := (\tilde{f}_1, \dots, \tilde{f}_{\bar{n}}) \cdot \tilde{\mathbf{c}}_j - D\tilde{g}_j \in K$$

for all $j \in \{1, \dots, \tilde{n}\}$ and solve the parametric elementary integration problem over (K, D) for $\hat{f}_1, \dots, \hat{f}_{\hat{n}} \in K$ in order to obtain $\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n \in C^{\tilde{n}}$ and $\hat{g}_1, \dots, \hat{g}_n$ from some elementary extension of (K, D) such that for all $j \in \{1, \dots, n\}$

$$D\hat{g}_j = (\hat{f}_1, \dots, \hat{f}_{\hat{n}}) \cdot \hat{\mathbf{c}}_j.$$

Finally, we compute

$$g_j := ((\bar{g}_1, \dots, \bar{g}_{\bar{n}}) \cdot (\tilde{c}_1, \dots, \tilde{c}_{\tilde{n}}) + (\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}})) \cdot \hat{\mathbf{c}}_j + \hat{g}_j$$

for all $j \in \{1, \dots, n\}$ and

$$(\mathbf{c}_1, \dots, \mathbf{c}_n) := (\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}) \cdot (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot (\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n).$$

Now, for these $\mathbf{c}_1, \dots, \mathbf{c}_n$ and $g_1, \dots, g_n$ we verify the claims made in the statements above. To this end, we fix a $\mathbf{c} \in C^{m+1}$ such that $f := (f_0, \dots, f_m) \cdot \mathbf{c}$ has an elementary integral over $(K(t), D)$. By construction of $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}$ there exists $\tilde{c} \in C^{\bar{n}}$ such that $\mathbf{c} = (\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}) \cdot \tilde{\mathbf{c}}$ and $\tilde{f} := f - (D\bar{g}_1, \dots, D\bar{g}_{\bar{n}}) \cdot \tilde{\mathbf{c}} \in K(t)_{red}$ has an elementary integral over $(K(t), D)$. Hence, by construction there is a $\hat{\mathbf{c}} \in C^{\tilde{n}}$ such that $\tilde{\mathbf{c}} = (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot \hat{\mathbf{c}}$ and $\hat{f} := \tilde{f} - (\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}}) \cdot \hat{\mathbf{c}} \in K$ has an elementary integral over $(K(t), D)$. Now, from Theorem 3.15 it follows that $\hat{f} \in K$ has an elementary integral over (K, D). Consequently, $\hat{\mathbf{c}} \in \mathrm{span}_C\{\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n\}$. Hence, we obtain $\mathbf{c} \in \mathrm{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ by construction of $\mathbf{c}_1, \dots, \mathbf{c}_n$. From the construction above it is also straightforward to verify $Dg_j = (f_0, \dots, f_m) \cdot \mathbf{c}_j$ for all $j \in \{1, \dots, n\}$. □

3.5 Generalizations to other functions

In this section t is not a monomial over (K, D) anymore, but we consider how other cases can be treated either heuristically but also algorithmically.

One very basic class of functions which has been explicitly excluded from our considerations so far is the class of algebraic functions. The main obstacle being that there is no algorithmic equivalent for Theorem 3.9 known for this case so far. Nevertheless also for fields where generators of the field satisfy algebraic relations much work has been done regrading integration, see [Tra79, Bro90b, Bro98] and references therein for example, as well as differential equations [Sin91]. Even without incorporating additional machinery into the algorithm presented here we can treat the case of radicals at least heuristically. If $c \in \mathbb{Q} \setminus \mathbb{Z}$ and $f \in K$, then $t = f^c$ still is hyperexponential over (K, D), it just does not satisfy the condition given in Theorem 2.51. So we still can apply our algorithm heuristically in this case. If an integral is found and it does not involve a hidden division by zero, then the result is valid. It just may be that not all elementary integrals are found.

3.5.1 Differentially transcendental extensions

In strong contrast to what we have considered in all the previous sections so far, the kind of t we consider in this section will not be such that $Dt \in K[t]$ nor even $Dt \in K(t)$, so the differential field generated by K and t will not be $K(t)$. Instead, we assume that all derivatives $t, Dt, D^2t, \ldots$ are algebraically independent over K, i.e., t is *differentially transcendental* over (K, D). The differential field generated by adjoining such a t to K is commonly denoted by $(K\langle t\rangle, D) := (K(t, Dt, D^2t, \ldots), D)$.

Examples: Gamma function, Psi function, Riemann Zeta function, etc.

We will, however, consider a more flexible representation of the field $K(t, Dt, D^2t, \ldots)$ by choosing $t_0, t_1, \ldots \in K\langle t\rangle$ such that

$$t_0 = t \tag{3.7}$$

$$Dt_n = a_n t_{n+1} + b_n \tag{3.8}$$

with $a_n, b_n \in K(t_0, \ldots, t_n)$ for $n \in \mathbb{N}$. Then, the following lemma shows that automatically $a_n \neq 0$ and $K(t, Dt, \ldots, D^n t) = K(t_0, \ldots, t_n)$ for all $n \in \mathbb{N}$. Note that this type of field extensions can also be used to at least heuristically treat any given function for which no other algorithm applies, even if the function satisfies some algebraic differential equation with coefficients from K. The flexibility in the representation introduced by (3.8) also allows to represent the polylogarithms Li_n for symbolic n in a convenient way, see also the example on page 69.

Lemma 3.20. *Let t be differentially transcendental over (K, D) and let $t_0, t_1, \ldots \in K\langle t\rangle$ such that* (3.7) *and* (3.8)*. Then*

1. *$a_n \neq 0$ for all $n \in \mathbb{N}$ and*
2. *for all $n \in \mathbb{N}$ there are $\tilde{a}_n, \tilde{b}_n \in K(t, Dt, \ldots, D^n t)$ such that $t_{n+1} = \tilde{a}_n D^{n+1}t + \tilde{b}_n$ and $\tilde{a}_n \neq 0$.*

Proof. We prove both statements in parallel by induction, for which we artificially include the case $n = -1$.
For $n = -1$ we define $a_{-1} := 1$, $\tilde{a}_{-1} := 1$, and $\tilde{b}_{-1} := 0$, then trivially $a_{-1}, \tilde{a}_{-1} \in K^*$ and $t_0 = \tilde{a}_{-1}t + \tilde{b}_{-1}$ by definition.
For $n \in \mathbb{N}$ we assume that for all $i \in \{-1, 0, \ldots, n-1\}$ there are $\tilde{a}_i, \tilde{b}_i \in K(t, Dt, \ldots, D^i t)$ such that $t_{i+1} = \tilde{a}_i D^{i+1}t + \tilde{b}_i$ and $\tilde{a}_i \neq 0$. Then, from the assumptions we obtain that $a_n t_{n+1} = Dt_n - b_n = D(\tilde{a}_{n-1}D^n t + \tilde{b}_{n-1}) - b_n = \tilde{a}_{n-1}D^{n+1}t + (D\tilde{a}_{n-1})D^n t + D\tilde{b}_{n-1} - b_n$. If we had $a_n = 0$, then this and $\tilde{a}_{n-1} \neq 0$ would imply $D^{n+1}t = \frac{(D\tilde{a}_{n-1})D^n t + D\tilde{b}_{n-1} - b_n}{-\tilde{a}_{n-1}}$

where the right hand side is in $K(t,\dots,D^n t)$ by induction hypothesis, which would be in contradiction to $t,\dots,D^{n+1}t$ being algebraically independent over K. Hence, $a_n \neq 0$ and we set $\tilde{a}_n := \frac{\tilde{a}_{n-1}}{a_n}$ and $\tilde{b}_n := \frac{(D\tilde{a}_{n-1})D^n t + D\tilde{b}_{n-1} - b_n}{a_n}$, which both are in $K(t,\dots,D^n t)$ by the induction hypothesis. $\square$

The second statement of the lemma above has some important immediate consequences, which we emphasize by stating the following corollary. The proof is trivial and so we omit it.

Corollary 3.21. *Let t be differentially transcendental over (K,D) and let $t_0,t_1,\dots \in K\langle t\rangle$ such that* (3.7) *and* (3.8)*. Then*

1. *$t_0,t_1,\dots$ are algebraically independent over K, and*
2. *$K(t,Dt,\dots,D^n t) = K(t_0,\dots,t_n)$ for all $n \in \mathbb{N}$.*

In the following we will formalize the ideas of Campbell [Cam88] into this framework. In this context we define the coefficient lifting $\kappa_D : K[t_0,t_1,\dots] \to K[t_0,t_1,\dots]$ of D by $\kappa_D(\sum_\alpha f_\alpha t^\alpha) := \sum_\alpha (Df_\alpha)t^\alpha$, where we used multiindex notation for brevity, and extend this to a derivation κ_D on $K(t_0,t_1,\dots)$ in the natural way by the quotient rule. In analogy to Lemma 2.4 it is easy to see that

$$Df = \kappa_D f + \sum_{k=0}^{\infty} \frac{\partial f}{\partial t_k} Dt_k. \tag{3.9}$$

Note that the sum contains only finitely many nonzero summands since $\frac{\partial f}{\partial t_k} = 0$ from some point on. Generalizing the definition of κ_D above, for each $n \in \mathbb{N}$ we define the derivation $\kappa_{D,n}$ on $K(t_0,t_1,\dots)$ by

$$\kappa_{D,n} f := \kappa_D f + \sum_{k=0}^{n-1} \frac{\partial f}{\partial t_k} Dt_k. \tag{3.10}$$

These derivations obey $\kappa_{D,n} f + \frac{\partial f}{\partial t_n} Dt_n = \kappa_{D,n+1} f$ for $f \in K(t_0,t_1,\dots)$ with $\kappa_{D,0} = \kappa_D$. For $f \in K(t_0,\dots,t_{n-1})$ we have in particular $\kappa_{D,n} f = Df$. An important measure on the elements of $K(t_0,t_1,\dots)$ is the highest index of any of the generators needed to represent the particular element of the field.

Definition 3.22. *Let t be differentially transcendental over (K,D), then we define the* differential degree *of $f \in K(t,Dt,\dots)$ by*

$$\operatorname{ddeg}_t(f) := \begin{cases} \min\left\{k \in \mathbb{N} \mid f \in K(t,\dots,D^k t)\right\} & \textit{if } f \notin K \\ -\infty & \textit{if } f \in K. \end{cases}$$

Note that for $t_0,t_1,\dots \in K\langle t\rangle$ with (3.7) and (3.8) the above corollary implies

$$\operatorname{ddeg}_t(f) = \begin{cases} \min\{k \in \mathbb{N} \mid f \in K(t_0,\dots,t_k)\} & \text{if } f \notin K \\ -\infty & \text{if } f \in K. \end{cases}$$

So we can say that $\frac{\partial f}{\partial t_k} = 0$ for $k > \operatorname{ddeg}_t(f)$ in (3.9). The differential degree obeys the following properties with respect to the operations of a differential field.

Lemma 3.23. *Let t be differentially transcendental over (K,D) and let $t_0,t_1,\ldots\in K\langle t\rangle$ such that (3.7) and (3.8). Then for $f,g\in K(t_0,t_1,\ldots)^*$ we have*

$$\begin{aligned}\operatorname{ddeg}_t(f+g) &\leq \max(\operatorname{ddeg}_t(f),\operatorname{ddeg}_t(g)),\\ \operatorname{ddeg}_t(f/g) &\leq \max(\operatorname{ddeg}_t(f),\operatorname{ddeg}_t(g)),\\ \operatorname{ddeg}_t(1/f) &= \operatorname{ddeg}_t(f),\\ \operatorname{ddeg}_t(Df) &= \operatorname{ddeg}_t(f)+1\end{aligned}$$

with equality in the first two relations if $\operatorname{ddeg}_t(f)\neq\operatorname{ddeg}_t(g)$.

Proof. By Corollary 3.21 the first three properties are trivial. If $f\in K$, then $Df\in K$ and hence $\operatorname{ddeg}_t(Df)=-\infty=\operatorname{ddeg}_t(f)+1$. If $f\notin K$, then (3.9) implies $\operatorname{ddeg}_t(Df)=\operatorname{ddeg}_t(f)+1$ by (3.8). □

In particular, the last property stated in the previous lemma has the important implication that the field of constants is not extended. Furthermore, with the following corollary we emphasize the structure of derivatives implied by (3.8) and (3.9).

Corollary 3.24. *Let t be differentially transcendental over (K,D), let $t_0,t_1,\ldots\in K\langle t\rangle$ such that (3.7) and (3.8), and let $F:=K(t_0,t_1,\ldots)$. Then* $\operatorname{Const}_D(F)=\operatorname{Const}_D(K)$ *and for all $f\in F$ and any $k\in\mathbb{N}$ with $k\geq\operatorname{ddeg}_t(f)$ there exist $a,b\in K(t_0,\ldots,t_k)$ with $a=a_k\frac{\partial f}{\partial t_k}$ and*

$$Df=at_{k+1}+b.$$

Based on these properties we now are ready to prove a refinement of Liouville's theorem for this situation. This and the following results present the ideas from [Cam88] in a more precise way.

Theorem 3.25. *Let t be differentially transcendental over (K,D) and let $t_0,t_1,\ldots\in K\langle t\rangle$ such that (3.7) and (3.8). Let $f\in F:=K(t_0,t_1,\ldots)$ such that f has an elementary integral over (F,D). Then with $k:=\operatorname{ddeg}_t(f)$ there are $v\in K(t_0,\ldots,t_{k-1})$, $c_1,\ldots,c_n\in\overline{\operatorname{Const}_D(K)}$, and $u_1,\ldots,u_n\in K(c_1,\ldots,c_n,t_0,\ldots,t_{k-1})^*$ such that (2.28), i.e.,*

$$f=Dv+\sum_{i=1}^n c_i\frac{Du_i}{u_i}.$$

In particular, if $k\geq 1$, we can also write this as

$$f=a_{k-1}\left(\frac{\partial v}{\partial t_{k-1}}+\sum_{i=1}^n c_i\frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i}\right)t_k+b$$

for some $b\in K(c_1,\ldots,c_n,t_0,\ldots,t_{k-1})$.

Proof. By Liouville's theorem (Theorem 2.58) we know that there are $v\in F$, $c_1,\ldots,c_n\in\overline{\operatorname{Const}_D(F)}$, and $u_1,\ldots,u_n\in\overline{F}(c_1,\ldots,c_n)^*$ such that (2.28) and by Corollary 3.24 we deduce $c_1,\ldots,c_n\in\overline{\operatorname{Const}_D(K)}$. Define

$$m:=\max(\operatorname{ddeg}_t(v),\operatorname{ddeg}_t(u_1),\ldots,\operatorname{ddeg}_t(u_n)).$$

If $m < 0$, then $f \in K$ and the statement is trivially fulfilled. So assume $m \geq 0$ now and assume that $v, c_1, \dots, c_n, u_1, \dots, u_n$ are chosen such that $u_1, \dots, u_n$ are pairwise relatively prime polynomials from $K(c_1, \dots, c_n, t_0, \dots, t_{m-1})[t_m]$. Then, applying Corollary 3.24 to each summand in (2.28) implies that

$$f = a_m \left(\frac{\partial v}{\partial t_m} + \sum_{i=1}^{n} c_i \frac{\frac{\partial u_i}{\partial t_m}}{u_i} \right) t_{m+1} + b$$

for some $b \in K(c_1, \dots, c_n, t_0, \dots, t_m)$ and by Lemma 3.23 we obtain $k \leq m+1$. Next, Lemma 3.20 implies that $a_m \neq 0$ and Corollary 3.21 implies that $t_0, t_1, \dots$ are algebraically independent. If we had $m > k-1$, then we could conclude $\tilde{f} := \frac{\partial v}{\partial t_m} + \sum_{i=1}^{n} c_i \frac{\frac{\partial u_i}{\partial t_m}}{u_i} = 0$. From this we would obtain $\max(\operatorname{ddeg}_t(u_1), \dots, \operatorname{ddeg}_t(u_n)) < m$ by applying Lemma 2.23 in the differential field $(K(t_0, \dots, t_m), \frac{\partial}{\partial t_m})$ to it at all possible p and noting that $u_1, \dots, u_n \in K(c_1, \dots, c_n)[t_0, \dots, t_m]$ are pairwise relatively prime. Therefore, the definitions of m and $\tilde{f}$ would imply $\operatorname{ddeg}_t(v) = m$ and $\tilde{f} = \frac{\partial v}{\partial t_m}$, respectively, which would give $\tilde{f} \neq 0$ altogether in contradiction to $\tilde{f} = 0$. Hence we have $m = k-1$. □

In particular, this theorem directly provides results analogous to Corollary 2.60 and Theorem 3.15. We state these special cases explicitly in the form of a corollary.

Corollary 3.26. *Let t be differentially transcendental over (K, D), let $t_0, t_1, \dots \in K\langle t \rangle$ with (3.7) and (3.8) and let $F := K(t_0, t_1, \dots)$. If $f \in K(t_0)$ has an elementary integral over (F, D), then $f \in K$. If $f \in K$ has an elementary integral over (F, D), then it has an elementary integral over (K, D).*

Theorem 3.25 also suggests that we should look at the following problem in order to compute elementary integrals over $(K(t_0, t_1, \dots), D)$.

Problem 3.27. *Given: t differentially transcendental over (K, D), $t_0, t_1, \dots \in K\langle t \rangle$ with (3.7) and (3.8), $k \in \mathbb{N}^+$, and $f_0, \dots, f_m \in K(t_0, \dots, t_{k-1})$.*

Find: a basis $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$, where $C := \operatorname{Const}_D(K)$, of the C-vector space of all $\mathbf{c} \in C^{m+1}$ such that there exist $v \in K(t_0, \dots, t_{k-1})$, $d_1, \dots, d_l \in \overline{C}$, and $u_1, \dots, u_l \in K(d_1, \dots, d_l, t_0, \dots, t_{k-1})^*$ with

$$(f_0, \dots, f_m) \cdot \mathbf{c} = \frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^{l} d_i \frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i}$$

as well as corresponding $v_j \in K(t_0, \dots, t_{k-1})$, $d_{j,1}, \dots, d_{j,l_j} \in \overline{C}$, and $u_{j,1}, \dots, u_{j,l_j} \in K(d_{j,1}, \dots, d_{j,l_j}, t_0, \dots, t_{k-1})^*$ for each $j \in \{1, \dots, n\}$.

At first glance this problem may look like it was just parametric elementary integration over $(K(t_0, \dots, t_{k-1}), \frac{\partial}{\partial t_{k-1}})$ and we could solve it easily by well-known algorithms, but there is a subtle difference. Observe that we need linear combinations with coefficients in $C = \operatorname{Const}_D(K)$ instead of $\operatorname{Const}_{\frac{\partial}{\partial t_{k-1}}}(K(t_0, \dots, t_{k-1})) = K(t_0, \dots, t_{k-2})$. Furthermore, instead of allowing residues $d_i \in \overline{K(t_0, \dots, t_{k-2})}$ we restrict to $d_i \in \overline{\operatorname{Const}_D(K)}$ above.

It turns out that we can apply the ideas of Theorem 3.9 in order to solve this task algorithmically by modifying the theorem to make use of two derivations: D and $\frac{\partial}{\partial t_{k-1}}$. Before we make those modifications explicit in our next theorem we need to realize that if we apply Lemma 2.4 in a differential extension (F, D) of $(K(t_0, t_1, \dots), D)$ to some $p \in K(t_0, \dots, t_n)[z]$ and $f \in F$ then we obtain

$$D(p(f)) = \sum_{i=0}^{\deg_z(p)} (\kappa_{D,n+1} \operatorname{coeff}(p, z^i)) f^i + \frac{\partial p}{\partial z}(f) Df. \tag{3.11}$$

Theorem 3.28. *Let t be differentially transcendental over (K, D), let $t_0, t_1, \dots \in K\langle t\rangle$ such that (3.7) and (3.8), and let $C := \operatorname{Const}_D(K)$. Assume that for any $k \in \mathbb{N}^+$ we can find a basis for the constant solutions of linear systems with coefficients from $K(t_0, \dots, t_{k-1})$ (cf. Section 2.4.3). Let $k \in \mathbb{N}^+$, let $a_0, \dots, a_m, b \in K(t_0, \dots, t_{k-2})[t_{k-1}]$ with $b \neq 0$ and $\gcd(b, \frac{\partial b}{\partial t_{k-1}}) = 1$, and let z be an indeterminate. Then using the half-extended Euclidean algorithm in $K(t_0, \dots, t_{k-2})[z]$ we can compute linear independent $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ such that:*

1. *If $\mathbf{c} \in C^{m+1}$ is such that there exist $v \in K(t_0, \dots, t_{k-1})$, $d_1, \dots, d_l \in \overline{C}$, and $u_1, \dots, u_l \in K(d_1, \dots, d_l, t_0, \dots, t_{k-1})^*$ with*
$$\frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^{l} d_i \frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i} = \frac{(a_0, \dots, a_m) \cdot \mathbf{c}}{b},$$
then $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.

2. *$\forall j \in \{1, \dots, n\} \exists r \in C[z] : \frac{(a_0, \dots, a_m) \cdot \mathbf{c}_j}{b} - \sum_{r(\alpha)=0} \alpha \frac{\frac{\partial g_\alpha}{\partial t_{k-1}}}{g_\alpha} \in K(t_0, \dots, t_{k-2})[t_{k-1}]$, where $g_\alpha := \gcd((a_0, \dots, a_m) \cdot \mathbf{c}_j - \alpha \frac{\partial b}{\partial t_{k-1}}, b)) \in K(\alpha, t_0, \dots, t_{k-2})[t_{k-1}]$ for all roots $\alpha \in \overline{C}$ of r.*

Proof. Define $\tilde{K} := K(t_0, \dots, t_{k-2})$ for brevity and let $\tilde{a}_0, \dots, \tilde{a}_m, \tilde{b} \in \tilde{K}[z]$ such that $\tilde{a}_i(t_{k-1}) = a_i$ and $\tilde{b}(t_{k-1}) = b$. First, we want to construct $q_0, \dots, q_m \in \tilde{K}[t_{k-1}][z]$ such that $\deg_z(q_i) < \deg_{t_{k-1}}(b)$ and

$$\forall \beta \in \overline{\tilde{K}}, \tilde{b}(\beta) = 0 : q_i(\beta) = D\left(\frac{\tilde{a}_i(\beta)}{\frac{d\tilde{b}}{dz}(\beta)}\right) \tag{3.12}$$

for all $i \in \{0, \dots, m\}$. To this end, by the half-extended Euclidean algorithm in $\tilde{K}[z]$ we compute for each $i \in \{0, \dots, m\}$ polynomials $p_i, \tilde{p}_{i,0}, \tilde{p}_{i,1} \in \tilde{K}[z]$ with $\deg_z(p_i) < \deg_z(\tilde{b})$, $\deg_z(\tilde{p}_{i,0}) < \deg_z(\tilde{b})$, and $\deg_z(\tilde{p}_{i,1}) < \deg_z(\tilde{b})$ such that

$$\begin{aligned}
\tilde{a}_i &\equiv p_i \frac{d\tilde{b}}{dz} \pmod{\tilde{b}},\\
\frac{dp_i}{dz} \cdot \sum_{j=0}^{\deg_z(\tilde{b})} \operatorname{coeff}(\kappa_{D,k-1} \tilde{b}_j, t_{k-1}^0) z^j &\equiv \tilde{p}_{i,0} \frac{d\tilde{b}}{dz} \pmod{\tilde{b}}, \text{ and}\\
\frac{dp_i}{dz} \cdot \sum_{j=0}^{\deg_z(\tilde{b})} \operatorname{coeff}(\kappa_{D,k-1} \tilde{b}_j, t_{k-1}^1) z^j &\equiv \tilde{p}_{i,1} \frac{d\tilde{b}}{dz} \pmod{\tilde{b}}
\end{aligned}$$

where $\tilde{b}_j := \operatorname{coeff}(\tilde{b}, z^j)$. Note that $\kappa_{D,k-1}$ introduces t_{k-1} linearly by (3.10) and (3.9), this applies also to the coefficients of q_i below. Since $\gcd(b, \frac{\partial b}{\partial t_{k-1}}) = 1$ implies $\gcd(\tilde{b}, \frac{d\tilde{b}}{dz}) = 1$ such p_i and $\tilde{p}_i$ exist. Now we compute

$$q_i := \sum_{j=0}^{\deg_z(p_i)} (\kappa_{D,k-1} \operatorname{coeff}(p_i, z^j))z^j - (\tilde{p}_{i,1}t_{k-1} + \tilde{p}_{i,0})$$

having $\deg_z(q_i) < \deg_z(\tilde{b})$ and arrange the coefficients in a matrix $A \in \tilde{K}^{2\deg_z(\tilde{b})\times(m+1)}$ by

$$A := \begin{pmatrix} \operatorname{coeff}(q_i, t_{k-1}^0 z^j))_{j,i} \\ \operatorname{coeff}(q_i, t_{k-1}^1 z^j))_{j,i} \end{pmatrix},$$

where $j \in \{0, \ldots, \deg(\tilde{b})-1\}$ and $i \in \{0, \ldots, m\}$. Finally, we compute a C-vector space basis $\mathbf{c}_1, \ldots, \mathbf{c}_n \in C^{m+1}$ of $\ker(A) \cap C^{m+1}$.

For verifying (3.12) we take $\beta \in \overline{\tilde{K}}$ such that $\tilde{b}(\beta) = 0$ and calculate

$$p_i(\beta) = \frac{\tilde{a}_i(\beta)}{\frac{d\tilde{b}}{dz}(\beta)}$$

by the definition of p_i. From the definition of $\tilde{p}_{i,0}$ and $\tilde{p}_{i,1}$ by (3.11) we also get

$$\tilde{p}_{i,1}(\beta)t_{k-1} + \tilde{p}_{i,0}(\beta) = \frac{\frac{dp_i}{dz}(\beta)\cdot \sum_{j=0}^{\deg_z(\tilde{b})}(\kappa_{D,k-1}\tilde{b}_j)\beta^j}{\frac{d\tilde{b}}{dz}(\beta)} = \frac{dp_i}{dz}(\beta)\frac{D(\tilde{b}(\beta)) - \frac{d\tilde{b}}{dz}(\beta)\cdot D\beta}{\frac{d\tilde{b}}{dz}(\beta)} = -\frac{dp_i}{dz}(\beta)\cdot D\beta.$$

Therefore, using (3.11) again we obtain

$$q_i(\beta) = \sum_{j=0}^{\deg_z(p_i)} (\kappa_{D,k-1} \operatorname{coeff}(p_i, z^j))\beta^j + \frac{dp_i}{dz}(\beta)\cdot D\beta = D(p_i(\beta)) = D\left(\frac{\tilde{a}_i(\beta)}{\frac{d\tilde{b}}{dz}(\beta)}\right).$$

Now let $\mathbf{c} \in C^{m+1}$ be fixed and we define $q := (q_0, \ldots, q_m) \cdot \mathbf{c} \in \tilde{K}[t_{k-1}][z]$. Then, by construction $\deg_z(q) < \deg_z(b)$. The roots of $r := \operatorname{res}_{t_{k-1}}((a_0, \ldots, a_m)\cdot\mathbf{c} - z\frac{\partial b}{\partial t_{k-1}}, b) \in \tilde{K}[z]$ are those $\alpha \in \overline{\tilde{K}}$ such that there exists a $\beta \in \overline{\tilde{K}}$ with $\tilde{b}(\beta) = 0$ and $(\tilde{a}_0(\beta), \ldots, \tilde{a}_m(\beta)) \cdot \mathbf{c} - \alpha\cdot\frac{d\tilde{b}}{dz}(\beta) = 0$. Hence if $\beta \in \overline{\tilde{K}}$ ranges over the roots of b then $\alpha = \frac{(a_0(\beta),\ldots,a_m(\beta))\cdot\mathbf{c}}{\frac{d\tilde{b}}{dz}(\beta)}$ ranges over the roots of r. By (3.12) this implies

$$\{q(\beta) \mid \beta \in \overline{\tilde{K}}, b(\beta) = 0\} = \{D\alpha \mid \alpha \in \overline{\tilde{K}}, r(\alpha) = 0\}. \tag{3.13}$$

For verifying the first part of the statement of the theorem assume that there exist $v \in K(t_0, \ldots, t_{k-1})$, $d_1, \ldots, d_l \in \overline{C}$, and $u_1, \ldots, u_l \in K(d_1, \ldots, d_l, t_0, \ldots, t_{k-1})^*$ with $\frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^l d_i \frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i} = \frac{(a_0,\ldots,a_m)\cdot\mathbf{c}}{b}$. Let $\alpha \in \overline{\tilde{K}}$ be such that $r(\alpha) = 0$. By Lemma 3.7 applied in $\tilde{K}(d_1, \ldots, d_l)[t_{k-1}]$ there exists an irreducible $s \in \tilde{K}(d_1, \ldots, d_l)[t_{k-1}]$ such that $\operatorname{res}_s(\frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b}) = \pi_s(\frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{\frac{\partial b}{\partial t_{k-1}}}) = \alpha$. Hence by Lemma 2.23 we obtain $\alpha = \operatorname{res}_s(\frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b}) = \sum_i d_i\nu_s(u_i) \in \overline{C}$. Therefore, we have that $\alpha \in \overline{C}$ for all roots of r, i.e., $q(\beta) = 0$ for all roots $\beta \in \overline{\tilde{K}}$ of $\tilde{b}$ by (3.13). Since $\tilde{b}$ is squarefree it has $\deg_z(\tilde{b})$ distinct roots in $\overline{\tilde{K}}$ and it follows that $q = 0$. Consequently, by definition we have $A \cdot \mathbf{c} = 0$, i.e., $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1,\dots,\mathbf{c}_n\}$ as required. For verifying the second part of the statement we fix some $j \in \{1,\dots,n\}$ and assume $\mathbf{c} = \mathbf{c}_j$ above. Then $q = (q_0,\dots,q_m)\cdot\mathbf{c}_j = (1, z, \dots, z^{\deg_z(\tilde{b})-1}, t_{k-1}, t_{k-1}z, \dots, t_{k-1}z^{\deg_z(\tilde{b})-1}) \cdot A \cdot \mathbf{c}_j = 0$. So by (3.13) all roots $\alpha \in \overline{K}$ of r lie in $\overline{C}$. Therefore $\frac{r}{\operatorname{lc}_z(r)} \in C[z]$ and it fulfils the statement by Theorem 3.8.1. □

Based on this it is straightforward to solve Problem 3.27 above, keeping in mind that linear combinations and constant solutions of systems refer to coefficients from C not from $K(t_0,\dots,t_{k-2})$.

Corollary 3.29. *Let t be differentially transcendental over (K, D), let $t_0, t_1, \dots \in K\langle t\rangle$ such that* (3.7) *and* (3.8)*, and let $C := \operatorname{Const}_D(K)$. Assume that for any $k \in \mathbb{N}^+$ we can find a basis for the constant solutions of linear systems with coefficients from $K(t_0,\dots,t_{k-1})$ (cf. Section 2.4.3). Let $k \in \mathbb{N}^+$ and let $f_0,\dots,f_m \in K(t_0,\dots,t_{k-1})$. Then we can solve Problem 3.27.*

The following theorem is the main point of this section showing that we can do parametric elementary integration over $(K(t_0,t_1,\dots), D)$ and presents the (non-parametric) algorithm stated in [Cam88] in a more formal way.

Theorem 3.30. *Let t be differentially transcendental over (K, D), let $t_0, t_1, \dots \in K\langle t\rangle$ with* (3.7) *and* (3.8) *and let $F := K(t_0,t_1,\dots)$ and $C := \operatorname{Const}_D(F)$. Assume we can find a basis for the constant solutions of linear systems with coefficients from K, solve Problem 3.27 for any $k \in \mathbb{N}^+$, and solve the parametric elementary integration problem over (K, D). Then, for any $f_0,\dots,f_m \in F$ we can compute $\mathbf{c}_1,\dots,\mathbf{c}_n \in \operatorname{Const}(K)^{m+1}$ and $g_1,\dots,g_n$ from some elementary extension of (F, D) such that:*

1. *If $(f_0,\dots,f_m)\cdot\mathbf{c} \in F$ has an elementary integral over (F, D) for some $\mathbf{c} \in C^{m+1}$, then $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1,\dots,\mathbf{c}_n\}$.*
2. *$\forall j \in \{1,\dots,n\} : Dg_j = (f_0,\dots,f_m)\cdot\mathbf{c}_j$.*

Proof. Note that $t_0, t_1, \dots$ are algebraically independent over K by Corollary 3.21 and $C = \operatorname{Const}(K)$ by Corollary 3.24. We prove the theorem by induction on $k := \max_i(\operatorname{ddeg}_t(f_i))$.
$k < 0$: In this case we just solve the parametric elementary integration problem over (K, D) for $f_0,\dots,f_m \in K$. The $\mathbf{c}_1,\dots,\mathbf{c}_n \in C^{m+1}$ and $g_1,\dots,g_n$ obtained satisfy the statements by Corollary 3.26.
$k = 0$: First, we set up a matrix $A \in K^{l\times m+1}$ such that $A\cdot\mathbf{c} = 0$ is equivalent to $(f_0,\dots,f_m)\cdot\mathbf{c} \in K$ for all $\mathbf{c} \in K^{m+1}$. We can do this by clearing the denominator

$$b := \operatorname{lcm}_{t_0}(\operatorname{den}_{t_0}(f_0),\dots,\operatorname{den}_{t_0}(f_m))$$

and constructing the rows of A by coefficient extraction

$$(\operatorname{coeff}(f_i b \div b, t_0^j))_{i=0,\dots,m}$$

for $j \in \{1, \dots, \max_i(\deg_{t_0}(f_i b)) - \deg_{t_0}(b)\}$ and

$$(\operatorname{coeff}(f_i b \bmod b, t_0^j))_{i=0,\dots,m}$$

for $j \in \{0, \dots, \deg_{t_0}(b) - 1)\}$. Alternatively, we can construct a (possibly) different matrix A based on partial fraction decomposition instead of computing b. In any case we compute a basis $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}} \in C^{m+1}$ of $\ker(A) \cap C^{m+1}$ and set

$$\tilde{f}_j := (f_0, \dots, f_m) \cdot \bar{\mathbf{c}}_j \in K$$

for $j \in \{1, \dots, \bar{n}\}$. Next, for $\tilde{f}_0, \dots, \tilde{f}_m \in K$ we solve the parametric elementary integration problem over (K, D) yielding some $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_n \in C^{\bar{n}}$ and $g_1, \dots, g_n$ from some elementary extension of (K, D). Finally, we compute

$$(\mathbf{c}_1, \dots, \mathbf{c}_n) := (\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}) \cdot (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_n).$$

For verifying the first statement of the theorem we fix a $\mathbf{c} \in C^{m+1}$ such that $f := (f_0, \dots, f_m) \cdot \mathbf{c} \in K(t_0)$ has an elementary integral over (F, D). By Corollary 3.26 we have $f \in K$ and hence by construction of $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}$ there exists a $\tilde{\mathbf{c}} \in C^{\bar{n}}$ such that $\mathbf{c} = (\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}) \cdot \tilde{\mathbf{c}}$. Now, by invoking Corollary 3.26 again we get $\tilde{\mathbf{c}} \in \operatorname{span}_C\{\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_n\}$ and therefore $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. The second statement is verified easily by just plugging in the definitions of g_j and $\mathbf{c}_j$.
$k > 0$: The case $k > 0$ works analogously to the case $k = 0$. Define $\tilde{K} := K(t_0, \dots, t_{k-1})$ for the sake of brevity. First, we set up a matrix $A \in K^{l \times m+1}$ such that $A \cdot \mathbf{c} = 0$ is equivalent to $(f_0, \dots, f_m) \cdot \mathbf{c} \in \tilde{K}[t_k]$ with $\deg_{t_k} \leq 1$ for all $\mathbf{c} \in K^{m+1}$. This can be done again by extracting appropriate coefficients. Then, we compute a basis $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}} \in C^{m+1}$ of $\ker(A) \cap C^{m+1}$ and set

$$\tilde{f}_{j,1} t_k + \tilde{f}_{j,0} := (f_0, \dots, f_m) \cdot \bar{\mathbf{c}}_j$$

with $\tilde{f}_{j,0}, \tilde{f}_{j,1} \in \tilde{K}$ for $j \in \{1, \dots, \bar{n}\}$. Next, for $\frac{\tilde{f}_{1,1}}{a_{k-1}}, \dots, \frac{\tilde{f}_{\bar{n},1}}{a_{k-1}} \in \tilde{K}$ we solve Problem 3.27 by Corollary 3.29 to obtain $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}} \in C^{\bar{n}}$ and corresponding $v_j, d_{j,i}, u_{j,i}$. We can interpret

$$\tilde{g}_j := v_j + \sum_{i=1}^{l_j} d_{j,i} \log(u_{j,i})$$

for $j \in \{1, \dots, \tilde{n}\}$ as being from some elementary extension of (F, D). We set

$$\tilde{f}_j := (\tilde{f}_{1,1} t_k + \tilde{f}_{1,0}, \dots, \tilde{f}_{\bar{n},1} t_k + \tilde{f}_{\bar{n},0}) \cdot \tilde{\mathbf{c}}_j - D\tilde{g}_j$$

for $j \in \{1, \dots, \tilde{n}\}$ and by Corollary 3.24 we verify $\operatorname{ddeg}_t(\tilde{f}_j) < k$ for all j by calculating

$$\begin{aligned}
\tilde{f}_j &= (\tilde{f}_{1,1} t_k + \tilde{f}_{1,0}, \dots, \tilde{f}_{\bar{n},1} t_k + \tilde{f}_{\bar{n},0}) \cdot \tilde{\mathbf{c}}_j - \left(Dv_j + \sum_i d_{j,i} \frac{Du_{j,i}}{u_{j,i}} \right) \\
&= \left((\tilde{f}_{1,1}, \dots, \tilde{f}_{\bar{n},1}) \cdot \tilde{\mathbf{c}}_j - a_{k-1} \left(\frac{\partial v_j}{\partial t_{k-1}} + \sum_i d_{j,i} \frac{\frac{\partial u_{j,i}}{\partial t_{k-1}}}{u_{j,i}} \right) \right) t_k + h_j
\end{aligned}$$

for some $h_j \in \tilde{K}$. Since we have $\frac{\partial v_j}{\partial t_{k-1}} + \sum_i d_{j,i} \frac{\frac{\partial u_{j,i}}{\partial t_{k-1}}}{u_{j,i}} = \frac{(\tilde{f}_{1,1},\dots,\tilde{f}_{\tilde{n},1})\cdot\tilde{\mathbf{c}}_j}{a_{k-1}}$ by definition, this implies $\tilde{f}_j = h_j \in \tilde{K}$. Then, by induction hypothesis we solve the parametric elementary integration problem over (F, D) for $\tilde{f}_1, \dots, \tilde{f}_{\tilde{n}} \in \tilde{K}$ obtaining $\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n \in C^{\tilde{n}}$ and $\hat{g}_1, \dots, \hat{g}_n$ from some elementary extension of (F, D). Finally, we compute

$$(\mathbf{c}_1, \dots, \mathbf{c}_n) := (\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}) \cdot (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot (\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n)$$

and

$$g_j := (\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}}) \cdot \hat{\mathbf{c}}_j + \hat{g}_j.$$

For verifying the first statement of the theorem we fix a $\mathbf{c} \in C^{m+1}$ such that $f := (f_0, \dots, f_m) \cdot \mathbf{c} \in \tilde{K}(t_k)$ has an elementary integral over (F, D). By Theorem 3.25 there are $v, b \in \tilde{K}$, $d_1, \dots, d_N \in \overline{C}$, and $u_1, \dots, u_N \in \tilde{K}(d_1, \dots, d_N)$ such that $f = a_{k-1}\left(\frac{\partial v}{\partial t_{k-1}} + \sum_{i=1}^{N} d_i \frac{\frac{\partial u_i}{\partial t_{k-1}}}{u_i}\right) t_k + b$. Hence by construction of $\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}$ and $\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}$ there is a $\hat{\mathbf{c}} \in C^{\tilde{n}}$ such that $\mathbf{c} = (\bar{\mathbf{c}}_1, \dots, \bar{\mathbf{c}}_{\bar{n}}) \cdot (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_{\tilde{n}}) \cdot \hat{\mathbf{c}}$. Now, by the calculation above we verify $f - D((\tilde{g}_1, \dots, \tilde{g}_{\tilde{n}}) \cdot \hat{\mathbf{c}}) = (\tilde{f}_1, \dots, \tilde{f}_{\tilde{n}}) \cdot \hat{\mathbf{c}} \in \tilde{K}$. So by construction of $\hat{\mathbf{c}}_1, \dots, \hat{\mathbf{c}}_n$ we obtain $\mathbf{c} \in \operatorname{span}_C\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. The second statement is easily verified based on the construction. □

3.6 Examples

Bessel functions

Example Formula 6.539.3 from the integral table [GR] contains the indefinite integral

$$\int \frac{1}{xJ_n(x)Y_n(x)}\,dx = \frac{\pi}{2}\ln\left(\frac{Y_n(x)}{J_n(x)}\right).$$

According to the framework discussed in Section 2.6.2 we can represent the integrand in the admissible differential field $(F, D) := (C(t_1, t_2, t_3, t_4), D)$ where $C := \mathbb{Q}(\pi, n)$ and the derivation is defined by

$$\begin{aligned} Dt_1 &= 1, \\ Dt_2 &= t_2^2 - \tfrac{2(n+1)}{t_1}t_2 + 1, \\ Dt_3 &= (-t_2 + \tfrac{n}{t_1})t_3, \\ Dt_4 &= \frac{2}{\pi t_1 t_3^2}. \end{aligned}$$

Note that the constant π has to appear in the differential field because it is part of the Wronskian, see Section A.2. Then we have the following correspondences.

$$t_1 \leftrightarrow x \qquad t_2 \leftrightarrow \frac{J_{n+1}(x)}{J_n(x)} \qquad t_3 \leftrightarrow J_n(x) \qquad t_4 \leftrightarrow \frac{Y_n(x)}{J_n(x)}$$

So the integrand is represented by $f := \frac{1}{t_1 t_3^2 t_4}$. With $K := C(t_1, t_2, t_3)$ we have that t_4 is primitive over (K, D) so we skip the Hermite reduction as the denominator $\operatorname{den}_{t_4}(f) = t_4$

is normal already. Then by the generalization of Czichowski's algorithm presented in Theorem 3.10 we compute the Gröbner basis

$$\{z - \tfrac{\pi}{2}, t_4\}$$

of the ideal $\langle \frac{1}{t_1 t_3^2} - z\frac{2}{\pi t_1 t_3^2}, t_4 \rangle$ to obtain $Q(z) = z - \frac{\pi}{2}$ and $S(z, t_4) = t_4$. Since $Q \in C[z]$ this yields the logarithmic part

$$\frac{\pi}{2}\ln(t_4).$$

It turns out that $f - \frac{\pi}{2}\frac{Dt_4}{t_4} \in K[t_4]$ is even zero. This means that we already obtained the integral above. Although the factor $\frac{\pi}{2}$ may seem surprising at first glance, it actually is a simple consequence of the structure of the differential field and found without effort by the generalization of Czichowski's algorithm.

Example Also complicated looking integrals can be solved by our algorithm such as the constructed example

$$\int \frac{x(x^2+1+2J_n(x))J_{n+1}(x) - ((n-2)x^2+n+2nJ_n(x))J_n(x)}{xJ_n(x)^2}\,dx,$$

which is not covered by the symbolic integration procedures in the current versions of *Mathematica* or *Maple*. With the same definition of the derivation as in the previous example we use the admissible differential field $(F, D) := (C(t_1, t_2, t_3), D)$ where now $C := \mathbb{Q}(n)$. Then the integrand can be represented by

$$f := 2t_2 - \tfrac{2n}{t_1} + \left((t_1^2+1)t_2 - (n-2)t_1 - \tfrac{n}{t_1}\right)\frac{1}{t_3}.$$

Let $K := C(t_1, t_2)$ and focus on the hyperexponential monomial t_3. First, we recognize $f \in K(t_3)_{red}$ and hence by Theorem 3.12 we need to solve the following Risch differential equation in (K, D).

$$Dy - (-t_2 + \tfrac{n}{t_1})y = \operatorname{coeff}(f, t_3^{-1})$$

By Theorem 6.1.2 and Lemma 6.3.5 of [Bro] we deduce that any solution $y \in K$ even has to satisfy $y \in C(t_1)$, from which it is easy to obtain the solution $y = t_1^2 + 1$ by comparing coefficients of t_2. This means that

$$\frac{t_1^2+1}{t_3}$$

is part of the integral and the remaining integrand is $2t_2 - \frac{2n}{t_1} \in K$. Now, for an elementary integral over $(K(t_3), D)$ by Theorem 3.15 we need to find $c \in C$ such that

$$2t_2 - \tfrac{2n}{t_1} - c\tfrac{Dt_3}{t_3} \in K$$

has an elementary integral over (K, D). This integrand is even in $C(t_1)[t_2]$ with degree less than $\deg_{t_2}(Dt_2)$ so by Corollary 3.14 we have to choose $c = -2$. This generates the term

$$-2\ln(t_3)$$

in the integral and makes the remaining integrand vanish. Altogether, we obtained the following closed form for the integral above.

$$\int \frac{x(x^2+1+2J_n(x))J_{n+1}(x) - ((n-2)x^2+n+2nJ_n(x))J_n(x)}{xJ_n(x)^2}\,dx = \frac{x^2+1}{J_n(x)} - 2\ln(J_n(x))$$

Integration by parts

When computing integrals by hand integration by parts often is a useful tool provided one finds the appropriate factors of the integrand. Also Hermite reduction is based on this principle and computes a suitable splitting of the integrand. Nevertheless, symbolic integration procedures sometimes cannot compute an integral even if it is just one step of integration by parts away from an integral they could compute. This may happen even in simple cases and is illustrated by the following integral. One step of integration by parts computes

$$\int e^{-x}\mathrm{Ei}(x)\,dx = \ln(x) - e^{-x}\mathrm{Ei}(x)$$

whereas version 12 of *Maple* did not find a closed form, this was fixed with version 13. Along these lines many similar examples can be constructed following the pattern

$$\int f(x)g'(x)\,dx = f(x)g(x) + \ln(h(x)) \tag{3.14}$$

where $-f'(x)g(x) = \frac{h'(x)}{h(x)}$. This pattern can be used as a guideline to construct examples which cause existing software to fail. Apart from the next two examples we will see others of this type in this section as well. Sometimes f and g may even be chosen such that a logand appears in the integral which does not show up in the denominator of the integrand due to cancellation like above, one might call this an "unexpected" logarithm. This can happen in particular if the presentation of the integrand is not unique due to algebraic relations among its constituents, which was investigated in [Boe10] for example, but we do not consider this here.

Example With the complete elliptic integrals $E(x)$ and $K(x)$, defined by the definite integrals $\int_0^{\pi/2}(1-x^2\sin(t)^2)^{\pm 1/2}\,dt$, we may choose $f(x) = E(x)$, $g(x) = \frac{1}{E(x)-K(x)}$, and $h(x) = \frac{1}{x}$. In other words,

$$\int \frac{xE(x)^2}{(1-x^2)(E(x)-K(x))^2}\,dx = \frac{E(x)}{E(x)-K(x)} - \ln(x),$$

which cannot be computed by the current versions of *Mathematica* or *Maple*. We make use of $C := \mathbb{Q}$ and the generators t_1 and t_2 with $Dt_1 = 1$ and $Dt_2 = \frac{1}{t_1}t_2^2 - \frac{2}{t_1}t_2 - \frac{1}{t_1(t_1^2-1)}$ corresponding to x and $\frac{K(x)}{E(x)}$. The integrand can be represented in the differential field $(C(t_1,t_2),D)$ by

$$f := \frac{t_1}{(1-t_1^2)(1-t_2)^2}.$$

Following the Hermite reduction we write $\mathrm{num}_{t_2}(f)$ as $\frac{t_1}{1-t_1^2} = a\cdot(-D(1-t_2)) + b\cdot(1-t_2)$ with $a = 1$ and $b = \frac{t_2-1}{t_1}$ to obtain the remaining integrand $f - D\frac{1}{1-t_2} = -\frac{1}{t_1}$. This is easily integrated as $-\log(t_1)$, so we obtain

$$\frac{1}{1-t_2} - \log(t_1)$$

as elementary integral of f over $(C(t_1,t_2),D)$ in accordance with above.

Example We consider the polylogarithms for indeterminate n and set $f(x) = \mathrm{Li}_n(x)$ and $h(x) = x$, which gives $g(x) = -\frac{1}{\mathrm{Li}_{n-1}(x)}$. Note that *Mathematica* and *Maple* cannot compute the integral

$$\int \frac{\mathrm{Li}_{n-2}(x)\mathrm{Li}_n(x)}{x\mathrm{Li}_{n-1}(x)^2}\,dx = \ln(x) - \frac{\mathrm{Li}_n(x)}{\mathrm{Li}_{n-1}(x)}$$

even for specific $n \in \{3, 4, \dots\}$. For computing the integral by our methods we use $C := \mathbb{Q}(n)$ and consider t differentially transcendental over $(K, D) := (C(x), \frac{d}{dx})$, cf. Section 3.5.1. We set $a_k = \frac{1}{x}$ and $b_k = 0$ in (3.8), so t_k from $(F, D) := (C(x, t_0, t_1, \dots), D)$ corresponds to $\mathrm{Li}_{n-k}(x)$. The integrand is represented by

$$f := \frac{t_0 t_2}{x t_1^2},$$

which has $\mathrm{ddeg}_t(f) = 2$ and even is of the form $\tilde{f}_1 t_2 + \tilde{f}_0$ with $\tilde{f}_1 = \frac{t_0}{x t_1^2} \in K(t_0, t_1)$ and $\tilde{f}_0 = 0$. So by Theorem 3.30 we just need to solve Problem 3.27 for $x\tilde{f}_1$. We obtain $v = -\frac{t_0}{t_1}$ and $f - Dv = \frac{1}{x}$, which is easily dealt with by Corollary 3.24. Altogether, we obtain the following elementary integral of f over (F, D).

$$-\frac{t_0}{t_1} + \log(x)$$

Inverse functions

Example As discussed in Section 2.6.3 the Lambert W function can be treated by change of variable. Our implementation handles the occurrence of $W(x^c)$ in the integrand and computes the following two integrals, for example.

$$\int \frac{x^2 + (x^2+2)W(x^2)}{x(W(x^2)+1)^2}\,dx = \ln(W(x^2)+1) + \frac{x^2}{2W(x^2)}$$

$$\int xW(x^{2/3})\,dx = \frac{x^2\left(27W(x^{2/3})^4 - 9W(x^{2/3})^3 + 9W(x^{2/3})^2 - 6W(x^{2/3}) + 2\right)}{54W(x^{2/3})^3}$$

Neither of them is directly computed by the current versions of *Mathematica* or *Maple* without change of variable. The first of the two has been considered by Bronstein in the context of the Risch-Norman algorithm, see [Bro, Bro07].

Example Also for the Jacobian elliptic functions we mention several suitable changes of variable in Section A.3. However, they lead to square roots among the generators of the differential field, which can only be treated heuristically by our algorithm. This is done in our implementation for $x = F(\arcsin(u), k)$ and $x = F(u, k)$ and allows to find integrals of many entries in Section 5.13 of [GR] such as

$$\int \frac{1}{\mathrm{dn}(x,k)}\,dx = \frac{1}{2\sqrt{k^2-1}} \ln\left(\frac{\mathrm{cn}(x,k) + \sqrt{k^2-1}\,\mathrm{sn}(x,k)}{\mathrm{cn}(x,k) - \sqrt{k^2-1}\,\mathrm{sn}(x,k)}\right),$$

which were also considered by Boettner [Boe10] in the context of the Risch-Norman algorithm. However, our method does not always give the result in the nice form displayed here. In addition, also the following integrals are treated by our implementation, for example.

$$\int \frac{1}{\mathrm{cn}(x,k)\mathrm{dn}(x,k)}\,dx = \frac{1}{2(k^2-1)}\ln\left(\frac{\mathrm{sn}(x,k)-1}{\mathrm{sn}(x,k)+1}\right) + \frac{k}{2(k^2-1)}\ln\left(\frac{\mathrm{sn}(x,k)+\frac{1}{k}}{\mathrm{sn}(x,k)-\frac{1}{k}}\right)$$

$$\int \frac{\mathrm{sn}(x,k)\mathrm{dn}(x,k)(\mathrm{am}(x,k)\mathrm{sn}(x,k)-\mathrm{cn}(x,k))}{\mathrm{am}(x,k)^2\mathrm{cn}(x,k)^2}\,dx = \frac{\mathrm{sn}(x,k)}{\mathrm{am}(x,k)\mathrm{cn}(x,k)} - \ln(\mathrm{am}(x,k))$$

The last integral is not handled by the current versions of *Mathematica* or *Maple* without change of variable and was constructed by pattern (3.14) with the choice $f(x) = \mathrm{sn}(x,k)$, $g(x) = \frac{1}{\mathrm{am}(x,k)\mathrm{cn}(x,k)}$, and $h(x) = \frac{1}{\mathrm{am}(x,k)}$.

Example Likewise the change of variable for dealing with Weierstraß elliptic functions, mentioned in Section A.3, leads to a square root among the generators of the differential field, which has to be treated in a heuristic way by our algorithm. The implementation includes also the related functions ζ and σ and can compute the following integrals among many others.

$$\int \wp(x)^3\,dx = \frac{1}{10}\wp(x)\wp'(x) + \frac{g_3}{10}x - \frac{3g_2}{20}\zeta(x)$$

$$\int \frac{x(12\wp(x)^2-g_2)}{\wp'(x)^2}\,dx = \sum_{(g_2^3-27g_3^2)\alpha^3-3g_2\alpha+2=0} \alpha\ln\left(\wp(x)+\frac{g_2^3-27g_3^2}{18g_3}(g_2\alpha+1)\alpha-\frac{g_2^2}{9g_3}\right) - \frac{2x}{\wp'(x)}$$

$$\int x\wp(x)\,dx = \ln(\sigma(x)) - x\zeta(x)$$

The first integral is a different form of Equation 5.141.3 in [GR], the second is related to Example 8.5 in [Boe10], and the last integral follows the pattern (3.14). The last two integrals are not found by the current versions of *Mathematica* or *Maple* without change of variable.

Chapter 4

Linear ordinary differential equations

In this chapter let (K, D) be a differential field of characteristic 0 and let t be a monomial over (K, D). Our aim is to solve linear ODEs, most importantly the Risch differential equation, with coefficients in $(K(t), D)$ and parametric inhomogeneous part. On the one hand we consider the parametric Risch differential equation in Section 4.1 because it arises as subproblem in the integration algorithm, on the other hand it is also important on its own right as it covers the limited integration problem. The problems discussed in the other sections of this chapter mainly serve the purpose of solving the Risch differential equation in our context. The algorithm presented in Section 4.3.1 can also be used for checking the condition given in Theorem 2.51 and likewise the algorithms discussed in Sections 4.2 and 4.3.2 are also relevant for Theorem 2.53 for instance. We are mainly interested to find the solutions of differential equations in a given differential field, which is enough for our needs. Apart from hyperexponential solutions considered in Section 4.3 we do not need to consider algorithms looking for more general types of solutions, such as Liouvillian solutions mentioned in Section 2.5 or even more general solutions.

One main contribution to this chapter is a complete algorithm for the parametric logarithmic derivative problem in admissible differential fields presented in Section 4.3.1. Risch had an algorithm in the elementary differential fields he considered, which he sketched in [Ris69]. Bronstein gave a heuristic method in monomial extensions of differential fields in Section 7.3 of his book [Bro]. To our knowledge no complete algorithm has been presented in this generality in the literature.

Another main contribution is the joint work with Moulay A. Barkatou on systems of differential equations presented in Section 4.4 and in [BR12]. It contains a generalization of Carole El Bacha's algorithm [ElB11, BE12] to monomial extensions as well as a generalization of Barkatou's algorithm [Bar99] to hyperexponential extensions of $(C(x), \frac{d}{dx})$ under some conditions on the system matrix.

A contribution to solving linear ODEs in towers of monomial extensions is made in form of Theorem 4.33 and a heuristic degree bound given in Section 4.3.2 which is still very useful for the cases which cannot be dealt with algorithmically. So far we were unable to find a counterexample for the bound given.

4.1 Parametric Risch differential equation

Let (F, D) be a differential field and $C := \mathrm{Const}(F)$, then the problem of solving the parametric Risch differential equation in (F, D), which is nothing but solving linear first order differential equations in their coefficient field, can be viewed as a generalization of the limited integration problem in (F, D).

Problem 4.1 (parametric Risch differential equation)**.** *Given:* $b, f_0, \dots, f_m \in F$.
Find: $\mathbf{c}_1, \dots, \mathbf{c}_n \in C^{m+1}$ *and* $g_1, \dots, g_n \in F$ *such that* $\{(g_j, \mathbf{c}_j) \mid j \in \{1, \dots, n\}\}$ *is a basis of the* C*-vector space of all solutions* $(g, \mathbf{c}) = (g, c_0, \dots, c_m) \in F \times C^{m+1}$ *of*

$$Dg + bg = \sum_{i=0}^{m} c_i f_i.$$

Risch was led to consider this problem as part of his main theorem in [Ris69]. There is a rich literature on this problem, so we do not present any details here and instead refer to the presentation in Chapters 6 and 7 of [Bro] and the references therein, in particular to the works of Bronstein and Davenport.

We just briefly outline the structure of the algorithm in monomial extensions $F = K(t)$, which basically will be the same for all algorithms discussed in this chapter. First the normal part of the denominator of the solutions is determined based on the denominators of b and f_i and also the special part of the denominator needs to be bounded by solving some subproblem, the parametric logarithmic derivative problem in this case. Then the polynomial solutions of the equation for the numerator are computed which—special to the Risch differential equation—can be reduced to the same problem but b and f_i being polynomials. Those solutions are computed by bounding the degree and comparing coefficients of the powers of t.

The results given in [Bro] do not provide a complete algorithm in admissible differential fields. First, the procedure for solving the parametric logarithmic derivative problem (used for bounding the solutions in hyperexponential extensions) is not complete, we will complete this in Section 4.3.1. Second, the computation of the coefficients of polynomial solutions in nonlinear extensions is not complete for all cases relevant to us. In certain cases also systems of differential equations arise as illustrated below, which can be reduced to scalar equations of higher order by the algorithms mentioned in Section 4.3.3. These equations then can be solved by methods given in [Sin91, Bro92], which we discuss in Section 4.2. Alternatively, for reasons of efficiency one may want to solve these systems directly without uncoupling. Some new results for this are given in Section 4.4.

Based on Chapter 7 of [Bro] and the algorithms discussed in Sections 4.2 and 4.3 one can prove the following theorem. The restriction on the ordering of the generators t_i comes from solving Problem 4.12 based on Theorem 4.28.

Theorem 4.2. *Let* $(F, D) = (C(t_1, \dots, t_n), D)$ *be an admissible differential field with the restriction that for any two non-Liouvillian monomials* t_i *and* t_j, $i < j$, *none of the monomials* $t_{i+1}, \dots, t_{j-1}$ *in between is allowed to be a hyperexponential monomial.*
Then we can solve the parametric Risch differential equation in (F, D).

In the remaining part of the section we show how one is naturally led to consider systems of differential equation resp. scalar differential equations of arbitrary order. Consider the situation of t being nonlinear and let $d := \deg_t(Dt) \geq 2$. Assume $b \in K[t]$ has $\deg_t(b) = d-1 > 0$ and $\operatorname{lc}_t(b) = -n \operatorname{lc}_t(Dt)$ for some $n \in \mathbb{N}^+$ and also $f_i \in K[t]$. When looking for polynomial solutions $g = g_k t^k + \dots + g_0$, $g_i \in K$, in most cases we can determine g_k by comparing the coefficients of t^{k+d-1} in the differential equation yielding $(\operatorname{lc}_t(b) + k \operatorname{lc}_t(Dt))g_k = c_0 \operatorname{coeff}(f_0, t^{k+d-1}) + \dots + c_m \operatorname{coeff}(f_m, t^{k+d-1})$. After that we plug g_k in and proceed with g_{k-1} down to g_0 the same way. In the unlucky case $k = n$ the left hand side of this equation is zero and we cannot determine g_n from it. Proceeding to comparing coefficients of the next lower power t^{n+d-2} in the Risch differential equation and so on down to t^0 we obtain a coupled differential algebraic system of $n+d-1$ equations for $g_0, \dots, g_n \in K$, which in full generality has the following shape.

$$D\begin{pmatrix} g_0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} * & * & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ * & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & & & & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & & & & \ddots & * \\ 0 & \cdots & 0 & * & \cdots & \cdots & \cdots & * \\ 0 & \cdots & \cdots & 0 & * & \cdots & \cdots & * \\ \vdots & & & & \ddots & \ddots & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & * & * \end{pmatrix} \begin{pmatrix} g_0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ g_n \end{pmatrix} = c_0 \begin{pmatrix} * \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ * \end{pmatrix} + \dots + c_m \begin{pmatrix} * \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ * \end{pmatrix} \tag{4.1}$$

More precisely, the system matrix $B = (b_{i,j})_{i \in \{0,\dots,n+d-2\}, j \in \{0,\dots,n\}}$ is from $K^{(n+d-1)\times(n+1)}$ and has a bandwidth of $d+1$ diagonals (one superdiagonal and $d-1$ subdiagonals). We will discuss how to reduce this system to scalar differential equations in Section 4.3.3. In practice $d = 2$ is the relevant case and we have a purely differential system with tridiagonal matrix, which we either solve directly in some cases or from which we can compute a single differential equation of order $n+1$ for g_n by solving the last n equations for $g_{n-1}, \dots, g_0$ explicitly in terms of $g_n, c_0, \dots, c_m$.

Exactly this situation occurs often in computing integrals of non-Liouvillian functions discussed in Section 2.6.2. Consider a solution $(\varphi_1, \varphi_2)^T$ of (2.20), $a_{i,j} \in K$, and let $t = \frac{\varphi_2}{\varphi_1}$ satisfy (2.21)

$$Dt = -a_{12}t^2 + (a_{22} - a_{11})t + a_{21}$$

and the conditions given in Theorem 2.53. Then by Corollary 2.54 φ_1 is a hyperexponential monomial over $(K(t), D)$ with $\frac{D\varphi_1}{\varphi_1} = a_{12}t + a_{11}$, see (2.24). In practice we often encounter integrands which are homogeneous polynomials in $K[\varphi_1, \varphi_2]$, in other words the integrand is of the form

$$f = (f_n t^n + f_{n-1}t^{n-1} + \dots + f_0)\varphi_1^n$$

for some $f_i \in K$ and $n \in \mathbb{N}^+$, where for the sake of a simpler presentation we now neglect that in general we need to treat linear combinations of such integrands. By Theorem 3.12 we see that any elementary integral of f over $(K(t,\varphi_1),D)$ is of the form $g\varphi_1^n$ for some $g \in K(t)$ which is the solution of the Risch differential equation

$$Dg + n(a_{12}t + a_{11})g = f_n t^n + \cdots + f_0.$$

By Theorem 6.1.2 and Lemma 6.3.5 in [Bro] it follows that $g \in K[t]$ and $\deg_t(g) \leq n$. Note that $b = n(a_{12}t + a_{11})$ satisfies $\deg_t(b) = \deg_t(Dt) - 1$ and $\operatorname{lc}_t(b) = -n \operatorname{lc}_t(Dt)$ as discussed above. Hence the coefficients of the polynomial g are found as the solution $\mathbf{g} = (g_0, \ldots, g_n)^T \in K^{n+1}$ of the system

$$D\mathbf{g} + B\mathbf{g} = \mathbf{f} \tag{4.2}$$

where $B = (b_{i,j})_{i,j\in\{0,\ldots,n\}}$ is tridiagonal with entries $b_{j-1,j} = ja_{21}$, $b_{j,j} = ja_{22} + (n-j)a_{11}$, and $b_{j+1,j} = (n-j)a_{12}$ and inhomogeneous part $\mathbf{f} = (f_0, \ldots, f_n)$. These systems are similar to the ones arising in [PB84] and [Chy00] and are used as examples in [BR12].

4.2 Equations of arbitrary order

Problem 4.3 (parametric linear ODEs)**.** *Given:* $a_0, \ldots, a_{d-1}, f_0, \ldots, f_m \in F$.

Find: $\mathbf{c}_1, \ldots, \mathbf{c}_n \in C^{m+1}$, *where* $C := \operatorname{Const}(F)$, *and corresponding* $g_1, \ldots, g_n \in F$ *such that* $\{(g_j, \mathbf{c}_j) \mid j \in \{1, \ldots, n\}\}$ *is a basis of the* C*-vector space of all solutions* $(g, \mathbf{c}) = (g, c_0, \ldots, c_m) \in F \times C^{m+1}$ *of*

$$D^d g + a_{d-1}D^{d-1}g + \cdots + a_0 g = (f_0, \ldots, f_m) \cdot \mathbf{c}_j.$$

Note that the parametric Risch differential equation discussed in Section 4.1 is a special case of this problem, where the order of the equation is $d = 1$. In the case where t is Liouvillian the recursive solving of Risch differential equations generates Risch differential equations in K and related problems. If t is nonlinear, however, by this process we also run into parametric linear ODEs of higher order in K.

Abramov [Abr89] gave an algorithm to solve this problem in $(F,D) = (C(x), \frac{d}{dx})$. We mainly rely on the algorithms for Liouvillian differential fields given by Singer in [Sin91] and their partial generalizations to monomial extensions by Bronstein [Bro92]. Some related work was also done by Fredet [BF99, Fre01, Fre04]. In the following we will outline many parts of the algorithm which results in Theorem 4.19.

In analogy to the degree ω_p of the derivation at p we define $\omega_{L,p}$ for differential operators $L \in K(t)[D]$ as follows, cf. Definition 5.1 in [Bro92].

Definition 4.4. *Let* $a_0, \ldots, a_d \in K(t)$, $a_d \neq 0$, *and let* $p \in K[t]$ *squarefree (or* $p = \frac{1}{t}$*). For the differential operator* $L := \sum_{k=0}^{d} a_k D^k$ *we define*

$$\omega_{L,p} := \min_{i\in\{0,\ldots,d\}} (\nu_p(a_k) + k\omega_p).$$

Note that this definition extends $\omega_{D,p} = \omega_p$ and that Lemma 2.14 implies

$$\nu_p(L(y)) \geq \nu_p(y) + \omega_{L,p}.$$

If for a solution $g \in K(t)$ of $L(g) = \sum_i c_i f_i$ we have equality in $\nu_p(L(g)) \geq \nu_p(g) + \omega_{L,p}$, then by $\nu_p(L(g)) \geq \min_i \nu_p(f_i)$ we obtain the lower bound

$$\nu_p(g) \geq \min_i \nu_p(f_i) - \omega_{L,p}. \tag{4.3}$$

In order to have a general lower bound on $\nu_p(g)$ covering all cases we need to identify the cases where the inequality $\nu_p(L(g)) \geq \nu_p(g) + \omega_{L,p}$ is strict, which is done in the following by a suitable definition of indicial equations for various types of p. The following short notation will be useful in defining these equations.

Definition 4.5. *For $k \in \mathbb{N}$ and $n \in \mathbb{Z} \setminus \{0\}$ we define $P_n^k \in \mathbb{Z}[z]$ by $P_n^k(z) := \prod_{i=0}^{k-1}(z + in)$.*

Obviously we have $\deg(P_n^k) = k$ by definition. Using the Pochhammer symbol we can write $P_n^k(z) = n^k \left(\frac{z}{n}\right)_k$, in particular $P_1^k(z) = (z)_k$.

4.2.1 The normal part of the denominator

Carefully analyzing the derivatives of arbitrary $y \in K(t)$ at normal polynomials p as was done in [Sin91] and [Bro92] we can prove the following lemma on the orders and local leading coefficients, which is the key to computing general bounds on the (normal part of the) denominators of solutions of ODEs.

Lemma 4.6. *Let $p \in K[t] \setminus K$ normal, let $k \in \mathbb{N}$, and let $y \in K(t)$. Then we have $\nu_p(D^k y) \geq \nu_p(y) - k$ and $\pi_p(p^{k-\nu_p(y)} D^k y) = \operatorname{llc}_p(y) P_{-1}^k(\nu_p(y)) \pi_p(Dp)^k$ with equality in the first relation if and only if $\nu_p(y) \notin \{0, \ldots, k-1\}$.*

For a differential operator $L := \sum_{k=0}^d a_k D^k \in K(t)[D]$ this means that any $y \in K(t)$ either satisfies $\nu_p(L(y)) = \nu_p(y) + \omega_{L,p}$ or

$$\operatorname{llc}_p(y) \sum_{k=0}^d \delta_{\nu_p(a_k)-k,\omega_{L,p}} \operatorname{llc}_p(a_k) P_{-1}^k(\nu_p(y)) \pi_p(Dp)^k = 0, \tag{4.4}$$

where $\delta_{i,j}$ denotes the Kronecker delta. This equation gives us a hint how the indicial equation should look like in this case. If $\operatorname{llc}_p(y)$ is invertible in K_p, which can be ensured by requiring p to be irreducible, then we immediately arrive at the indicial equation resp. indicial polynomial in $K_p[z]$ given by the following lemma. From this we easily obtain an indicial polynomial in $K[z]$ by (4.5) below.

Lemma 4.7. *Let $a_0, \ldots, a_d \in K(t)$, $a_d \neq 0$, and let $p \in K[t]$ be irreducible and normal. Define the operator $L := \sum_{k=0}^d a_k D^k$ and let $y \in K(t)$. Then $\nu_p(L(y)) > \nu_p(y) + \omega_{L,p}$ if and only if $\nu_p(y)$ is a root of*

$$\sum_{k=0}^d \delta_{\nu_p(a_k)-k,\omega_{L,p}} \operatorname{llc}_p(a_k) P_{-1}^k(z) \pi_p(Dp)^k \in K_p[z].$$

Since the summands of the polynomial $P \in K_p[z]$ in the previous lemma have pairwise distinct degree in z and not all summands are identically zero, there are only finitely many zeros of P. Computationally we take the canonical representants in $K[t]$ of the coefficients of the polynomial $P \in K_p[z]$ given in the lemma above and form the indicial polynomial in $K[z]$ by

$$P_{L,p}(z) := \gcd_z(\operatorname{coeff}(P, t^0), \ldots, \operatorname{coeff}(P, t^{\deg_t(p)-1})) \in K[z], \tag{4.5}$$

which has the same integer roots as P.

Following Bronstein we can drop the assumption of p being irreducible, in which case there exists a factor q of p such that $\operatorname{llc}_p(y)$ is invertible in K_q. Hence starting from (4.4) we have to use the resultant in order to form an indicial polynomial with coefficients from K, cf. Lemmas 5.2 and 5.3 in [Bro92].

Lemma 4.8. *Let $a_0, \ldots, a_d \in K(t)$, $a_d \neq 0$, and let $p \in K[t] \setminus K$ be normal and balanced w.r.t. $\{a_0, \ldots, a_d\}$. Define the operator $L := \sum_{k=0}^{d} a_k D^k$ and let $y \in K(t)$ such that $\nu_p(L(y)) > \nu_p(y) + \omega_{L,p}$. Then $\nu_p(y)$ is a root of*

$$\operatorname{res}_t\left(\sum_{k=0}^{d} \delta_{\nu_p(a_k)-k,\omega_{L,p}} \operatorname{llc}_p(a_k) P_{-1}^k(z)\pi_p(Dp)^k, p\right) \in K[z].$$

Based on Lemma 4.7 we can compute the normal part of the universal denominator of the solutions of a linear differential equation, if we can factor polynomials from $K[t]$ into irreducibles. Theorem 4.9 below is a straightforward generalization of parts of Lemma 3.2 in [Sin91]. If we just rely on balanced factorizations, then a similar algorithm can be given based on Lemma 4.8, but we do not give the details here and refer to [Bro92] instead.

Theorem 4.9. *Assume $C := \operatorname{Const}(K(t)) = \operatorname{Const}(K)$ and assume that we can determine the integer roots of polynomials with coefficients in K (see Section 2.4.5) and that we can factor such polynomials into irreducibles. Let $a_0, \ldots, a_{d-1}, f_0, \ldots, f_m \in K(t)$ and define the operator $L : K(t) \to K(t)$ by $L(y) := D^d y + a_{d-1}D^{d-1}y + \cdots + a_0 y$.*
Then we can compute $b \in K(t)^*$ such that for any solution $(g, \mathbf{c}) \in K(t) \times C^{m+1}$ of $L(g) = (f_0, \ldots, f_m) \cdot \mathbf{c}$ we have that

$$bg \in K(t)_{red}.$$

Proof. By assumption we can determine all monic irreducible normal $p \in K[t]$ such that $\nu_p(a_k) \leq k - d$ for some $k \in \{0, \ldots, d-1\}$ or $\nu_p(f_i) < -d$ for some $i \in \{0, \ldots, m\}$. These are finitely many and for each of them we compute $\omega_{L,p} = \min\limits_{k \in \{0,\ldots,d-1\}} (\nu_p(a_k) - k)$ and $\beta_p := \min_i \nu_p(f_i)$ as well as

$$\begin{aligned} \mu_p &:= \inf\{z \in \mathbb{Z} \mid P_{L,p}(z) = 0\} \text{ and} \\ \lambda_p &:= \min(\beta_p - \omega_{L,p}, \mu_p), \end{aligned}$$

where $P_{L,p} \in K[z]$ is given by (4.5). Then we compute the following product over all those p

$$b := \prod_p p^{-\lambda_p} \in K(t)^*.$$

For verifying the desired property of b we fix a solution $(g, \mathbf{c}) \in K(t) \times C^{m+1}$ of $L(g) = (f_0, \dots, f_m) \cdot \mathbf{c}$ and we prove $\nu_p(g) \geq -\nu_p(b)$ for any monic irreducible normal $p \in K[t]$. First, assume that $\nu_p(a_k) \leq k - d$ for some $k \in \{0, \dots, d-1\}$ or $\nu_p(f_i) < -d$ for some $i \in \{0, \dots, m\}$. If $\nu_p(L(g)) = \nu_p(g) + \omega_{L,p}$, then $\nu_p(g) \geq \beta_p - \omega_{L,p} \geq -\nu_p(b)$. Otherwise $\nu_p(g)$ is a root of $P_{L,p}(z) = 0$ by Lemma 4.7 and hence $\nu - p(g) \geq \mu_p \geq -\nu_p(b)$ again. Now, assume that $\nu_p(a_k) > k - d$ for all $k \in \{0, \dots, d-1\}$ and $\nu_p(f_i) \geq -d$ for all $i \in \{0, \dots, m\}$ instead. If we had $\nu_p(g) < 0$, then we would have $\nu_p(D^d g) = \nu_p(g) - d < \nu_p(L(g) - D^d g)$ by Lemma 4.6 and hence $\nu_p(L(g)) = \nu_p(g) - d < -d \leq \min_i \nu_p(f_i)$ in contradiction to $\nu_p(L(g)) \geq \min_i \nu_p(f_i)$. Therefore $\nu_p(g) \geq 0 = -\nu_p(b)$. $\square$

Theorem 4.10. *([Bro92, Thm 5.5]) Assume $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$ and assume that we can determine the integer roots of polynomials with coefficients in K (see Section 2.4.5) and that we can compute balanced factorizations of such polynomials (see Section 2.4.2). Let $a_0, \dots, a_{d-1}, f_0, \dots, f_m \in K(t)$ and define the operator $L : K(t) \to K(t)$ by $L(y) := D^d y + a_{d-1} D^{d-1} y + \dots + a_0 y$.*
Then we can compute $b \in K(t)^*$ such that for any solution $(g, \mathbf{c}) \in K(t) \times C^{m+1}$ of $L(g) = (f_0, \dots, f_m) \cdot \mathbf{c}$ we have that

$$bg \in K(t)_{red}.$$

4.2.2 Degree bounds

With the normal part of the universal denominator constructed in the previous theorems we are now left with determining bounds on the orders of $\tilde{g} = bg$ at special polynomials p and at $p = \frac{1}{t}$. Note that $\tilde{g} = bg$ are exactly the solutions of a parametric linear ODE which can be obtained from the original parametric ODE by plugging in $y = \tilde{y}/b$ and multiplying by b, cf. Lemma 4.29 later with $u = -\frac{Db}{b}$.

In admissible differential fields we have $K(t)_{red} = K[t]$, except when t is hyperexponential and we have $K(t)_{red} = K[t, \frac{1}{t}]$. So we will focus on bounds on the order at $p = \frac{1}{t}$, which corresponds to bounds on the degree of polynomial solutions in $K[t]$, since we can compute bounds on the order at t for the Laurent polynomial solutions by the same methods if t is hyperexponential. We distinguish the three cases of t being a primitive, a hyperexponential, or a nonlinear monomial. For primitive t we refer to the algorithm given in [Sin91] and for nonlinear t a bound can be obtained by the same methods as we computed the normal part of the denominator, cf. [Bro92]. Our main concern will be the case of a hyperexponential t.

Primitive extensions

In case of t being primitive over (K, D) a non-trivial indicial equation cannot be given directly in general. An iterative algorithm to obtain an equation for bounding the degree of solutions in $K[t]$ by not only looking at the leading terms but successively also at lower powers of t was presented by Singer in [Sin91]. This algorithm always terminates after finitely many steps, but no a-priori bound on the number of iterations needed is currently known. Without giving the details of the algorithm we just state the result.

Theorem 4.11. *([Sin91, Lemma 3.8]) Assume $Dt \in K$ and $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Assume further that we can solve parametric linear ODEs in (K, D). Let $a_0, \ldots, a_d, f_0, \ldots, f_m \in K(t)$, $a_d \neq 0$, and define the operator $L : K(t) \to K(t)$ by $L(y) := a_d D^d y + \cdots + a_0 y$.*
Then we can compute $n \in \mathbb{Z}$ such that for any solution $(g, \mathbf{c}) \in K[t] \times C^{m+1}$ of $L(g) = (f_0, \ldots, f_m) \cdot \mathbf{c}$ we have that

$$\deg_t(g) \leq n.$$

Hyperexponential extensions

For determining the upper and lower degree bounds of Laurent polynomial solutions $g \in K[t, \frac{1}{t}]$ the corresponding equation to solve in general is not a purely algebraic equation, but is of the following form.

Problem 4.12. *Given: $L \in K[D]$ and a hyperexponential monomial t over (K, D).*

Find: a finite set $\Lambda \subset \mathbb{Z}$ such that $n \in \Lambda$ for all $(y, n) \in K^* \times \mathbb{Z}$ with

$$L(yt^n) = 0.$$

Singer solves this problem in Lemma 3.3 of [Sin91] by computing a basis of all solutions of the form $e^{\int u}$, $u \in K$, of $L(e^{\int u}) = 0$, which amounts to solving the associated Riccati equation in K (see Section 4.3.2), and then extracting the candidates for n from this basis by solving homogeneous Risch differential equations in $(K(t), D)$. A more direct approach to compute a candidate set for n was taken by Bronstein and Fredet [BF99] for the case $(K, D) = (C(x), \frac{d}{dx})$, which was extended to more general (K, D) by Fredet under the condition that $(K(t), D)$ is what she called a *well-defined exponential extension* of (K, D) [Fre01, Fre04].

The following lemma and theorem present the approach taken by Singer in Lemma 3.3 of [Sin91], except that we make use of the parametric logarithmic derivative problem for extracting the candidates for n from the basis of hyperexponential solutions in the lemma.

Lemma 4.13. *Assume that we can compute a basis for the hyperexponential solutions over (K, D) of ODEs with coefficients in K (see Section 4.3.2) and that we can solve the parametric logarithmic derivative problem in (K, D) (see Section 4.3.1). Then we can solve Problem 4.12 over (K, D).*

Proof. Fix an operator $L \in K[D]$. By assumption we can compute $u_1, \ldots, u_m \in K$ such that any $u \in K$ with $L(e^{\int u}) = 0$ satisfies $e^{\int u} = ve^{\int u_j}$ for some $v \in K$ and $j \in \{1, \ldots, m\}$. For each of the u_j we solve the parametric logarithmic derivative problem $\frac{Dg}{kg} = -u_j + c\frac{Dt}{t}$ in (K, D). Finally, based on the solutions $(g_j, k_j, c_j) \in K^* \times \mathbb{Z} \times \mathbb{Q}$ obtained we set

$$\Lambda := \left\{ n \in \mathbb{Z} \,\middle|\, \exists j \in \{1, \ldots, m\} \exists g \in K : \tfrac{Dg}{g} = -u_j + n\tfrac{Dt}{t} \right\}.$$

Since t satisfies the properties given in Theorem 2.51 the set Λ is finite. Now, fix $y \in K^*$ and $n \in \mathbb{Z}$ such that $L(yt^n) = 0$. By definition there exist $j \in \{1, \ldots, m\}$ and $v \in K$ such that $yt^n = ve^{\int u_j}$. Consequently $g = \frac{v}{y}$ satisfies $\frac{Dg}{g} = -u_j + n\frac{Dt}{t}$ and hence $n \in \Lambda$. □

Theorem 4.14. *Assume $\frac{Dt}{t} \in K$ and $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Assume further that we can solve Problem 4.12 over (K, D). Let $a_0, \ldots, a_d, f_0, \ldots, f_m \in K(t)$, $a_d \neq 0$, and define the operator $L : K(t) \to K(t)$ by $L(y) := a_d D^d y + \cdots + a_0 y$.*
Then we can compute $n_0, n_1 \in \mathbb{Z}$ such that for any solution $(g, \mathbf{c}) \in K[t, \frac{1}{t}] \times C^{m+1}$ of $L(g) = (f_0, \ldots, f_m) \cdot \mathbf{c}$ we have that

$$\nu_t(g) \geq n_0 \qquad \textit{and} \qquad \nu_{1/t}(g) \geq n_1.$$

Proof. Let $p_0 := t$ and $p_1 := \frac{1}{t}$. Compute $\omega_{L,p_j} = \min_k \nu_{p_j}(a_k)$ and $\beta_j := \min_i \nu_{p_j}(f_i)$ for $j \in \{0, 1\}$ as well as $L_j := \sum_{k=0}^{d} \pi_{p_j}(p_j^{-\omega_{L,p_j}} a_k) D^k \in K[D]$. For L_0 and L_1 we compute corresponding sets $\Lambda_0, \Lambda_1 \subset \mathbb{Z}$ by solving Problem 4.12. After that we compute $\mu_j := \inf((-1)^j \Lambda_j)$ and $n_j := \min(\beta_j - \omega_{L,p_j}, \mu_j)$ for $j \in \{0, 1\}$.

Now, fix a solution $(g, \mathbf{c}) \in K[t, \frac{1}{t}] \times C^{m+1}$ of $L(g) = (f_0, \ldots, f_m) \cdot \mathbf{c}$. If $\nu_{p_j}(L(g)) = \nu_{p_j}(g) + \omega_{L,p_j}$, then $\nu_{p_j}(g) \geq \beta_j - \omega_{L,p_j} \geq n_j$ by (4.3). Otherwise $\nu_{p_j}(L(g)) > \nu_{p_j}(g) + \omega_{L,p_j}$ implies $0 = \pi_{p_j}(p_j^{-\nu_{p_j}(g) - \omega_{L,p_j}} L(g)) = p_j^{-\nu_{p_j}(g)} L_j(\mathrm{llc}_{p_j}(g) p_j^{\nu_{p_j}(g)})$, which in turn implies that $(-1)^j \nu_{p_j}(g) \in \Lambda_j$ by definition of Λ_j. Hence we have $\nu_{p_j}(g) \geq \mu_j \geq n_j$ also in this case. □

Nonlinear extensions

As mentioned above this case is analogous to computing the normal part of the universal denominator. The following theorem is a variant of Theorem 6.5 from [Bro92].

Theorem 4.15. *Assume t is a nonlinear monomial and $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Assume further that we can determine the integer roots of polynomials with coefficients in K (see Section 2.4.5). Let $a_0, \ldots, a_d, f_0, \ldots, f_m \in K(t)$, $a_d \neq 0$, and define the operator $L : K(t) \to K(t)$ by $L(y) := a_d D^d y + \cdots + a_0 y$.*
Then we can find $n \in \mathbb{Z}$ such that for any solution $(g, \mathbf{c}) \in K[t] \times C^{m+1}$ of $L(g) = (f_0, \ldots, f_m) \cdot \mathbf{c}$ we have that

$$\deg_t(g) \leq n.$$

Proof. Let $j := \deg_t(Dt) - 1 \in \mathbb{N}^+$ and compute $\omega_{L,1/t} = \min_k(\nu_{1/t}(a_k) - jk)$ and $\beta := \min_i \nu_{1/t}(f_i)$. In addition, compute

$$\begin{aligned} P_{L,1/t}(z) &:= \sum_{k=0}^{d} \delta_{\nu_{1/t}(a_k) - jk, \omega_{L,1/t}} \mathrm{llc}_{1/t}(a_k) \,\mathrm{lc}_t(Dt)^k P_j^k(z) \in K[z], \\ \mu &:= \sup\left\{ z \in \mathbb{Z} \;\middle|\; P_{L,1/t}(z) = 0 \right\}, \text{ and} \\ n &:= \max(\omega_{L,1/t} - \beta, \mu). \end{aligned}$$

Since the summands of $P_{L,1/t}$ have pairwise distinct degree in z and not all summands are identically zero, there are only finitely many zeros of $P_{L,1/t}$.

Now, let $(g, \mathbf{c}) \in K[t] \times C^{m+1}$ such that $L(g) = (f_0, \ldots, f_m) \cdot \mathbf{c}$. From (2.3) we obtain $\deg_t(D^k g) \leq \deg_t(g) + jk$. Moreover, for $g \neq 0$ we have $\mathrm{coeff}(D^k g, t^{\deg_t(g) + jk}) = P_j^k(\deg_t(g)) \,\mathrm{lc}_t(g) \,\mathrm{lc}_t(Dt)^k$. Note that $\nu_{1/t}(L(g)) \geq \min_k \nu_{1/t}(a_k D^k g) \geq \omega_{L,1/t} - \deg_t(g)$.

Either we have $\nu_{1/t}(L(g)) = \omega_{L,1/t} - \deg_t(g)$, which includes the case $g = 0$ and implies $\deg_t(g) \le \omega_{L,1/t} - \beta \le n$ by (4.3), or we have

$$\begin{aligned} 0 &= \pi_{1/t}(t^{\omega_{L,1/t}-\deg_t(g)} L(g)) \\ &= \sum_{k=0}^{d} \delta_{\nu_{1/t}(a_k)-jk,\omega_{L,1/t}} \operatorname{llc}_{1/t}(a_k) \operatorname{coeff}(D^k g, t^{\deg_t(g)+jk}) \\ &= \sum_{k=0}^{d} \delta_{\nu_{1/t}(a_k)-jk,\omega_{L,1/t}} \operatorname{llc}_{1/t}(a_k) \operatorname{lc}_t(g) P_j^k(\deg_t(g)) \operatorname{lc}_t(Dt)^k \\ &= \operatorname{lc}_t(g) P_{L,1/t}(\deg_t(g)), \end{aligned}$$

which implies $P_{L,1/t}(\deg_t(g))$ and hence again $\deg_t(g) \le n$. □

4.2.3 Main recursive step

The following theorems summarize what can be done in the different cases of monomials t, the proofs are similar to Lemma 3.2 in [Sin91].

Theorem 4.16. *Assume $Dt \in K$ and $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Assume further that we can determine the integer roots of polynomials with coefficients in K (see Section 2.4.5), that we can factor such polynomials into irreducibles, and that we can solve parametric linear ODEs in (K, D).*
Then we can solve parametric linear ODEs in $(K(t), D)$.

Theorem 4.17. *Assume $\frac{Dt}{t} \in K$ and $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Assume that we can determine the integer roots of polynomials with coefficients in K (see Section 2.4.5), that we can factor such polynomials into irreducibles, and that we can solve parametric linear ODEs in (K, D). Assume further that we can compute a basis for the hyperexponential solutions over (K, D) of ODEs with coefficients in K (see Section 4.3.2) and that we can solve the parametric logarithmic derivative problem in (K, D) (see Section 4.3.1).*
Then we can solve parametric linear ODEs in $(K(t), D)$.

Theorem 4.18. *Assume t is nonlinear and $C := \mathrm{Const}(K(t)) = \mathrm{Const}(K)$. Assume further that we can determine the integer roots of polynomials with coefficients in K (see Section 2.4.5), that we can factor such polynomials into irreducibles, and that we can solve parametric linear ODEs in (K, D).*
Then we can solve parametric linear ODEs in $(K(t), D)$.

Combining these with the relevant results of the other sections we can prove the following, where the restriction on the order of the generators of the field comes from solving Problem 4.12 based on Theorem 4.28.

Theorem 4.19. *Let $(F, D) = (C(t_1, \ldots, t_n), D)$ be an admissible differential field with the restriction that for any non-Liouvillian monomial t_i none of the monomials $t_{i+1}, \ldots, t_n$ above is allowed to be a hyperexponential monomial.*
Then we can solve parametric linear ODEs in (F, D).

4.3 Related problems

4.3.1 Parametric logarithmic derivative problem

Let (F, D) be a differential field, then the parametric logarithmic derivative problem in (F, D) can be seen as a variant of parametric elementary integration where in addition the integrals are required to be expressible as logarithms of radicals of elements from F.

Problem 4.20 (parametric logarithmic derivative problem)**.** *Given:* $f_0, \ldots, f_m \in F$.

Find: $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{Q}^{m+1}$ *and corresponding* $g_1, \ldots, g_n \in F^*$ *and* $k_1, \ldots, k_n \in \mathbb{Z} \setminus \{0\}$ *such that*

$$\frac{Dg_j}{k_j g_j} = (f_0, \ldots, f_m) \cdot \mathbf{c}_j$$

and $\{\mathbf{c}_j \mid j \in \{1, \ldots, n\}\}$ *is a basis of the* $\mathbb{Q}$*-vector space of all* $\mathbf{c} = (c_0, \ldots, c_m) \in \mathbb{Q}^{m+1}$ *for which* $\exists g \in F^* \exists k \in \mathbb{Z} \setminus \{0\} : \frac{Dg}{kg} = \sum\limits_{i=0}^{m} c_i f_i$.

It is easy to see that the set of solutions indeed is a $\mathbb{Q}$-vector space: note that for $g_j \in F^*$, $k_j \in \mathbb{Z} \setminus \{0\}$, and $\mathbf{c} \in \mathbb{Q}^n$ by the logarithmic derivative identity (2.1) we have

$$\left(\frac{Dg_1}{k_1 g_1}, \ldots, \frac{Dg_n}{k_n g_n}\right) \cdot \mathbf{c} = \frac{Dg}{kg} \tag{4.6}$$

with $g := \prod_{j=1}^{n} g_j^{c_j k / k_j}$ and $k \in \mathbb{Z} \setminus \{0\}$ such that $\frac{c_j}{k_j} k \in \mathbb{Z}$ for all $j \in \{1, \ldots, n\}$.

The solution of this problem in $(K(t), D)$ will be split across several lemmas reflecting the phases of the integration algorithm corresponding to the Sections 3.1 through 3.4. Lemma 4.22 is a variant of Exercise 7.1 in [Bro]. The main idea of Lemma 4.23 and Lemma 4.24 was already sketched in [Ris69, p. 187] for the case t being exponential over (K, D). The algorithm is summarized in Theorem 4.25. If the differential field (F, D) is admissible, then we can use this theorem for recursively solving the parametric logarithmic derivative problem in (F, D). This theorem resp. Lemma 4.23 is the reason for including factorization in the requirements of admissible differential fields. The base case of the problem in the constant field is trivial since $\frac{Dg}{kg} = 0$ for any $g \in \mathrm{Const}(F)^*$ and $k \in \mathbb{Z} \setminus \{0\}$. Now we state the main result of this section, which is an immediate consequence of Theorem 4.25.

Theorem 4.21. *Let* $(F, D) = (C(t_1, \ldots, t_n), D)$ *be an admissible differential field. Then we can solve the parametric logarithmic derivative problem over* (F, D).

Proof. The proof proceeds by induction on $n \in \mathbb{N}$.
$n = 0$: For $F = C$ we compute a basis $\mathbf{c}_1, \ldots, \mathbf{c}_l \in \mathbb{Q}^{m+1}$ of the $\mathbb{Q}$-vector space $\{\mathbf{c} \in \mathbb{Q}^{m+1} \mid (f_0, \ldots, f_m) \cdot \mathbb{c} = 0\}$ and set $g_j := 1$ and $k_j := 1$ for all $j \in \{1, \ldots, l\}$. This trivially solves the problem since $\frac{Dg}{kg} = 0$ for any $g \in C^*$ and $k \in \mathbb{Z} \setminus \{0\}$.
$n > 0$: Let $K := C(t_1, \ldots, t_{n-1})$, so $(F, D) = (K(t_n), D)$ fulfils the assumptions of Theorem 4.25 by Definition 3.3. Then by induction hypothesis Theorem 4.25 implies that we can solve the parametric logarithmic derivative problem over (F, D). $\square$

Lemma 4.22. *Let $d := \deg_t(Dt)$. Assume that we can find a basis for the rational solutions of linear systems with coefficients from K (see Section 2.4.4). Then for any $a_0, \dots, a_m, b \in K[t]$, $b \neq 0$, we can compute $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{Q}^{m+1}$ and $w_1, \dots, w_n \in K(t)$ simple such that:*

1. *If for some $\mathbf{c} \in \mathbb{Q}^{m+1}$ there exist $g \in K(t)^*$ and $k \in \mathbb{Z} \setminus \{0\}$ with $\frac{Dg}{kg} = \frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b}$, then $\mathbf{c} \in \operatorname{span}_{\mathbb{Q}}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.*

2. *$\forall j \in \{1, \dots, n\} : w_j = \frac{(a_0,\dots,a_m)\cdot\mathbf{c}_j}{b}$ and $\nu_{1/t}(w_j) \geq -\max(0, d-1)$ (if $\operatorname{Const}(K) = K$ and $d = 0$ we even require $\nu_{1/t}(w_j) > 0$).*

Proof. First compute $\tilde{b} := \gcd(b, Db)$. Then, consider the linear system $A \cdot \mathbf{c} = 0$ with entries in K obtained from simultaneously comparing the coefficients of powers of t in

$$\sum_{i=0}^{m} (a_i \div t^{\deg(b)+N}) c_i = 0, \tag{4.7}$$

where $N := 0$ if $\operatorname{Const}(K) = K$ and $d = 0$ or $N := \max(1, d)$ otherwise, and

$$\sum_{i=0}^{m} (a_i \bmod \tilde{b}) c_i = 0. \tag{4.8}$$

Next, compute a $\mathbb{Q}$-vector space basis $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{Q}^{m+1}$ of $\ker(A) \cap \mathbb{Q}^{m+1}$ and finally set

$$w_j := \frac{(a_0, \dots, a_m) \cdot \mathbf{c}_j}{b}$$

for each $j \in \{1, \dots, n\}$.

Now we check that these $\mathbf{c}_1, \dots, \mathbf{c}_n$ and $w_1, \dots, w_n$ satisfy the statement of the theorem. Note that [Bro, Lemma 3.4.4] implies that $\frac{b}{\tilde{b}}$ is the product of all normal irreducible factors of b. For each $j \in \{1, \dots, n\}$ we have that $\operatorname{den}(w_j) = \frac{b}{\operatorname{lc}(b)\gcd((a_0,\dots,a_m)\cdot\mathbf{c}_j, b)}$ divides $\frac{b}{\tilde{b}}$ by virtue of (4.8) and hence is normal. This means w_j is simple. Moreover, from (4.7) we see that $\nu_{1/t}(w_j) = \nu_{1/t}((a_0, \dots, a_m)\cdot\mathbf{c}_j) - \nu_{1/t}(b) > -(\deg(b)+N)+\deg(b) = N$ as required. So all properties of $w_1, \dots, w_n$ requested are fulfilled. Now, let $\mathbf{c} \in \mathbb{Q}^{m+1}$ be fixed and assume that there exist $g \in K(t)^*$ and $k \in \mathbb{Z} \setminus \{0\}$ with $\frac{Dg}{kg} = \frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b}$. Let $p \in K[t]$ be an irreducible factor of b, then utilizing Theorem 2.16 we obtain $\nu_p\left(\frac{Dg}{g}\right) \geq -1$ and if $p|Dp$ even $\nu_p\left(\frac{Dg}{g}\right) \geq 0$. Hence $\frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b}$ is simple and (4.8) holds. On the other hand by Theorem 2.17 it follows that $\nu_{1/t}\left(\frac{Dg}{g}\right) \geq -\max(0, d-1)$. In case of $\operatorname{Const}(K) = K$ and $d = 0$ we have $\deg(Dq) < \deg(q)$ for any $q \in K[t]$, which implies $\nu_{1/t}\left(\frac{Dg}{g}\right) > 0$. Altogether, $\nu_{1/t}\left(\frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b}\right) > -N$ and hence (4.7) holds. Now, since $\mathbf{c} \in \mathbb{Q}^{m+1}$ satisfies (4.7) and (4.8) it also satisfies $\mathbf{c} \in \operatorname{span}_{\mathbb{Q}}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ by construction of $\mathbf{c}_1, \dots, \mathbf{c}_n$. $\square$

Enforcement of (4.7) could be dropped from the computations above, since the restriction of $\nu_{1/t}(w)$ is not a necessary requirement for the subsequent lemmas. However, in practice it may be desirable to narrow the space of solution candidates as much and as early as possible.

Lemma 4.23. *Assume that we can find a basis for the rational solutions of linear systems with coefficients from K (see Section 2.4.4) and that we can factor polynomials in $K[t]$ into irreducibles. Then for any $a_0, \dots, a_m, b \in K[t]$ with $b \neq 0$ and $\gcd(b, Db) = 1$ we can compute $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{Q}^{m+1}$ and $b_1, \dots, b_n \in K[t]$ as well as $g_1, \dots, g_n \in K(t)^*$ and $k_1, \dots, k_n \in \mathbb{Z} \setminus \{0\}$ such that:*

1. *If for some $\mathbf{c} \in \mathbb{Q}^{m+1}$ there exist $g \in K(t)^*$ and $k \in \mathbb{Z} \setminus \{0\}$ with $\frac{Dg}{kg} = \frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b}$, then $\mathbf{c} \in \operatorname{span}_{\mathbb{Q}}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$.*
2. *$\forall j \in \{1, \dots, n\} : \frac{Dg_j}{k_j g_j} = \frac{(a_0,\dots,a_m)\cdot\mathbf{c}_j}{b} - b_j$.*

Proof. If $\deg(b) = 0$ then we can choose $\mathbf{c}_1, \dots, \mathbf{c}_{m+1} \in \mathbb{Q}^{m+1}$ to be the canonical basis of $\mathbb{Q}^{m+1}$. Then the statements are trivially fulfilled by $b_j := \frac{a_j}{b}$, $g_j := 1$, and $k_j := 1$. From now we assume that $\deg(b) > 0$. First, compute a factorization of b into irreducibles in $K[t]$ and let $p_1, \dots, p_l \in K[t]$ be the irreducible factors of b. Then consider the linear system $A \cdot (\mathbf{c}, \mathbf{r})^T = 0$ with coefficients in K obtained from comparing the coefficients of the powers of t in

$$\sum_{i=0}^{m} (a_i \bmod p_j) c_i + (Db \bmod p_j) r_j = 0 \tag{4.9}$$

for all $j \in \{1, \dots, l\}$ and compute a $\mathbb{Q}$-vector space basis $(\mathbf{c}_1, \mathbf{r}_1), \dots, (\mathbf{c}_n, \mathbf{r}_n) \in \mathbb{Q}^{m+1} \times \mathbb{Q}^l$ of $\ker(A) \cap \mathbb{Q}^{m+1+l}$. Finally, for each $j \in \{1, \dots, n\}$ compute

$$b_j := \frac{(a_0, \dots, a_m) \cdot \mathbf{c}_j}{b} + \left(\frac{Dp_1}{p_1}, \dots, \frac{Dp_l}{p_l} \right) \cdot \mathbf{r}_j \in K(t),$$

write $\mathbf{r}_j = -\frac{1}{k_j} \tilde{\mathbf{r}}_j$ for some $k_j \in \mathbb{Z} \setminus \{0\}$ and $\tilde{\mathbf{r}}_j \in \mathbb{Z}^l$, and using the multiindex notation set

$$g_j := (p_1, \dots, p_l)^{\tilde{\mathbf{r}}_j} \in K(t).$$

Now we verify that these $\mathbf{c}_1, \dots, \mathbf{c}_n$ and $b_1, \dots, b_n$ satisfy the statement of the theorem. For all $\mathbf{c} \in \mathbb{Q}^{m+1}$ and each $j \in \{1, \dots, l\}$ define $r_j(\mathbf{c}) := -\operatorname{res}_{p_j}\left(\frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b} \right) \in K[t]$. Let $d \in K[t]$ be an inverse of Db modulo b, i.e., $dDb \equiv 1 \pmod{b}$. Then for all $\mathbf{c} \in \mathbb{Q}^{m+1}$, $j \in \{1, \dots, l\}$ and any $r_j \in K[t]$ by Lemma 2.22 we have modulo p_j

$$\sum_{i=0}^{m} a_i c_i + r_j Db \equiv \left(\sum_{i=0}^{m} d a_i c_i + r_j \right) Db \equiv \left(\sum_{i=0}^{m} \operatorname{res}_{p_j}\left(\frac{a_i}{b}\right) c_i + r_j \right) Db \pmod{p_j}.$$

From this we obtain that (4.9) implies $r_j = r_j(\mathbf{c})$ and for $r_j(\mathbf{c}) \in K$ also the converse implication is true. The rest of the proof is based on this important fact. Consequently, by construction of $\mathbf{c}_1, \dots, \mathbf{c}_n$ and $\mathbf{r}_1, \dots, \mathbf{r}_n$ we have that for all $\mathbf{c} \in \mathbb{Q}^{m+1}$ the condition $\forall j \in \{1, \dots, l\} : r_j(\mathbf{c}) \in \mathbb{Q}$ is equivalent to $\mathbf{c} \in \operatorname{span}_{\mathbb{Q}}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. Now, take $\mathbf{c} \in \mathbb{Q}^{m+1}$ fixed such that there exist $g \in K(t)^*$ and $k \in \mathbb{Z} \setminus \{0\}$ with $\frac{Dg}{kg} = \frac{(a_0,\dots,a_m)\cdot\mathbf{c}}{b}$. Then for all $j \in \{1, \dots, l\}$ we verify that $r_j(\mathbf{c}) = -\frac{1}{k} \operatorname{res}_{p_j}\left(\frac{Dg}{g}\right) = -\frac{\nu_{p_j}(g)}{k} \in \mathbb{Q}$ by Lemma 2.21. Therefore, we have $\mathbf{c} \in \operatorname{span}_{\mathbb{Q}}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ by above reasoning. To conclude the proof take

$j \in \{1, \ldots, n\}$ and using the fact above note that for all $i \in \{1, \ldots, l\}$ by construction the i-th component of $\mathbf{r}_j$ is $r_i(\mathbf{c}_j)$ and we have $\operatorname{res}_{p_i}(b_j) = \operatorname{res}_{p_i}\left(\frac{(a_0,\ldots,a_m)\cdot\mathbf{c}_j}{b}\right) + r_i(\mathbf{c}_j) = 0$, which implies $\nu_{p_i}(b_j) \geq 0$. Hence we have $b_j \in K[t]$ and by construction we also have $\frac{Dg_j}{k_j g_j} = \frac{1}{k_j}(\frac{Dp_1}{p_1}, \ldots, \frac{Dp_l}{p_l}) \cdot \tilde{\mathbf{r}}_j = \frac{(a_0,\ldots,a_m)\cdot\mathbf{c}_j}{b} - b_j$. □

Lemma 4.24. *Let $f \in K[t]$, $g \in K(t)^*$ and $k \in \mathbb{Z} \setminus \{0\}$ such that $\frac{Dg}{kg} = f$. Then the following statements hold.*

1. *If t is primitive over (K, D) and* $\operatorname{Const}(K(t)) = \operatorname{Const}(K)$*, then $f, g \in K$.*
2. *If t is hyperexponential over (K, D) and* $\operatorname{Const}(K(t)) = \operatorname{Const}(K)$*, then $f \in K$ and $\frac{D\tilde{g}}{k\tilde{g}} = f - c\frac{Dt}{t}$ with $c := \frac{\nu_{1/t}(g)}{k} \in \mathbb{Q}$ and $\tilde{g} := gt^{ck} \in K^*$.*
3. *If t is a nonlinear monomial over (K, D) and $S^{irr} = \emptyset$, then $f, g \in K$.*

Proof. Let $p \in K[t]$ be irreducible and assume $\nu_p(g) \neq 0$ and $\gcd(p, Dp) = 1$. Then from Theorem 2.16 we would obtain $\nu_p(f) = \nu_p\left(\frac{Dg}{g}\right) = -1$. Hence for any irreducible $p \in K[t]$ with $\nu_p(g) \neq 0$ we have $p \in S^{irr}$. This means there are $\tilde{g} \in K^*$, $p_1, \ldots, p_n \in S^{irr}$, and $k_1, \ldots, k_n \in \mathbb{Z} \setminus \{0\}$ such that $g = \tilde{g}p_1^{k_1} \ldots p_n{}^{k_n}$. From this the statements follow easily by Theorems 2.50 and 2.51. □

Gluing together the previous lemmas we obtain the following theorem. The proof is completely analogous to the proofs of Theorems 3.16 through 3.19 so we omit it. We just refer to (4.6) for the necessary bookkeeping of intermediate expressions.

Theorem 4.25. *Assume t is either Liouvillian over (K, D) with* $\operatorname{Const}(K(t)) = \operatorname{Const}(K)$ *or nonlinear with $S^{irr} = \emptyset$ and that we can solve the parametric logarithmic derivative problem in (K, D). Assume further that we can find a basis for the rational solutions of linear systems with coefficients from K (see Section 2.4.4). If we can factor polynomials in $K[t]$ into irreducibles, then we can solve the parametric logarithmic derivative problem in $(K(t), D)$ as well.*

Remark Avoiding solving in $\mathbb{Q}$ we at least can solve a modified version of the parametric logarithmic derivative problem, where coefficients can be from $C := \operatorname{Const}(K)$. More precisely, we might consider the problem of finding a C-vector space basis of all $\mathbf{c} \in C^{m+1}$ such that there exist $g_1, \ldots, g_l \in K(t)^*$ and $r_1, \ldots, r_l \in C$ with

$$\sum_{j=1}^{l} r_j \frac{Dg_j}{g_j} = \frac{(a_0, \ldots, a_m) \cdot \mathbf{c}}{b}.$$

Analogous results can be obtained by modifying the lemmas above in the obvious way. □

4.3.2 Associated Riccati equations

In Lemma 4.13 we saw that we can solve Problem 4.12 over some differential field (F, D), which is needed for solving linear ODEs in a hyperexponential extension of (F, D), by solving the following more general problem over (F, D).

Problem 4.26 (hyperexponential solutions)**.** *Given:* (F, D) *and* $a_0, \dots, a_{d-1} \in F$.
Find: $u_1, \dots, u_n \in F$ *and* $v_{1,1}, \dots, v_{1,n_1}, \dots, v_{n,1}, \dots, v_{1,n_n} \in F$ *such that if* $y = e^{\int u}$ *is a solution of*

$$D^d y + a_{d-1} D^{d-1} y + \cdots + a_0 y = 0$$

then there are $i \in \{1, \dots, n\}$ *and* $c_1, \dots, c_{n_i} \in \mathrm{Const}(F)$ *with* $y = \sum_{j=1}^{n_i} c_j v_{i,j} e^{\int u_i}$.

Using the procedure given by Singer in Lemma 2.4 of [Sin91] we can reduce the problem of finding a basis of all hyperexponential solutions to finding only one hyperexponential solution at a time. For Lemma 4.13 we only need the u_i and not the $v_{i,j}$ as well.

Problem 4.27 (one hyperexponential solution)**.** *Given:* (F, D) *and* $a_0, \dots, a_{d-1} \in F$.
Find: $u \in F$ *such that* $y = e^{\int u}$ *is a solution of*

$$D^d y + a_{d-1} D^{d-1} y + \cdots + a_0 y = 0.$$

This problem is also interesting on its own as it is the basic building block for computing a basis of all *d'Alembertian* solutions over (F, D) of linear ODEs as explained in [AP94]. D'Alembertian solutions over (F, D) are a subclass of Liouvillian solutions over (F, D), but we do not discuss this further.

Recall Definition 2.33 and Lemma 2.34, for finding hyperexponential solutions the associated Riccati equation defined there is a crucial tool in the algorithm given by Singer in [Sin91]. Note that for $d > 1$ the associated Riccati equation is not a linear equation anymore. Based on Proposition 2.3 and Lemma 2.4 in [Sin91] and Theorems 4.16 and 4.17 we can prove the following theorem.

Theorem 4.28. *Let* $(F, D) = (C(t_1, \dots, t_n), D)$ *be an admissible differential field with the restriction that any monomial* t_i *is required to be a Liouvillian monomial.*
Then we can solve Problems 4.27 and 4.26 over (F, D).

It would be nice to allow also non-Liouvillian monomials among the generators. Bronstein gave a partial result in Theorem 8.4 of [Bro92], but currently there is no complete algorithm for this situation. Based on this result and Lemma 2.2 in [Fre04] we can at least give a heuristic method to compute hyperexponential solutions over any admissible differential field. The only heuristic step will be to solve Problem 4.32 below. Like Singer's algorithm after bounding the orders of the solutions we proceed by producing finitely many candidates for the local leading coefficients of the solutions $u \in K(t)$ and then removing them by the following lemma such that the remaining part of the solution has bigger order. Then again we determine candidates for the next local coefficients and so on. Note that this means that we have to keep track of a series of case distinctions for the subexpressions of the solutions.

Lemma 4.29. *Let $a_0, \dots, a_n \in F$ and let $L(y) := \sum_{i=0}^{n} a_i D^i y$. Furthermore, let $u \in F$ and $\tilde{L}(y) := \sum_{i=0}^{n} \tilde{a}_i D^i y$, where $\tilde{a}_i := \sum_{j=0}^{n-i} \frac{(j+1)_i}{i!} a_{i+j} P_j(u) \in F$. Then for $y \in F^*$ we have*

$$L(ye^{\int u}) = 0 \quad \Longleftrightarrow \quad \tilde{L}(y) = 0.$$

Note that in the above lemma we have $\tilde{a}_0 = R(u)$ and $\tilde{a}_n = a_n$. Let t be a nonlinear monomial over (K, D) with $S^{irr} = \emptyset$. We will outline the steps of the heuristic for $(F, D) = (K(t), D)$. Bronstein's result mentioned above allows us to reduce Problem 4.27 to finding solutions $u \in K(t)$ of the form

$$u = s + \frac{Dp}{p}$$

for some $p, s \in K[t]$. We will not care about the additional information that p is free of d-th powers and relatively prime to each $\operatorname{den}_t(a_i)$. The only heuristic step will be to bound the degree of p, but first we focus on s.

If $\deg_t(s) \geq \deg_t(Dt)$, then $\nu_{1/t}(u) = -\deg_t(s)$ and $\operatorname{llc}_{1/t}(u) = \operatorname{lc}_t(s)$. Hence we have $\nu_{1/t}(P_i(u)) = -i \deg_t(s)$ and $\operatorname{llc}_{1/t}(P_i(u)) = \operatorname{lc}_t(s)^i$ by Lemma 2.2 in [Fre04]. For bounding the degree of s we can prove the following analogue of Lemma 8.3 from [Bro92].

Lemma 4.30. *Let t be a nonlinear monomial over (K, D) and let $a_0, \dots, a_d \in K(t)$. If $u \in K(t)$ satisfies $\sum_{i=0}^{d} a_i P_i(u) = 0$, then*

$$\nu_{1/t}(u) \geq \min\left(1 - \deg_t(Dt), \inf\left(\mathbb{Z} \cap \left\{ \tfrac{\nu_{1/t}(a_j) - \nu_{1/t}(a_i)}{i-j} \;\middle|\; i, j \in \{0, \dots, d\}, i < j \right\}\right)\right).$$

This enables us to reduce the problem to the case where $\deg_t(s) \leq \deg_t(Dt) - 1$. Without loss of generality we can assume p to be monic, then we have $\tilde{u} := \pi_{1/t}(ut^{1-\deg_t(Dt)}) = \deg_t(p) \operatorname{lc}_t(Dt) + \operatorname{coeff}(s, t^{\deg_t(Dt)-1})$. Furthermore, we can prove the following refinement of Lemma 2.2 in [Fre04].

Lemma 4.31. *Let t be a nonlinear monomial over (K, D) and let $u \in K(t)$ as well as $k := -\nu_{1/t}(u)$ and $i \in \mathbb{N}$.*

1. *If $k = \deg_t(Dt) - 1$, then $\pi_{1/t}(P_i(u)) \geq -ik$ and*

$$\pi_{1/t}(P_i(u)t^{-ik}) = \prod_{j=0}^{i-1} (\operatorname{llc}_{1/t}(u) + jk \operatorname{lc}_t(Dt)).$$

2. *If $k \in \{1, \dots, \deg_t(Dt) - 2\}$ and $i > 0$, then $\nu_{1/t}(P_i(u)) = -k - (i-1)(\deg_t(Dt) - 1)$ and*

$$\pi_{1/t}(P_i(u)t^{-k-(i-1)(\deg_t(Dt)-1)}) = P^{i-1}_{\deg_t(Dt)-1}(k) \operatorname{lc}_t(Dt)^{i-1} \operatorname{llc}_{1/t}(u).$$

Based on this we can determine candidates for the polynomial part of u and reduce the problem to finding solutions of the form $u = s + \frac{Dp \bmod p}{p}$ with $s \in K$, in other words $e^{\int u} = pe^{\int (s-(Dp\div p))}$. In practice we have $\deg_t(Dt) = 2$, which means

$$e^{\int u} = pe^{\int (u_0 - \deg_t(p)(Dt\div t))}$$

with $u_0 = s + \operatorname{coeff}(p, t^{\deg_t(p)-1}) \operatorname{lc}_t(Dt)$. Assume we know $\deg_t(p)$ then by Lemma 4.29 we need to find solutions of the form

$$y = pe^{\int u_0},$$

with $u_0 \in K$ and $p \in K[t]$ with given degree, of the transformed equation $\tilde{L}(y) = 0$ with $\tilde{L} \in K(t)[D]$. This can be done in the following way. If $\deg_t(p) = 0$ then this can be reduced to Problem 4.27 over (K, D) by taking the coefficients $\delta_{\nu_{1/t}(a_i),\min_j \nu_{1/t}(a_j)} \operatorname{llc}_{1/t}(a_i)$, where a_i are the coefficients of $\tilde{L}$. So assume $\deg_t(p) > 0$ now. First, note that the derivatives $e^{-\int u_0} D^i y \in K[t]$ have degree $\deg_t(p)$ for all i and leading coefficient $P^i_{\deg_t(Dt)-1}(\deg_t(p)) \operatorname{lc}_t(Dt)^i \operatorname{lc}_t(p)$. Hence by solving Problem 4.26 over (K, D) for the equation with coefficients $\delta_{\nu_{1/t}(a_i),\min_j \nu_{1/t}(a_j)} \operatorname{llc}_{1/t}(a_i) \in K$, where a_i are the coefficients of $\tilde{L}$, we obtain finitely many candidates for u_0. Based on these we construct the transformed equation of $\tilde{L}$ having p as solution. We determine the solutions in $K(t)$ of this equation by Theorem 4.18 and pick the polynomials $p \in K[t]$ with the given degree among them. Then we would be done, we only need to worry about how to determine $\deg_t(p)$ in the first place. If we could determine a bound on this degree then we just could try all values up to this bound and check whether they lead to a solution.

Problem 4.32. *Given: a monomial t over (K, D), with $S^{irr} = \emptyset$ and $\deg_t(Dt) = 2$, and $a_0, \dots, a_d \in K(t)$*

Find: $n \in \mathbb{N}$ such that for all $p \in K[t]$ and $u_0 \in K$ with

$$\sum_{i=0}^{d} a_i D^i (pe^{\int (u_0 - \deg_t(p)(Dt\div t))}) = 0$$

we have $\deg_t(p) < n$.

Altogether, we can summarize this procedure with the following theorem.

Theorem 4.33. *Assume t satisfies $\deg_t(Dt) = 2$ and $S^{irr} = \emptyset$. Assume that we can factor polynomials in $K[t]$ into irreducibles, that we can solve linear ODEs in (K, D), and that we can solve Problems 4.27 and 4.26 over (K, D). If we can solve Problem 4.32, then we can solve Problem 4.27 over $(K(t), D)$.*

Finally, we give a heuristic solution to Problem 4.32. Without loss of generality we assume $a_0, \dots, a_d \in K[t]$, then we choose

$$n := d\cdot(\max_{i\in\{0,\dots,d\}} \deg_t(a_i) + 1).$$

This can be motivated by the following considerations. Let

$$y = (t^m + p_{m-1}t^{m-1} + \dots + p_0)e^{\int (u_0 - m(Dt\div t))},$$

$p_j \in K$, be given. The derivatives are of the form $D^i y = (p_{i,m}t^m + \cdots + p_{i,0})e^{\int (u_0 - m(Dt \div t))}$ for some $p_{i,j} \in K$ again. Now we try to find $a_0, \ldots, a_d \in K[t]$ with $\deg_t(a_i) \leq k$ such that

$$e^{-\int (u_0 - m(Dt \div t))} \sum_{i=0}^{d} a_i D^i y = 0$$

where each summand has degree at most $k + m$. Comparing coefficients of the powers of t yields $k + m + 1$ equations for the $(d + 1)(k + 1)$ coefficients in $a_0, \ldots, a_d$. Then the condition

$$m < d \cdot (k + 1)$$

ensures that we have more coefficients than equations and can find such $a_0, \ldots, a_d \in K[t]$ for the given y. Reversing the roles to known a_i and unknown y we arrive at the heuristic bound stated above.

4.3.3 Reduction of systems of ODEs to scalar ODEs

First of all we consider the situation described in Section 4.1 where we encountered differential systems for the first time. In the setting of (4.1) the lowest, i.e., $(d - 1)$-st, subdiagonal of the system matrix $B \in K^{(n+d-1)\times(n+1)}$ is guaranteed to have all entries $b_{j+d-1,j} = (j - n)\,\mathrm{lc}_t(Dt)$ nonzero, which can be exploited to reduce the system to $d - 1$ scalar differential equations of order $\leq \lfloor \frac{n}{d-1} \rfloor + 1$ for g_n by solving the last n equations for $g_{n-1}, \ldots, g_0$ explicitly in terms of $g_n, c_0, \ldots, c_m$. More precisely, there is one equation exactly of order $\lfloor \frac{n}{d-1} \rfloor + 1$ and one equation each of order at most $\lfloor \frac{n+1}{d-1} \rfloor, \lfloor \frac{n+2}{d-1} \rfloor, \ldots, \lfloor \frac{n+d-2}{d-1} \rfloor$. For $d = 2$ this means that we obtain exactly one differential equation for g_n, which is of order $n + 1$ and can be solved by the algorithm discussed in Section 4.2. For $d > 2$ this means that we end up with more than one equation for g_n and chances are that we can solve for g_n by mere elimination, even without the need of solving Problem 4.3.

Apart from this special situation there are general algorithms to uncouple differential systems of the form

$$D\mathbf{y} + A\mathbf{y} = \mathbf{f}$$

with a square matrix A and reduce them to one or more scalar ODEs. We just briefly mention several of those algorithms. A method for transforming systems into several uncoupled scalar equations was given in [Poo]. The cyclic vector method can be used if we want to compute a single differential equation which is equivalent to the original system. An algorithm to compute a companion block diagonal form was given in [Bar93], where also the cyclic vector method is recalled. Each of the blocks directly corresponds to a scalar ODE, in practice the matrix is just transformed into a companion matrix in most cases. The algorithm given in [AZ96] has the same objective. Similarly one can also compute a suitable normal form like the Hermite or Jacobson normal form of the system, corresponding to a row echelon form or a diagonal form respectively.

4.4 Systems of ODEs

Problem 4.34. *(system of ODEs) Given:* (F, D), $A \in F^{n\times n}$, *and* $\mathbf{f}_0, \ldots, \mathbf{f}_m \in F^n$

Find: a C-vector space basis of all solutions $(\mathbf{y}, \mathbf{c}) \in F^n \times C^{m+1}$ such that

$$D\mathbf{y} + A\mathbf{y} = \sum_{i=0}^{m} c_i \mathbf{f}_i. \tag{4.10}$$

This problem has first been solved in a direct manner without reducing it to scalar ODEs by Barkatou [Bar99] for the case $(F, D) = (C(x), \frac{d}{dx})$. It also has been investigated for systems in $C(x)[t, t^{-1}]$, with t hyperexponential, in Chapter 7 of [Fre01] even for several hyperexponential generators, but no direct algorithm to compute the solutions in $C(x, t)$ nor even in $C(x)[t, t^{-1}]$ was provided there.

The following theorem summarizes the results of the joint work with Moulay A. Barkatou we presented in [BR12]. We will detail the algorithm again in what follows and state some parts in more generality.

Theorem 4.35. *Let $(K, D) = (C(x), \frac{d}{dx})$ and let t be a hyperexponential monomial over (K, D) with $\operatorname{Const}(K(t)) = C$. Let $\mathbf{f}_0, \ldots, \mathbf{f}_m \in K(t)^n$ and $A \in K(t)^{n \times n}$ such that*

1. *if $\nu_t(A) < 0$, then the matrix $\operatorname{llc}_t(A) \in K^{n \times n}$ is invertible and*
2. *if $\nu_{1/t}(A) < 0$, then the matrix $\operatorname{llc}_{1/t}(A) \in K^{n \times n}$ is invertible.*

Then we can compute a C-vector space basis of all solutions $(\mathbf{y}, \mathbf{c}) \in K(t)^n \times C^{m+1}$ such that

$$D\mathbf{y} + A\mathbf{y} = \sum_{i=0}^{m} c_i \mathbf{f}_i.$$

Also for systems the algorithm follows the same three major steps that already have been used for scalar ODEs. First, compute the normal part of the universal denominator, then determine degree bounds and compute the solutions by comparing coefficients and recursively solving the same problem in smaller fields. The important ingredient for computing the solutions is to obtain lower bounds on the possible values of $\nu_p(\mathbf{y})$ for all p. If no cancellation occurs, i.e. $\nu_p(D\mathbf{y} + A\mathbf{y}) = \min(\nu_p(D\mathbf{y}), \nu_p(A\mathbf{y}))$, then in analogy to (4.3) we can bound $\nu_p(\mathbf{y})$ from below based on ω_p and $\nu_p(A)$, since then we have

$$\nu_p(\mathbf{y}) = \nu_p(D\mathbf{y} + A\mathbf{y}) - \min(\omega_p, \nu_p(A)). \tag{4.11}$$

The remaining cases again need to be determined via some suitable indicial equation. For systems it is more difficult than for scalar equations to compute indicial equations and a main tool for computing them in this case is super-reduction [Bar04], see also [Bar99, Pfl97]. From super-reduced systems we can compute all the integer slopes of the Newton-polygon and determine the corresponding characteristic polynomials [Pfl00]. If we are interested in one particular polynomial corresponding to a given integer slope k then k-simple systems as introduced in [Pfl00] are just what we need and the condition of being super-reduced is too strong. Recently an algorithm for directly computing k-simple forms of first-order differential systems at $x = 0$ with coefficients from $C((x))$ was developed in Chapter 4 of [ElB11], eliminating the need to compute a super-reduced form first. We present a rational version of this algorithm for systems in $K(t)$ for quite general K and t.

4.4.1 k-simple systems

For computing indicial and characteristic polynomials we consider the following setting. Let $k \in \mathbb{N}$ and $p \in K[t]$ irreducible or $p = \frac{1}{t}$ and recall that ω_p denotes the degree of the derivation D at p. Furthermore, let $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $\tilde{M}, N \in K(t)^{n\times n}$ such that $\nu_p(\tilde{M}), \nu_p(N) \geq 0$, then for $M := p^{\operatorname{diag}(\boldsymbol{\alpha})}(I_n + p\tilde{M})$ we consider the operator

$$L(\mathbf{y}) = Mp^{k-\omega_p}D\mathbf{y} + N\mathbf{y}. \tag{4.12}$$

Note that the definition of M implies $\nu_p(M) \geq 0$. Operators of the the form $D\mathbf{y} + A\mathbf{y}$ with $A \in K(t)^{n\times n}$ have to be multiplied from the left by an appropriate factor to match this form.

Let $\mathbf{y} = \mathbf{g}e^{\int w}$ with $\mathbf{g} \in K(t)^n$ and $w \in K(t)$ such that $\nu_p(\mathbf{g}) = 0$ and $\nu_p(w) \geq \omega_p - k$, then the equation $L(\mathbf{y}) = 0$ reads

$$L(\mathbf{y}) = (Mp^{k-\omega_p}D\mathbf{g} + wMp^{k-\omega_p}\mathbf{g} + N\mathbf{g})e^{\int w} = 0.$$

With $M_0 := \pi_p(p^{\operatorname{diag}(\boldsymbol{\alpha})})$, $N_0 := \pi_p(N)$, $\mathbf{g}_0 := \pi_p(\mathbf{g})$, and $w_0 := \pi_p(wp^{k-\omega_p})$ we have

$$\pi_p(Mp^{k-\omega_p}D\mathbf{g} + wMp^{k-\omega_p}\mathbf{g} + N\mathbf{g}) = M_0\pi_p(p^{k-\omega_p}D\mathbf{g}) + (w_0M_0 + N_0)\mathbf{g}_0.$$

Now, if $\nu_p(D\mathbf{g}) > \omega_p - k$, then from $\mathbf{g}_0 \neq 0$ we deduce that the polynomial

$$P_k(\mu) := \det(\mu M_0 + N_0)$$

has a root at $\mu = w_0$. If $P_k \not\equiv 0$, then L is called *k-simple* at p and P_k is called the characteristic polynomial ($k > 0$) or the indicial polynomial ($k = 0$) at p. In other words, if L is k-simple, then by computing P_k we get finitely many candidates for w_0. For $k = 0$ this is equivalent to the notion of simple systems used in [Bar99]. The general case was introduced in [Pfl00].

As not every L is k-simple at p we need to compute an equivalent operator

$$\tilde{L}(\mathbf{z}) = SL(T\mathbf{z}) = SMp^{k-\omega_p}TD\mathbf{z} + S(NT + Mp^{k-\omega_p}(DT))\mathbf{z} \tag{4.13}$$

with $S, T \in K(t)^{n\times n}$ invertible such that $\tilde{L}$ obeys the same structure above and is k-simple at p.

Then for general $\mathbf{g} \in K(t)^n$ and $w \in K(t)$ with $\nu_p(w) \geq \omega_p - k$ and $L(\mathbf{g}e^{\int w}) = 0$ we normalize to $\tilde{\mathbf{g}} := T^{-1}\mathbf{g}p^{-\nu_p(T^{-1}\mathbf{g})}$ and $\tilde{w} := w + \nu_p(T^{-1}\mathbf{g})\frac{Dp}{p}$. So we have $\nu_p(\tilde{\mathbf{g}}) = 0$ and again $\tilde{L}(\tilde{\mathbf{g}}e^{\int \tilde{w}}) = 0$. Hence with

$$\tilde{w}_0 = \pi_p(\tilde{w}) = \pi_p(wp^{k-\omega_p}) + \delta_{k,0}\nu_p(T^{-1}\mathbf{g})\pi_p((Dp)p^{-1-\omega_p}) \tag{4.14}$$

we have $P_k(\tilde{w}_0) = 0$, where $\delta_{k,0}$ is the Kronecker delta. Moreover, observe that

$$\nu_p(\mathbf{g}) \geq \nu_p(T^{-1}\mathbf{g}) + \nu_p(T) \tag{4.15}$$

and even $\nu_p(T) \leq \nu_p(\mathbf{g}) - \nu_p(T^{-1}\mathbf{g}) \leq -\nu_p(T^{-1})$.

Using the notation above we will show in the remaining part of this section, given any such operator $L(\mathbf{y}) = Mp^{k-\omega_p}D\mathbf{y} + N\mathbf{y}$, how to compute $S, T \in K(t)^{n\times n}$ invertible such that the operator $SL(T\mathbf{z})$ is k-simple at p, i.e., the corresponding $P_k(\mu)$ is not the zero polynomial. Carole El Bacha in Chapter 4 of her thesis [ElB11] developed an algorithm for the case $p = x$ and the field $(C((x)), \frac{d}{dx})$, see also [BE12]. The rational algorithm given below is a direct generalization of it. The main difference is in verifying that the term $SMp^{k-\omega_p}(DT)$ in the equivalence transformation (4.13) does not interfere, for which we impose the condition

$$\forall f \in O_p : \nu_p(Df) > \omega_p - k. \tag{4.16}$$

Note that this condition is trivially fulfilled if $K_p = \mathrm{Const}(K)$ or $k > 0$ holds. In the joint work with Barkatou [BR12] we give the generalization for the field $(C(x), \frac{d}{dx})$ and arbitrary irreducible $p \in C[x]$ or $p = \frac{1}{x}$ and hint its applicability to more general fields.

The algorithm repeats the same step again and again, at each step applying some equivalence transformation determined according to the three cases shown below. At the beginning and after each step of the algorithm we perform a normalizing transformation: we multiply each row of the current operator L by $p^{-\min(\alpha_i,\nu_p(\mathbf{n}_i))}$ from the left, where $\mathbf{n}_i$ is the i-th row of N, which can be summarized as some $S = p^{\mathrm{diag}(\boldsymbol{\beta})}$, and we apply a permutation matrix P such that the operator $PSL(P^{-1}\mathbf{y})$ has $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $\alpha_1 \leq \cdots \leq \alpha_n$. If the resulting operator is k-simple at p, then we collect all the transformations done so far into the overall transformation matrices S and T and stop, otherwise we proceed with the next step. If the input is of the form $D\mathbf{y} + A\mathbf{y}$, then in the initial normalization we multiply each row by $p^{-\min(\omega_p-k,\nu_p(\mathbf{a}_i))}$ from the left instead in order to obtain the form (4.12), where $\mathbf{a}_i$ is the i-th row of A.

By the normalization in between the steps the sum $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$ is either decreased or at least stays the same. As long as the operator is not k-simple, which happens for $\boldsymbol{\alpha} = 0$ at latest, the steps ensure that $|\boldsymbol{\alpha}|$ will be decreased eventually. The idea of the transformations applied below is to make one row of N_0 zero for which the corresponding α_i is greater than zero, since then at the first part of the subsequent normalizing transformation α_i will be decreased in this situation.

When constructing transformation matrices by elements from K_p we actually refer to canonical representatives from $K[t]$ (w.r.t. π_p). We also use r to denote the rank of the matrix $M_0 = \pi_p(M)$ and observe that

$$M_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

We subdivide the matrix $N_0 = \pi_p(N)$ into the same block sizes as M_0 above. Below we assume that the operator is not k-simple yet, in other words the rows of the matrix

$$\mu M_0 + N_0 = \begin{pmatrix} \mu I_r + N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \in K_p(\mu)^{n\times n}$$

are linearly dependent.

Case 1 We check whether the rows of the submatrix $(N_{21}\ N_{22})$ are linearly dependent. If they are not, we proceed with Case 2 below. If they are, then for some $i > r$ we can determine a vector $\mathbf{u} \in K_p^{n-i}$ such that $(0, \dots, 0, 1, \mathbf{u}) \cdot N_0 = 0$. Let $j \in \{i+1, \dots, n\}$ be maximal such that $\alpha_i = \alpha_j$ and define $\tilde{\mathbf{u}} = (-u_1, \dots, -u_{j-i}, 0, \dots, 0) \in K_p^{n-i}$. Then we apply the transformation

$$S = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & \mathbf{u} \\ 0 & 0 & I_{n-i} \end{pmatrix}, \quad T = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & \tilde{\mathbf{u}} \\ 0 & 0 & I_{n-i} \end{pmatrix}.$$

We have that $\nu_p(DT) \geq \omega_p$ and only its i-th row may be non-zero, which implies $\nu_p(Sp^{\operatorname{diag}(\boldsymbol{\alpha})}(I_n + p\tilde{M})p^{k-\omega_p}DT) \geq \alpha_i + k$, in particular $\nu_p(SMp^{k-\omega_p}DT) > 0$. So the new N_0 has all zeros in its i-th row and, since $\alpha_i > 0$, the subsequent normalizing transformation will decrease $|\boldsymbol{\alpha}|$.

Case 2 We refine the subdivision of N_0 from above by splitting off the first q rows and columns for the maximal $q \in \{0, \dots, r\}$ such that

$$N_0 = \begin{pmatrix} N_{11} & 0 & 0 \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{pmatrix}$$

and check whether the rows of the submatrix $(N_{32}\ N_{33})$ are linearly dependent. If they are not, we proceed with Case 3 below. If they are, then we apply the transformation

$$S = \begin{pmatrix} p^{-1}I_q & 0 \\ 0 & I_{n-q} \end{pmatrix}, \quad T = \begin{pmatrix} pI_q & -pM_{12} \\ 0 & I_{n-q} \end{pmatrix},$$

where M_{12} is the corresponding submatrix obtained from $\pi_p(\tilde{M})$ by deleting the first q columns of the first q rows

$$\pi_p(\tilde{M}) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

We have that $\nu_p(DT) \geq 1 + \omega_p$ and only its first q rows may be non-zero, which implies $\nu_p(Sp^{\operatorname{diag}(\boldsymbol{\alpha})}(I_n + p\tilde{M})p^{k-\omega_p}DT) \geq k$ where only the first q rows can have order ν_p exactly k. So at worst $SMp^{k-\omega_p}DT$ contributes to the first q rows of N_0, for $k = 0$, but does not interfere with the last $n-r$ rows in any case. This transformation does not change $\boldsymbol{\alpha}$, but the new N_0 has $(0\ N_{32}\ N_{33})$ as its last $n-r$ rows, which are linearly dependent. So $|\boldsymbol{\alpha}|$ will be decreased either now by normalizing or at latest after the next step, which will be Case 1 then.

Case 3 We apply a $n \times n$ permutation matrix P acting on the rows $\{q+1, \dots, r\}$ only in order to ensure that for the operator $PL(P^{-1}\mathbf{y})$ with the same subdivision of N_0 from above we can determine a vector $\mathbf{u} \in K_p^{n-q-1}$ such that $(0, \dots, 0, 1, \mathbf{u}) \cdot N_0 = 0$. Analogous to Case 1 we define $\tilde{\mathbf{u}} = (-u_1, \dots, -u_{r-q-1}, 0, \dots, 0) \in K_p^{n-q-1}$ and then we apply the transformation

$$S = \begin{pmatrix} I_q & 0 & 0 \\ 0 & 1 & \mathbf{u} \\ 0 & 0 & I_{n-q-1} \end{pmatrix}, \quad T = \begin{pmatrix} I_q & 0 & 0 \\ 0 & 1 & \tilde{\mathbf{u}} \\ 0 & 0 & I_{n-q-1} \end{pmatrix}.$$

Now condition (4.16) plays the important role to ensure that $\nu_p(DT) > \omega_p - k$, which implies $\nu_p(Sp^{\operatorname{diag}(\boldsymbol{\alpha})}(I_n + p\tilde{M})p^{k-\omega_p}DT) > 0$. So $SMp^{k-\omega_p}DT$ does not contribute to the new N_0 and the effect of this transformation is that the new N_0 has an increased value of q. The normalization will not change the operator and can be skipped immediately after this step. This will just result in Case 3 being applied until we are in one of the other two cases, which happens for $q = r$ at latest.

To see that we actually can construct the transformation described in Case 3 consider the following. The rows of

$$\mu M_0 + N_0 = \begin{pmatrix} \mu I_q + N_{11} & 0 & 0 \\ N_{21} & \mu I_{r-q} + N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{pmatrix} \in K_p(\mu)^{n\times n}$$

are linearly dependent. As $\mu I_q + N_{11}$ is invertible we see that the rows of the submatrix

$$\begin{pmatrix} \mu I_{r-q} + N_{22} & N_{23} \\ N_{32} & N_{33} \end{pmatrix}$$

are linearly dependent, which remains true after specializing $\mu = 0$. But the rows of $(N_{32}\ N_{33})$ are linearly independent since we would not have reached Case 3 otherwise. Therefore, $q < r$ and in particular we can find a transformation of the type described above after a suitable permutation.

Altogether, this proves the following theorem.

Theorem 4.36. *Let t be a monomial over (K, D) and let $A \in K(t)^{n\times n}$. Let $k \in \mathbb{N}$ and $p \in K[t]$ irreducible or $p = \frac{1}{t}$, then we can compute $S, T \in K(t)^{n\times n}$ such that the transformed operator $SL(T\mathbf{z})$ is k-simple at p with $\nu_p(T) \geq 0$ as well as $\det(S) = \pm p^a$ and $\det(T) = \pm p^b$ for some $a, b \in \mathbb{Z}$.*

4.4.2 The normal part of the denominator

The algorithm for computing k-simple systems described in the previous section can be applied successfully to operators $D\mathbf{y} + A\mathbf{y}$ with coefficients from $K(t)$ at any normal irreducible $p \in K[t]$ since Theorem 2.16 implies (4.16). Based on this we can prove the following theorem, which not only relies on the generic case (4.11) but also on the fact that for 0-simple operators (4.12) at such a p we have

$$\nu_p(Mp^{-1}D\mathbf{y} + N\mathbf{y}) = \nu_p(\mathbf{y})$$

as long as $\nu_p(\mathbf{y})\pi_p(Dp)$ is not a root of the indicial polynomial. By also allowing positive values of the bound λ_p for $\nu_p(\mathbf{y})$ in the computation below also parts of the numerator can be determined in some cases. The following theorem is the analogue of Theorem 4.9 for systems.

Theorem 4.37. *Assume $C := \operatorname{Const}(K(t)) = \operatorname{Const}(K)$ and assume that we can determine the integer roots of polynomials with coefficients in K (see Section 2.4.5) and that we can factor such polynomials into irreducibles. Let $A \in K(t)^{n\times n}$ and $\mathbf{f}_0, \dots, \mathbf{f}_m \in K(t)^n$.*

Then we can compute $b \in K(t)^*$ *such that for all* $\mathbf{y} \in K(t)^n$ *with* $L(\mathbf{y}) := D\mathbf{y} + A\mathbf{y} \in \operatorname{span}_C\{\mathbf{f}_0, \ldots, \mathbf{f}_m\}$ *we have*

$$b\mathbf{y} \in K(t)^n_{red}.$$

Proof. By assumption we can determine all monic irreducible normal $p \in K[t]$ with $\nu_p(A) < 0$ or $\min_i \nu_p(\mathbf{f}_i) < -1$. By Theorem 2.16 we have $\omega_p = -1$ and (4.16) for normal p. These are finitely many and for each of them by Theorem 4.36 we compute transformation matrices $S_p, T_p \in K(t)^{n\times n}$ such that the operator $S_pL(T_p\mathbf{z})$ is 0-simple at p and we also compute the corresponding indicial polynomial $P_{p,0}(\mu) \in K_p[\mu]$. From this we determine

$$\begin{aligned}
n_p &:= \min(\nu_p(S_p\mathbf{f}_0), \ldots, \nu_p(S_p\mathbf{f}_m)),\\
\mu_p &:= \inf\{\mu \in \mathbb{Z} \mid P_{p,0}(\mu\pi_p(Dp)) = 0\},\\
\lambda_p &:= \min(n_p, \mu_p) + \nu_p(T_p).
\end{aligned}$$

Then we compute the following product over all p considered

$$b := \prod_p p^{-\lambda_p} \in K(t)^*.$$

Now we fix $\mathbf{y} \in K(t)^n$ with $L(\mathbf{y}) \in \operatorname{span}_C\{\mathbf{f}_0, \ldots, \mathbf{f}_m\}$. For any normal irreducible $p \in K[t]$ we verify that $-\nu_p(b) \le \nu_p(\mathbf{y})$. First, assume p was considered in the computation above. So $\nu_p(S_pL(T_p\mathbf{z})) \ge n_p$ for $\mathbf{z} = T_p^{-1}\mathbf{y}$, which implies that either $P_{p,0}(\nu_p(\mathbf{z})\pi_p(Dp)) \ne 0$ and $\nu_p(\mathbf{z}) = \nu_p(S_pL(T_p\mathbf{z})) \ge n_p$ or $P_{p,0}(\nu_p(\mathbf{z})\pi_p(Dp)) = 0$. Therefore we have $-\nu_p(b) = \lambda_p \le \nu_p(\mathbf{z}) + \nu_p(T_p) \le \nu_p(\mathbf{y})$. Assuming p was not considered in the computation above instead, then in particular $\nu_p(A) \ge 0$ and $\min_i \nu_p(\mathbf{f}_i) \ge -1$. If $\nu_p(\mathbf{y}) \ne 0$ then $\min_i \nu_p(\mathbf{f}_i) \le \nu_p(L(\mathbf{y})) = \nu_p(D\mathbf{y}) = \nu_p(\mathbf{y}) - 1$ as in (4.11). Hence $-\nu_p(b) = 0 \le \nu_p(\mathbf{y})$. Altogether we obtain $b\mathbf{y} \in K(t)^n_{red}$. $\square$

4.4.3 Degree bounds

Once we have determined the normal part of the universal denominator we substitute $\mathbf{y} = \tilde{\mathbf{y}}/b$ in (4.10) and multiply by b to obtain the new system

$$D\tilde{\mathbf{y}} + \left(A - \frac{Db}{b}I_n\right)\tilde{\mathbf{y}} = \sum_{i=0}^{m} c_i\mathbf{f}_ib \tag{4.17}$$

for $\tilde{\mathbf{y}} \in K(t)^n_{red}$. Then we do the remaining computations based on this system. In particular we need to bound $\nu_{1/t}(\tilde{\mathbf{y}})$. As in the scalar case, if t is nonlinear, a degree bound for the solutions can be found with the same methods we used in Theorem 4.37 above. In the following we will assume that t is hyperexponential and satisfies the conditions of Theorem 2.51.

We now have to find bounds on the possible values of λ_0 and λ_1 in the solutions

$$\tilde{\mathbf{y}} = \sum_{i=\lambda_0}^{\lambda_1} \mathbf{y}_it^i$$

of (4.17). The system matrix $\tilde{A} = A - \frac{Db}{b} I_n$ has coefficients from $K(t)$, but the contribution from $\frac{Db}{b} I_n$ has the property that $\nu_t(\frac{Db}{b}) \geq 0$ and $\nu_{1/t}(\frac{Db}{b}) \geq 0$. So $\tilde{A}$ also satisfies both conditions stated in Theorem 4.35 if A does.

For computing the bounds on $\nu_p(\tilde{\mathbf{y}})$ for $p = t$ and $p = \frac{1}{t}$ we distinguish two cases each. If $\mu_p := \nu_p(\tilde{A}) < 0$, then we have that $\operatorname{llc}_p(A) \in K^{n\times n}$ is invertible hence (4.11) is always true and reads $\nu_p(\tilde{\mathbf{y}}) = \nu_p(D\tilde{\mathbf{y}} + \tilde{A}\tilde{\mathbf{y}}) - \mu_p$. If $\mu_p \geq 0$, then we have $\nu_p(\tilde{\mathbf{y}}) \leq \nu_p(D\tilde{\mathbf{y}} + \tilde{A}\tilde{\mathbf{y}})$ in general and strict inequality occurs iff $\pi_p((D\tilde{\mathbf{y}} + \tilde{A}\tilde{\mathbf{y}})p^{-\lambda}) = 0$ or equivalently

$$D(\mathbf{g}p^\lambda) + \pi_p(\tilde{A})\mathbf{g}p^\lambda = 0$$

for $\lambda = \nu_p(\tilde{\mathbf{y}})$ and $\mathbf{g} = \pi_p(\tilde{\mathbf{y}}p^{-\lambda})$. Consequently, we are reduced to the problem of finding solutions $\mathbf{g}t^\lambda$ with $\mathbf{g} \in K^n$ and $\lambda \in \mathbb{Z}$ of the homogeneous system

$$D\mathbf{y} + \pi_p(\tilde{A})\mathbf{y} = 0.$$

Note that $\pi_p(\tilde{A}) \in K^{n\times n}$. We present an algorithm computing a finite set of candidates for λ for the case $(K, D) = (C(x), \frac{d}{dx})$ similar to the approach in [BF99].

Theorem 4.38. *Let $A \in C(x)^{n\times n}$ and $Dt = at$ where $a \in C(x)$ is such that there are no $k \in \mathbb{Z} \setminus \{0\}$ and $g \in C(x)^*$ with $a = \frac{Dg}{kg}$. Then we can compute a finite set $\Lambda \subset \mathbb{Z}$ such that for any $\mathbf{y} = \mathbf{g}t^\lambda$ with $\mathbf{g} \in C(x)^n$, $\lambda \in \mathbb{Z}$, and $D\mathbf{y} + A\mathbf{y} = 0$ we have $\lambda \in \Lambda$.*

Proof. Choose $p \in C[x]$ irreducible (or $p = \frac{1}{x}$) such that $\nu_p(a) < \omega_p$ or $\operatorname{res}_p(a) \notin \mathbb{Q}$. We can do this because if for each irreducible $p \in C[x]$ we have $\nu_p(a) \geq -1$ and $\operatorname{res}_p(a) \in \mathbb{Q}$ then $a = \tilde{a} + \sum_{i=1}^N r_i \frac{Dp_i}{p_i}$ for some $\tilde{a} \in C[x]$, $r_i \in \mathbb{Q}$, and $p_i \in C[x]$ and by assumption on a it follows that $\tilde{a} \neq 0$, which implies $\nu_{1/x}(a) < 1$. Let $a_0 := \operatorname{llc}_p(a)$, $p_0 := \pi_p((Dp)p^{-1-\omega_p})$, and $\beta := \omega_p - \nu_p(a)$. For constructing Λ we distinguish two cases.
Case 1: $\beta > 0$
Compute a β-simple form of $D\mathbf{y} + A\mathbf{y}$ at p as well as the corresponding characteristic polynomial $P_\beta(\mu) \in (C[x]/\langle p\rangle)[\mu]$ (resp. $\in C[\mu]$). Determine the set

$$\tilde{\Lambda} := \{\lambda \in \mathbb{Z} \setminus \{0\} \mid P_\beta(\lambda a_0) = 0\}.$$

Next, compute a 0-simple form of $D\mathbf{y} + A\mathbf{y}$ at p as well as the corresponding transformation matrix $T \in C(x)^{n\times n}$ and indicial polynomial $P_0(\mu) \in (C[x]/\langle p\rangle)[\mu]$ (resp. $\in C[\mu]$). If $P_0(\nu p_0) = 0$ has a solution $\nu \in \mathbb{Z}$, then set $\Lambda := \tilde{\Lambda} \cup \{0\}$, otherwise set $\Lambda := \tilde{\Lambda}$.
Case 2: $\beta = 0$
Compute a 0-simple form of $D\mathbf{y} + A\mathbf{y}$ at p as well as the corresponding transformation matrix $T \in C(x)^{n\times n}$ and indicial polynomial $P_0(\mu) \in (C[x]/\langle p\rangle)[\mu]$ (resp. $\in C[\mu]$). Determine the set

$$\Lambda := \{\lambda \in \mathbb{Z} \mid \exists \nu \in \mathbb{Z} : P_0(\nu p_0 + \lambda a_0) = 0\},$$

which is finite since p_0 and a_0 are $\mathbb{Q}$-linearly independent because of $\frac{a_0}{p_0} = \operatorname{res}_p(a) \notin \mathbb{Q}$.
Now, we verify the desired property of Λ. For $\mathbf{y} = \mathbf{g}t^\lambda$ as above we have $0 = D\mathbf{y} + A\mathbf{y} = (D\mathbf{g} + \lambda a\mathbf{g} + A\mathbf{g})t^\lambda$, hence $D\mathbf{g} + \lambda a\mathbf{g} + A\mathbf{g} = 0$. Again we treat the two cases separately.
Case 1: $\beta > 0$
If $\lambda \neq 0$, then the term $\lambda a\mathbf{g}$ dominates and λa_0 is a root of the characteristic polynomial

P_β, hence $\lambda \in \Lambda$. If $\lambda = 0$, then $\nu_p(T^{-1}\mathbf{g})p_0$ is a root of the indicial polynomial P_0 and we deduce $\lambda \in \Lambda$.
Case 2: $\beta = 0$
In this case $\nu_p(T^{-1}\mathbf{g})p_0 + \lambda a_0$ is a root of the indicial polynomial P_0 and we have $\lambda \in \Lambda$. $\square$

We briefly describe how to proceed after computing λ_0 and λ_1. Either we focus on the place $p = t$ or $p = \frac{1}{t}$ in the third step. Starting from t^{λ_0} or t^{λ_1} respectively we successively proceed through the powers of t. If $\nu_p(A) \geq 0$, then we need to do the following. For each power t^i by comparing its coefficients, or, more precisely, multiplying (4.17) by t^{-i} and applying π_p, we obtain a differential system of the form

$$D\mathbf{y}_i + (\pi_p(A) + (ia - \tilde{b})I_n)\mathbf{y}_i = \sum_{j=0}^{m_i} \tilde{c}_{i,j}\tilde{\mathbf{f}}_{i,j} \tag{4.18}$$

with coefficients from $C(x)$, where $\tilde{b} = \pi_p(\frac{Db}{b})$. In order to compute all solutions $\mathbf{y}_i \in C(x)^n$ and $\tilde{c}_{i,j} \in C$ of these systems we apply the algorithm described in [Bar99] modified to obtain indicial equations based on the algorithm described in Section 4.4.1 instead of super-reduction. Plugging in the solutions in the ansatz made for $\tilde{\mathbf{y}}$ in (4.17) generates a new inhomogeneous part with higher ν_t or lower $\nu_{1/t}$ respectively and possibly with a different m. Then we proceed with the next power of t until we eventually consider t^{λ_1} or t^{λ_0}. If $\nu_p(A) < 0$ instead, then by multiplying the equation by $t^{-i-\nu_t(A)}$ or $t^{-i+\nu_{1/t}(A)}$ respectively and applying π_p as before we just need solve a linear system for each of the $\mathbf{y}_i$ with system matrix $\mathrm{llc}_p(A)$. After that in either case the remaining inhomogeneous part has to vanish, which provides conditions on the remaining free constants.

Chapter 5

Definite integration

After briefly discussing the use of parametric elementary integration and how to choose the integrands for evaluating given parameter integrals this chapter contains in Section 5.1 some examples to highlight several aspects of the algorithm and to show how it can be used in practice. We will not present our package `Integrator` in detail here, but apply it to the examples that follow and use its results. The last example had been evaluated before for general σ up to $n = 3$ only and for $\sigma = 0$ up to $n = 6$ by Olivier Oloa. The remaining evaluations of parameter integrals given in this chapter were known already and had been done by other techniques not necessarily in an automated fashion.

The importance of finding linear combinations of several integrands by algorithms like the one presented in Chapters 3 and 4 or other approaches mentioned in Chapter 1 lies in its application to (definite) parameter integrals. We can find linear relations among the corresponding definite integrals of the individual integrands as explained there. Finding recurrences or differential equations satisfied by the parameter integral is of particular importance in practice.

For finding a recurrence equation for the parameter integral $I(n) := \int_a^b f(n,x)\,dx$ we choose $f_i(x) := f(n+i,x)$ as input for our algorithm. Then an output

$$c_0 f_0 \dots + c_m f_m = Dg$$

corresponds to the recurrence

$$c_m(n)I(n+m) + \dots + c_0(n)I(n) = g(b) - g(a).$$

For finding a differential equation for the parameter integral $I(y) := \int_a^b f(y,x)\,dx$ we choose $f_i(x) := \frac{\partial^i f}{\partial y^i}(y,x)$ as input for our algorithm. Then an output

$$c_0 f_0 + \dots + c_m f_m = Dg$$

corresponds to the differential equation

$$c_m(y)I^{(m)}(y) + \dots + c_0(y)I(y) = g(b) - g(a).$$

Those relations can then be used to compute the value of the parameter integrals or deduce other properties, like asymptotic behaviour. Our algorithm only deals with computing the relations, for solving them other software will be useful such as the built-in functionality of computer algebra systems or additional packages, e.g., the *Mathematica* package `Sigma` [Sch01, Sch06] for solving recurrences. We will make use of this in the examples presented later. Also the computed relations may be interesting in their own right. For instance, from many of the integral definitions of special functions relations can be derived with this principle, which then may serve to describe the function within our framework given in Section 2.6.

For a full evaluation of parameter integrals along the method described above also definite integrals which do not involve additional parameters typically need to be evaluated in order to obtain initial conditions. Often it is possible to obtain these via an indefinite integral computed by our algorithm again, often it may be necessary to obtain their values by other means. For instance the following integrals and variants thereof are important as they sometimes show up in the computation.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \tag{5.1}$$

$$\int_{0}^{\infty} e^{-x}\ln(x)\,dx = -\gamma \tag{5.2}$$

Also some parameter integrals cannot be fully evaluated for all values of the parameters by this method.

$$\int_0^1 x^a(1-x)^b\,dx = B(a+1,b+1) \qquad \text{where } \Re(a),\Re(b) > -1 \tag{5.3}$$

$$\int_0^\infty \frac{x^a}{e^x-1}\,dx = \Gamma(a+1)\zeta(a+1) \qquad \text{where } \Re(a) > 0 \tag{5.4}$$

5.1 Examples

A first example

In the integral table [GH] as formula II325.12 we find the following parameter integral for $a \in \mathbb{R}$.

$$\int_0^a \frac{\ln(1+ax)}{1+x^2}dx$$

We want to use this example to underline the potential of performing parametric elementary integration in contrast to limited integration. As a second characteristic of this example we encounter a parameter dependent bound of the interval of integration. Below we abbreviate the integrand by $f(a,x)$ and the parameter integral by $I(a)$ and we want to compute a differential equation satisfied by $I(a)$. Let $C := \mathbb{Q}(a)$ and $(F,D) := (C(t_1,t_2),D)$ with derivation D defined by

$$Dt_1 = 1 \qquad \text{and} \qquad Dt_2 = \frac{a}{at_1+1},$$

which gives rise to the correspondences

$$t_1 \leftrightarrow x \qquad \text{and} \qquad t_2 \leftrightarrow \ln(1+ax).$$

Applying the parametric elementary integration over (F, D) to the partial derivatives $f_0(x) := f(a,x)$ and $f_1(x) := \frac{\partial f}{\partial a}(a,x)$ as mentioned above we obtain the output

$$\begin{aligned}
\frac{\partial f}{\partial a}(a,x) &= \frac{d}{dx} \sum_{4(a^2+1)^2\alpha^3+(a^2-3)\alpha+1=0} \alpha \ln\left(x + \frac{2(a^2+1)^2(8\alpha^2+\alpha)+3a^2-5}{a(a^2+9)}\right) \\
&= \frac{d}{dx}\left(-\frac{i\ln(x-i)}{2(a-i)} + \frac{i\ln(x+i)}{2(a+i)} - \frac{\ln(x+\frac{1}{a})}{a^2+1}\right) \\
&= \frac{d}{dx}\left(\frac{(1-ia)\ln(x-i)+(1+ia)\ln(x+i)-2\ln(1+ax)}{2(a^2+1)}\right)
\end{aligned}$$

from our software. With this rewriting of the right hand side and taking the boundary term $\int_0^a \frac{\partial f}{\partial a}(a,x)\,dx = I'(a) - f(a,a)$ into account we obtain the following differential equation.

$$I'(a) = \frac{(1-ia)\ln(a-i)+(1+ia)\ln(a+i)+\pi a}{2(a^2+1)}$$

In order to solve this differential equation we can use our integration procedure again. This time we integrate w.r.t. a, i.e., D models derivation w.r.t. a, and we construct the differential field from $C := \mathbb{Q}(i,\pi)$ with the generators

$$t_1 \leftrightarrow a \qquad t_2 \leftrightarrow \ln(a-i) \qquad t_3 \leftrightarrow \ln(a+i).$$

By our software we obtain the indefinite integral

$$\int \frac{(1-ia)\ln(a-i)+(1+ia)\ln(a+i)+\pi a}{2(a^2+1)}\,da = \frac{(\pi - i\ln(a-i)+i\ln(a+i))(\ln(a-i)+\ln(a+i))}{4}.$$

The initial value $I(0) = 0$ is trivial to obtain and in combination with the indefinite integral implies the evaluation

$$I(a) = \frac{(\pi - i\ln(a-i)+i\ln(a+i))(\ln(a-i)+\ln(a+i))}{4},$$

which can be simplified to the form $\arctan(a)\ln(1+a^2)/2$ given in the integral table. In contrast, if we use limited integration in (F, D) to obtain the differential equation for $I(a)$, then we realize that we do not find a relation of the derivatives f and $\frac{\partial f}{\partial a}$, not even when $\frac{\partial^2 f}{\partial a^2}$ is included. Only when we go up to the third derivative $\frac{\partial^3 f}{\partial a^3}$ we find a relation.

$$\frac{\partial^3 f}{\partial a^3}(a,x) + \frac{4a}{a^2+1}\frac{\partial^2 f}{\partial a^2}(a,x) + \frac{2}{a^2+1}\frac{\partial f}{\partial a}(a,x) = \frac{d}{dx}\frac{x^2-2ax-1}{(a^2+1)^2(1+ax)^2}$$

Including all the boundary terms generated from the parameter dependent upper bound we arrive at the following third-order differential equation for $I(a)$.

$$I^{(3)}(a) + \frac{4a}{a^2+1}I''(a) + \frac{2}{a^2+1}I'(a) = \frac{3-a^2}{(a^2+1)^3}$$

In order to obtain an evaluation of the integral from this equation we not only need to solve a higher order equation but we also need to compute more initial values to pick the correct solution of the differential equation. Generally speaking, this may be more difficult. In the present case it is not a big problem, however.

Laplace transform

We consider the following Laplace transform of Legendre polynomials.

$$L(n,s) := \mathcal{L}_x\left(P_n(\cos(x))\right)(s) = \int_0^\infty e^{-sx} P_n(\cos(x))\,dx$$

The result is given by formulas 7.243.3 and 7.243.4 in [GR]. In the following we show in several steps how the evaluation can be done based on our algorithm as well. More precisely, we will compute a recurrence for $L(n,s)$ and also the corresponding initial values. At the moment the implementation cannot handle this non-Liouvillian integrand. In order to represent the integrand $f(n,s,x) := e^{-sx}P_n(\cos(x))$ and its shifts in n we will use the following generators of the differential field.

$$t_1 \leftrightarrow e^{ix} \qquad t_2 \leftrightarrow \frac{P_{n+1}(\cos(x))}{P_n(\cos(x))} \qquad t_3 \leftrightarrow e^{-sx}P_n(\cos(x))$$

We define the constant field $C := \mathbb{Q}(i)(n,s)$ with indeterminate n and s. From this we construct the admissible differential field $(F,D) := (C(t_1,t_2,t_3),D)$, where the derivation is defined by

$$\begin{aligned}
Dt_1 &= it_1,\\
Dt_2 &= \frac{2i(n+1)}{t_1-\frac{1}{t_1}}\left(-t_2^2+(t_1+\tfrac{1}{t_1})t_2-1\right),\\
Dt_3 &= \left(\frac{2i(n+1)}{t_1-\frac{1}{t_1}}\left(t_2-\tfrac{1}{2}(t_1+\tfrac{1}{t_1})\right)-s\right)t_3.
\end{aligned}$$

This means we have two hyperexponential monomials t_1 and t_3 and one nonlinear monomial t_2. In this differential field the shifts $f(n,s,x), f(n+1,s,x), f(n+2,s,x)$ of the integrand are represented by

$$f_0 := t_3, \qquad f_1 := t_2t_3, \qquad \text{and} \qquad f_2 := \left(\frac{(2n+3)(t_1+1/t_1)}{2(n+2)}t_2 - \frac{n+1}{n+2}\right)t_3,$$

respectively, and we want to do the corresponding parametric elementary integration over (F,D). All integrands are multiples of t_3 by an element of $K := C(t_1,t_2)$, so by Theorem 3.12 we need to solve the following parametric Risch differential equation in (K,D) with $b := \frac{Dt_3}{t_3}$.

$$Dg + bg = \sum_{i=0}^{2} c_i \cdot \operatorname{coeff}(f_i,t_3)$$

Since both b and all $\operatorname{coeff}(f_i,t_3)$ are polynomials in $C(t_1)[t_2]$ we infer by Theorem 7.1.1 from [Bro] that any solution has to satisfy also $g \in C(t_1)[t_2]$. Furthermore, we have $\deg_{t_2}(b) = 1$ and $\operatorname{lc}_{t_2}(b) = -\operatorname{lc}_{t_2}(Dt_2)$ so we detect possible cancellation, cf. part iii of Lemma 6.5.1 in [Bro], and we obtain the degree bound $\deg_{t_2}(g) \le 1$. By comparing coefficients of t_2^0 and t_2^1 this leads to a coupled differential system for the coefficients $g_0, g_1 \in C(t_1)$ of $g = g_1t_2 + g_0$.

$$D\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} + \begin{pmatrix} -i(n+1)\frac{t_1+1/t_1}{t_1-1/t_1} - s & -\frac{2i(n+1)}{t_1-1/t_1} \\ \frac{2i(n+1)}{t_1-1/t_1} & i(n+1)\frac{t_1+1/t_1}{t_1-1/t_1} - s \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = c_0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -\frac{n+1}{n+2} \\ \frac{(2n+3)(t_1+1/t_1)}{2(n+2)} \end{pmatrix}$$

Without going into details, the only solution $(g_0, g_1, c_0, c_1, c_2) \in C(t_1)^2 \times C^3$ is given by

$$\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} \frac{(2n+3)s}{(n+2)(s^2+(n+2)^2)} \\ \frac{2n+3}{(n+2)(s^2+(n+2)^2)}\left(-\frac{s}{2}(t_1+1/t_1) + \frac{n+2}{2i}(t_1-1/t_1)\right) \end{pmatrix}$$

$$c_0 = -\frac{s^2+(n+1)^2}{s^2+(n+2)^2}, \qquad c_1 = 0, \qquad c_2 = 1$$

and its constant multiples. For example, after a suitable rewriting of the system this solution can be computed by the *Mathematica* package `HolonomicFunctions`, cf. [Kou09]. In other words we computed the relation

$$f_2 - \frac{s^2+(n+1)^2}{s^2+(n+2)^2} f_0 = D\left(\frac{2n+3}{(n+2)(s^2+(n+2)^2)}\left(\left(-\frac{s}{2}(t_1+1/t_1) + \frac{n+2}{2i}(t_1-1/t_1)\right)t_2 + s\right)t_3\right),$$

which translates back to the original functions as

$$f(n+2,s,x) - \frac{s^2+(n+1)^2}{s^2+(n+2)^2} f(n,s,x) = \frac{d}{dx}\left(\frac{(2n+3)e^{-sx}}{(n+2)(s^2+(n+2)^2)}\left((-s\cos(x) + (n+2)\sin(x))\, P_{n+1}(\cos(x)) + sP_n(\cos(x))\right)\right).$$

When integrating from 0 to ∞, by $P_n(1) = 1$ and the boundedness of $\cos(x)$ and $\sin(x)$, we obtain the recurrence

$$L(n+2,s) = \frac{s^2+(n+1)^2}{s^2+(n+2)^2} L(n,s)$$

for $s > 0$. Now we need to compute the initial values for $n = 0$ and $n = 1$. In both cases the corresponding indefinite integral is easily obtained for $f(0,s,x) = e^{-sx}$ and $f(1,s,x) = \cos(x)e^{-sx}$, e.g. by our software or even by hand.

$$\begin{aligned} \int f(0,s,x)\,dx &= -\frac{e^{-sx}}{s} \\ \int f(1,s,x)\,dx &= (\sin(x) - s\cos(x))\frac{e^{-sx}}{s^2+1} \end{aligned}$$

From these we obtain the initial values

$$L(0,s) = \frac{1}{s} \qquad \text{and} \qquad L(1,s) = \frac{s}{s^2+1}$$

and from them we can construct the following result for $n \in \mathbb{N}$ and $s > 0$ using the recurrence computed above.

$$L(n,s) = \begin{cases} \dfrac{1}{s} \displaystyle\prod_{k=1}^{n/2} \frac{s^2 + (2k-1)^2}{s^2 + (2k)^2} & n \text{ even} \\ \dfrac{1}{s} \displaystyle\prod_{k=0}^{(n-1)/2} \frac{s^2 + (2k)^2}{s^2 + (2k+1)^2} & n \text{ odd} \end{cases}$$

Fourier transform

Recently Oleksandr Pavlyk shared the following example of a Fourier transform with us.

$$F(a,\omega) := \mathcal{F}_x\left(\frac{1}{\sqrt{\cosh(x) + \cosh(a)}}\right)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\sqrt{\cosh(x) + \cosh(a)}}\, dx$$

He already had computed the result $F(a,\omega) = \frac{\sqrt{\pi}}{\cosh(\pi\omega)}\, {}_2F_1\left(\frac{1}{2} - i\omega, \frac{1}{2} + i\omega; 1; -\sinh(\frac{a}{2})^2\right)$ by a series expansion and a recurrence for its coefficients. In the following we will show how our program can help to evaluate this Fourier transform and we abbreviate the integrand by

$$f(a,\omega,x) := \frac{e^{i\omega x}}{\sqrt{\cosh(x) + \cosh(a)}}.$$

We also use this example to emphasize that the parameters need not occur rationally in the field of constants, both in theory and in practice. More specifically, in the present case the field of constants $C := \mathbb{Q}(i, \cosh(a), \sinh(a), \omega)$ contains $\cosh(a)$ and $\sinh(a)$, which satisfy the usual relations. In order to deal with this parameter integral we want to compute a differential equation w.r.t. the parameter a. For this purpose it is enough to use the two hyperexponential generators $t_1 = e^x$ and $t_2 = f(a,\omega,x)$ to generate the admissible differential field $(C(t_1,t_2), D)$, which avoids dealing with the square root directly. The partial derivatives $f, \frac{\partial f}{\partial a}, \frac{\partial^2 f}{\partial a}$ can be expressed as

$$f_0 := t_2, \qquad f_1 := -\frac{\sinh(a)}{t_1 + 1/t_1 + 2\cosh(a)}\, t_2, \qquad \text{and}$$

$$f_2 := \left(-\frac{\cosh(a)}{t_1 + 1/t_1 + 2\cosh(a)} + \frac{3\sinh(a)^2}{(t_1 + 1/t_1 + 2\cosh(a))^2}\right) t_2$$

respectively. With our program we compute the relation

$$\frac{\partial^2 f}{\partial a}(a,\omega,x) + \coth(a)\, \frac{\partial f}{\partial a}(a,\omega,x) + \left(\omega^2 + \tfrac{1}{4}\right) f(a,\omega,x) = \\ \frac{d}{dx}\left(-\left(i\omega + \frac{\sinh(x)}{2(\cosh(x) + \cosh(a))}\right) f(a,\omega,x)\right),$$

which for $\omega \in \mathbb{R}$ implies the following differential equation, since then $\lim\limits_{x\to\pm\infty} f(a,\omega,x) = 0$ converges exponentially fast and all integrals exist.

$$\frac{\partial^2 F}{\partial a^2}(a,\omega) + \coth(a)\frac{\partial F}{\partial a}(a,\omega) + \left(\omega^2 + \tfrac{1}{4}\right) F(a,\omega) = 0$$

The general solution of this differential equation can be written in terms of Legendre functions as

$$F(a,\omega) = c_1(\omega)P_{i\omega-\frac{1}{2}}(\cosh(a)) + c_2(\omega)Q_{i\omega-\frac{1}{2}}(\cosh(a)).$$

For determining c_1 and c_2 we analyze the properties at $a = 0$. Since the Legendre function of the first kind has a finite value at $\cosh(0) = 1$ and the Legendre function of the second kind has a (logarithmic) singularity there we know that $c_2(\omega)$ vanishes identically. Now, by definition of the Legendre function of the first kind we obtain $F(0,\omega) = c_1(\omega)$. We evaluate

$$F(0,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\sqrt{\cosh(x)+1}}\, dx = \frac{\sqrt{\pi}}{\cosh(\pi\omega)}$$

by *Mathematica* for example. Hence, we obtain the following closed form for the Fourier transform.

$$F(a,\omega) = \frac{\sqrt{\pi}}{\cosh(\pi\omega)} P_{i\omega-\frac{1}{2}}(\cosh(a))$$

A discrete Fourier transform related to stationary determinantal processes

The following integral arose in the analysis of the entropy of stationary determinantal processes [LS03].

$$\hat{F}(n) = \int_0^1 e^{-2n\pi i x} \ln(\sin(\tfrac{\pi}{2}x))\, dx$$

In [LPR02] this integral was treated for $n \in \mathbb{N}^+$ by a series expansion and termwise integration, which led to the evaluation of a double sum by a variant of the WZ method and Zeilberger's fast algorithm. With the real part appearing in standard integral tables, e.g. formula 4.384.3 in [GR], they proved the imaginary part to be

$$\Im(\hat{F}(n)) = \frac{1}{n\pi} \sum_{k=1}^{n} \frac{1}{2k-1}.$$

We show how our software can be used to evaluate the integral in a more direct way by computing a recurrence for $\hat{F}(n)$ and an initial value. To this end, let

$$f(n,x) := e^{-2n\pi i x} \ln(\sin(\tfrac{\pi}{2}x))$$

and try to compute a recurrence for $\hat{F}(n)$, which then can be solved by the package `Sigma`. First, our algorithm finds

$$f(n+1,x) - \frac{n}{n+1} f(n,x) = \frac{d}{dx} \frac{e^{-2(n+1)\pi i x}}{2(n+1)\pi i} \left(\frac{1}{4(n+1)} + \frac{e^{\pi i x}}{2n+1} + \frac{e^{2\pi i x}}{4n} + (e^{2\pi i x}-1) \ln(\sin(\tfrac{\pi}{2}x)) \right).$$

Note that the right hand side cannot be evaluated at $n = 0$. Integrating over $(0,1)$ yields the recurrence

$$\hat{F}(n+1) - \frac{n}{n+1}\hat{F}(n) = \frac{i}{(n+1)(2n+1)\pi}.$$

For a complete evaluation we still need to obtain an initial value for the recurrence. For $n = 1$ our program computes an antiderivative of $f(1, x)$, which can be written as

$$\int f(1,x)\,dx = \frac{e^{-\pi i x}}{2\pi i} + \frac{e^{-2\pi i x}}{8\pi i} - \frac{x}{4} + \frac{1-e^{-2\pi i x}}{2\pi i}\ln(\sin(\tfrac{\pi}{2}x)).$$

By the limit $\lim\limits_{x\to 0^+} \frac{1-e^{-2\pi i x}}{2\pi i}\ln(\sin(\frac{\pi}{2}x)) = 0$ this yields to

$$\hat{F}(1) = -\frac{1}{4} + \frac{i}{\pi}.$$

Altogether, we obtain the following solution of the recurrence by the package `Sigma`.

$$\hat{F}(n) = -\frac{1}{4n} + \frac{i}{n\pi}\sum_{k=1}^{n}\frac{1}{2k-1}$$

Now, we also have a brief look at the case $n = 0$, for which above recurrence does not hold. Specializing $n = 0$ in the integral our algorithm does not find an antiderivative of $f(0, x)$, which implies that $\int f(0,x)\,dx$ is not an elementary function. Indeed, *Mathematica* returns a result in terms of the dilogarithm.

A family of Binet-like integrals

First, recall Binet's first formula for $\ln\Gamma$, see equation 1.9(4) in volume 1 of [Bateman] for example, which can be written in the following form.

$$\int_0^1 \left(\frac{1}{\ln(x)} + \frac{1}{1-x} - \frac{1}{2}\right)\frac{x^{\sigma-1}}{\ln(x)}\,dx = -\ln\Gamma(\sigma) + (\sigma - \tfrac{1}{2})\ln(\sigma) - \sigma + \tfrac{1}{2}\ln(2\pi)$$

Olivier Oloa sent a family of Binet-like integrals to us [Olo11]. He was interested in the evaluation of

$$B_n(\sigma) := \int_0^1 f_n(\sigma, x)\,dx$$

where the the integrand is given by

$$f_n(\sigma,x) := \left(\frac{1}{\ln(x)} + \frac{1}{1-x}\right)^n x^\sigma.$$

Observe that $\frac{1}{\ln(x)} + \frac{1}{1-x}$ is positive on the interval $(0, 1)$ and has limits 1 and $\frac{1}{2}$ at $x = 0$ and $x = 1$ respectively, so the integral exists for $\sigma > -1$. We will show how our software can assist in finding evaluations of $B_n(\sigma)$ for specific $n \in \mathbb{N}$ recursively from evaluations of $B_0(\sigma)$ and $B_1(\sigma)$. First, by our program we compute the following relation of $\frac{\partial f_{n+2}}{\partial\sigma}, \frac{\partial f_{n+1}}{\partial\sigma}, f_{n+1}, f_n$.

$$\frac{\partial f_{n+2}}{\partial\sigma}(\sigma,x) + \frac{\sigma-n}{n+1}\frac{\partial f_{n+1}}{\partial\sigma}(\sigma,x) - \frac{2n+1}{n+1}f_{n+1}(\sigma,x) + f_n(\sigma,x) = \frac{d}{dx}\left(\frac{\ln(x)}{n+1}f_{n+1}(\sigma+1,x)\right)$$

When integrated from 0 to 1 this yields the mixed (difference-differential) relation

$$B'_{n+2}(\sigma) + \frac{\sigma - n}{n+1} B'_{n+1}(\sigma) - \frac{2n+1}{n+1} B_{n+1}(\sigma) + B_n(\sigma) = 0, \tag{5.5}$$

which will allow us to express B_{n+2} in terms of B_{n+1} and B_n. In order to do so we just need to compute the value of $B_{n+2}(\sigma)$ for one specific value of σ. To this end, by a change of variable we obtain $B_n(\sigma) = \frac{1}{\sigma+1} \int_0^1 f_n(0, x^{\frac{1}{\sigma+1}})\, dx$ from which it is easy to see that $B_n(\sigma)$ is asymptotically equivalent to $\frac{1}{2^n \sigma}$ for $\sigma \to \infty$ and to $\frac{1}{\sigma+1}$ for $\sigma \to -1$. In particular we have

$$\lim_{\sigma \to \infty} B_n(\sigma) = 0. \tag{5.6}$$

Now, with this additional information we can use (5.5) to express B_n as the following integral in terms of B_{n-1} and B_{n-2}:

$$B_n(\sigma) = \int_\sigma^\infty \frac{s-n+2}{n-1} B'_{n-1}(s) - \frac{2n-3}{n-1} B_{n-1}(s) + B_{n-2}(s)\, ds \tag{5.7}$$

On the other hand we have indefinite integrals for the special cases $n = 0$ and $n = 1$:

$$\begin{aligned} \int f_0(\sigma, x)\, dx &= \frac{x^{\sigma+1}}{\sigma+1} \\ \int f_1(\sigma, x)\, dx &= \mathrm{Ei}((\sigma+1)\ln(x)) + B_x(\sigma+1, 0) \end{aligned}$$

where $\mathrm{Ei}(x)$ is the exponential integral and $B_x(\alpha, \beta)$ is the incomplete Beta function. Based on these and the expansions $\mathrm{Ei}(x) = \ln(-x) + \gamma + \mathcal{O}(-x)$ for $x \to 0^-$ (see Equations 6.2.6 and 6.6.2 in [DLMF]) and $B_x(\alpha, 0) = -\ln(1-x) - \gamma - \psi(\alpha) + \mathcal{O}(1-x)$ for $x \to 1^-$ (e.g. from equation 5.9.16 in [DLMF]) we can compute the corresponding integrals on the interval $(0, 1)$ for $\sigma > -1$:

$$\begin{aligned} B_0(\sigma) &= \frac{1}{\sigma+1} \\ B_1(\sigma) &= \ln(\sigma+1) - \psi(\sigma+1) \end{aligned}$$

By virtue of (5.7) these initial values allow us to compute $B_n(\sigma)$ recursively. Based on repeated indefinite integration we obtain for example

$$\begin{aligned} B_2(\sigma) &= (\sigma+1)\ln(\sigma+1) + \sigma\psi(\sigma+1) - 2\psi^{(-1)}(\sigma+1) - 2\sigma + c_2 \\ B_3(\sigma) &= \tfrac{(\sigma+1)^2}{2}\ln(\sigma+1) - \tfrac{\sigma(\sigma-1)}{2}\psi(\sigma+1) + 3\sigma\psi^{(-1)}(\sigma+1) - \\ &\quad -6\psi^{(-2)}(\sigma+1) - \tfrac{\sigma(3\sigma+2)}{2} + \tfrac{3}{2}c_2\sigma + c_3 \\ B_4(\sigma) &= \tfrac{(\sigma+1)^3}{6}\ln(\sigma+1) + \tfrac{\sigma(\sigma-1)(\sigma-2)}{6}\psi(\sigma+1) - 2\sigma(\sigma-1)\psi^{(-1)}(\sigma+1) + \\ &\quad +2(5\sigma-1)\psi^{(-2)}(\sigma+1) - 20\psi^{(-3)}(\sigma+1) - \tfrac{\sigma(22\sigma^2+27\sigma+18)}{36} + c_2\sigma^2 + \tfrac{5}{3}c_3\sigma + c_4 \\ B_5(\sigma) &= \tfrac{(\sigma+1)^4}{24}\ln(\sigma+1) - \tfrac{\sigma(\sigma-1)(\sigma-2)(\sigma-3)}{24}\psi(\sigma+1) + \tfrac{5\sigma(\sigma-1)(\sigma-2)}{6}\psi^{(-1)}(\sigma+1) - \\ &\quad -\tfrac{5(9\sigma^2-12\sigma+2)}{6}\psi^{(-2)}(\sigma+1) + 5(7\sigma-3)\psi^{(-3)}(\sigma+1) - 70\psi^{(-4)}(\sigma+1) - \\ &\quad -\tfrac{\sigma(25\sigma^3+43\sigma^2+57\sigma+42)}{144} + \tfrac{5}{12}c_2\sigma^3 + \tfrac{1}{4}c_3\sigma(5\sigma+1) + \tfrac{7}{4}c_4\sigma + c_5 \end{aligned}$$

where the constants c_2, c_3, c_4, c_5 are determined by (5.6) and depend on the particular definition of the iterated integrals $\psi^{(-k)}$ of the ψ function, which are called *negapolygamma* functions. One possible definition, which is used in *Mathematica*, is $\psi^{(-1)}(x) := \ln\Gamma(x)$ and $\psi^{(-k)}(x) := \int_0^x \psi^{(-k+1)}(t)\,dt$, i.e., $\psi^{(-k)}(0) = 0$, for $k \in \{2, 3, \dots\}$. Another definition was introduced and studied by Espinosa and Moll, see [EM04]. Their balanced negapolygamma functions, which are used in *Maple*, are defined in terms of the Hurwitz ζ function as

$$\psi^{(-k)}(x) := \frac{1}{(k-1)!}(\zeta'(1-k,x) + H_{k-1}\zeta(1-k,x)),$$

where H_n are the harmonic numbers and the derivative in ζ' is taken w.r.t. the first argument of ζ. In particular $\psi^{(-1)}(x) = \zeta'(0,x) = \ln\Gamma(x) - \frac{\ln(2\pi)}{2}$. While the first definition of $\psi^{(-k)}$ yields constants in terms of the derivative of the Riemann ζ function at nonnegative integers

$$\begin{aligned}
c_2 &= \ln(2\pi) - \tfrac{3}{2},\\
c_3 &= 3\ln(2\pi) - \tfrac{19}{24} - 6\zeta'(-1),\\
c_4 &= 6\ln(2\pi) + \tfrac{83}{72} - 22\zeta'(-1) + \tfrac{5}{2\pi^2}\zeta(3),\\
c_5 &= \tfrac{125}{12}\ln(2\pi) + \tfrac{32833}{8640} - \tfrac{155}{3}\zeta'(-1) + \tfrac{85}{8\pi^2}\zeta(3) - \tfrac{35}{3}\zeta'(-3),
\end{aligned}$$

it appears that for the latter definition the constants lie all in $\mathbb{Q}$:

$$c_2 = -\frac{3}{2}, \quad c_3 = -\frac{31}{24}, \quad c_4 = -\frac{49}{72}, \quad c_5 = -\frac{2827}{8640}.$$

Appendix A

Common special functions

In order to facilitate the representation of common special functions via differential fields suitable for the algorithms presented in this thesis we highlight their properties in a way that fits the structures described in Sections 2.6 and 3.5. The formulas given here are not meant to be authoritative or comprehensive, they simply serve the purpose of making explicit how general the discussed structures are and should help the reader to match common special functions with them. For precise definitions and additional properties of the functions we refer to standard resources like [Bateman] and [DLMF].

A.1 Liouvillian functions

A.1.1 Elementary functions

For elementary functions we emphasize that when representing powers $f(x)^c$ in Liouvillian field extensions it is not necessary to decompose them into the two functions $\ln(f(x))$ and $\exp(c\ln(f(x)))$ first, as we would have to do in elementary extensions, but we can directly use the hyperexponential expression $f(x)^c = \exp\left(\int \frac{cf'(x)}{f(x)}\,dx\right)$. Trigonometric and hyperbolic functions are rewritten in terms of exponentials. Some of the inverses can be represented in regular Liouvillian extensions.

$$\begin{aligned}\arctan(x) &= \int_0^x \frac{1}{t^2+1}\,dt\\ \operatorname{arctanh}(x) &= \int_0^x \frac{1}{1-t^2}\,dt\end{aligned}$$

Other inverses, like $\arccos(x) = \int_x^1 \frac{1}{\sqrt{1-t^2}}\,dt$ or $\operatorname{arcsinh}(x) = \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt$, involving radicals may be treated by the methods discussed in Sections 2.6.3 or 3.5. Also the Gudermannian function and its inverse can be represented in regular Liouvillian extensions.

$$\begin{aligned}\operatorname{gd}(x) &= \int_0^x \frac{1}{\cosh(t)}\,dt\\ \operatorname{gd}^{-1}(x) &= \int_0^x \frac{1}{\cos(t)}\,dt\end{aligned}$$

For the Lambert W function and other functions, which are not elementary themselves but are the inverse of an elementary function, we refer to Section A.3.

A.1.2 Exponential integral and related functions

For generalized versions of the exponential, sine, and cosine integrals see the section on the incomplete Gamma function below.

$$
\begin{aligned}
\mathrm{Ei}(x) &= \int_{-\infty}^{x} \frac{e^t}{t}\,dt \\
\mathrm{li}(x) &= \int_{0}^{x} \frac{1}{\ln(t)}\,dt \\
\mathrm{Si}(x) &= \int_{0}^{x} \frac{\sin(t)}{t}\,dt \\
\mathrm{Ci}(x) &= -\int_{x}^{\infty} \frac{\cos(t)}{t}\,dt \\
\mathrm{Shi}(x) &= \int_{0}^{x} \frac{\sinh(t)}{t}\,dt \\
\mathrm{Chi}(x) &= \int_{0}^{x} \frac{\cosh(t)-1}{t}\,dt + \ln(x) + \gamma
\end{aligned}
$$

A.1.3 Error function and related functions

$$
\begin{aligned}
\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\,dt \\
\mathrm{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2}\,dt \\
\mathrm{erfi}(x) &= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{t^2}\,dt \\
\mathrm{F}(x) &= e^{-x^2} \int_{0}^{x} e^{t^2}\,dt \\
\mathrm{G}(x) &= e^{-x^2}\,\frac{\pi\,\mathrm{erfi}(x) - \mathrm{Ei}(x^2)}{2}
\end{aligned}
$$

A.1.4 Fresnel integrals

$$
\begin{aligned}
\mathcal{F}(x) &= \int_{x}^{\infty} e^{i\frac{\pi}{2}t^2}\,dt \\
\mathrm{C}(x) &= \int_{0}^{x} \cos\left(\frac{\pi}{2}t^2\right)\,dt \\
\mathrm{S}(x) &= \int_{0}^{x} \sin\left(\frac{\pi}{2}t^2\right)\,dt
\end{aligned}
$$

A.1.5 Polylogarithms and related functions

For specific $n \in \{2, 3, \dots\}$ the polylogarithms are Liouvillian functions and satisfy the following recursive relations.

$$\begin{aligned}\mathrm{Li}_2(x) &= -\int_0^x \frac{\ln(1-t)}{t}\,dt \\ \mathrm{Li}_n(x) &= \int_0^x \frac{\mathrm{Li}_{n-1}(t)}{t}\,dt\end{aligned}$$

For general values of n we refer to Section A.3.

A.1.6 Incomplete Gamma and related functions

$$\begin{aligned}\gamma(a,x) &= \int_0^x t^{a-1}e^{-t}\,dt \\ \Gamma(a,x) &= \int_x^\infty t^{a-1}e^{-t}\,dt \\ \mathrm{B}_x(a,b) &= \int_0^x t^{a-1}(1-t)^{b-1}\,dt \\ \mathrm{E}_a(x) &= x^{a-1}\int_x^\infty \frac{e^{-t}}{t^a}\,dt \\ \mathrm{Si}(a,x) &= \int_0^x t^{a-1}\sin(t)\,dt \\ \mathrm{Ci}(a,x) &= \int_0^x t^{a-1}\cos(t)\,dt \\ \mathrm{si}(a,x) &= \int_x^\infty t^{a-1}\sin(t)\,dt \\ \mathrm{ci}(a,x) &= \int_x^\infty t^{a-1}\cos(t)\,dt\end{aligned}$$

A.1.7 Chebyshev polynomials

For specific $n \in \mathbb{N}$ the Chebyshev polynomials are polynomials. For general n they, like most other classical orthogonal polynomial families, satisfy second-order differential equations (see Section A.2) but at the same time they are elementary functions.

$$\begin{aligned}T_n(x) &= \cos(n\arccos(x)) \\ U_n(x) &= \frac{\sin((n+1)\arccos(x))}{\sin(\arccos(x))} \\ V_n(x) &= \frac{\sin((n+\frac{1}{2})\arccos(x))}{\sin(\frac{1}{2}\arccos(x))} \\ W_n(x) &= \frac{\cos((n+\frac{1}{2})\arccos(x))}{\cos(\frac{1}{2}\arccos(x))}\end{aligned}$$

Hence, the relations among the trigonometric functions also give rise to a lot of relations among Chebyshev polynomials like doubling relations $T_{2n}(x) = 2T_n(x)^2 - 1$, conversion formulas

$$T_n(x) = (1 - 2x^2)U_n(x) + xU_{n+1}(x) \quad \text{and} \quad U_n(x) = \frac{T_n(x) - xT_{n+1}(x)}{1 - x^2},$$

and de Moivre's formula

$$(x + \sqrt{x^2 - 1})^n = T_n(x) + U_{n-1}(x)\sqrt{x^2 - 1}.$$

They also satisfy Cassini-type identities like the Fibonacci polynomials do.

$$\begin{aligned} T_{n+1}(x)^2 - T_n(x)T_{n+2}(x) &= 1 - x^2 \\ U_{n+1}(x)^2 - U_n(x)U_{n+2}(x) &= 1 \end{aligned}$$

A.1.8 Fibonacci and Lucas polynomials

Like the Chebyshev polynomials also the Fibonacci and Lucas families of polynomials are elementary functions even for general n. For the differential systems they satisfy see Section A.2. In order to have real-valued functions also for non-integer values of n we may write

$$\begin{aligned} F_n(x) &= \frac{1}{\sqrt{x^2+4}} \left(\frac{x + \sqrt{x^2+4}}{2} \right)^n - \frac{\cos(n\pi)}{\sqrt{x^2+4}} \left(\frac{x + \sqrt{x^2+4}}{2} \right)^{-n} \\ L_n(x) &= \left(\frac{x + \sqrt{x^2+4}}{2} \right)^n + \cos(n\pi) \left(\frac{x + \sqrt{x^2+4}}{2} \right)^{-n} \end{aligned}$$

In analogy to the Chebyshev polynomials, which are given by trigonometric functions, we may write the Fibonacci and Lucas polynomials in terms of hyperbolic functions.

$$\begin{aligned} F_n(x) &= \frac{(1 - \cos(n\pi))\cosh(n \operatorname{arcsinh}(\frac{x}{2})) + (1 + \cos(n\pi))\sinh(n \operatorname{arcsinh}(\frac{x}{2}))}{2\cosh(\operatorname{arcsinh}(\frac{x}{2}))} \\ L_n(x) &= (1 + \cos(n\pi))\cosh(n \operatorname{arcsinh}(\tfrac{x}{2})) + (1 - \cos(n\pi))\sinh(n \operatorname{arcsinh}(\tfrac{x}{2})) \end{aligned}$$

These two functions can be converted into one another by

$$F_n(x) = \frac{2L_{n+1}(x) - xL_n(x)}{x^2 + 4} \quad \text{and} \quad L_n(x) = 2F_{n+1}(x) - xF_n(x)$$

and satisfy many other identities, like Cassini-type identities for example.

$$\begin{aligned} F_{n+1}(x)^2 - F_n(x)F_{n+2}(x) &= \cos(n\pi) \\ L_{n+1}(x)^2 - L_n(x)L_{n+2}(x) &= -\cos(n\pi)(x^2 + 4) \end{aligned}$$

They also satisfy

$$\left(\frac{x + \sqrt{x^2+4}}{2} \right)^n = \frac{L_n(x) + F_n(x)\sqrt{x^2+4}}{2}.$$

A.2 Functions satisfying second-order equations

In this section we list special functions that satisfy a system of the form (2.10) or (2.16) as discussed in Section 2.6.2. For the sake of brevity we often give only the system matrix $A(x) = (a_{i,j}(x))_{i,j\in\{1,2\}}$ and a corresponding fundamental matrix solution $\Phi(x)$ as well as its Wronskian $\det \Phi(x)$ for the homogeneous systems, which is determined by $A(x)$ up to a factor independent of x.

Most of the functions listed involve additional parameters and the formulas given hold for symbolic parameters (ranging over a domain where the functions are defined) as well as for most specialized values in the domain. Note that the formulas given below might give $\det \Phi(x) = 0$ for some specific values of the parameters, in which case $\Phi(x)$ has linear dependent columns for these values of the parameters.

A.2.1 Orthogonal polynomials and related functions

We list the systems satisfied by many of the classical families of polynomials. The same systems apply also to the corresponding functions for non-integer values of n.

Legendre and associated Legendre functions

$$\begin{pmatrix} P_n(x) \\ P_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} \frac{(n+1)x}{1-x^2} & -\frac{n+1}{1-x^2} \\ \frac{n+1}{1-x^2} & -\frac{(n+1)x}{1-x^2} \end{pmatrix} \begin{pmatrix} P_n(x) \\ P_{n+1}(x) \end{pmatrix}$$

A second solution can be given in terms of Legendre functions of second kind.

$$\Phi(x) = \begin{pmatrix} P_n(x) & Q_n(x) \\ P_{n+1}(x) & Q_{n+1}(x) \end{pmatrix} \qquad \det \Phi(x) = \frac{1}{n+1}$$

The associated Legendre functions of first and second kind satisfy the following system.

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} \frac{(n+1)x}{1-x^2} & \frac{m-(n+1)}{1-x^2} \\ \frac{m+(n+1)}{1-x^2} & -\frac{(n+1)x}{1-x^2} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

$$\Phi(x) = \begin{pmatrix} P_n^m(x) & Q_n^m(x) \\ P_{n+1}^m(x) & Q_{n+1}^m(x) \end{pmatrix} \qquad \det \Phi(x) = -\frac{\Gamma(n+m+1)}{\Gamma(n-m+2)}$$

Hermite polynomials

$$\begin{pmatrix} H_n(x) \\ H_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} 2x & -1 \\ 2(n+1) & 0 \end{pmatrix} \begin{pmatrix} H_n(x) \\ H_{n+1}(x) \end{pmatrix}$$

A second solution can be given in terms of confluent hypergeometric functions.

$$\Phi(x) = \begin{pmatrix} H_n(x) & {}_1F_1(-\frac{n}{2}; \frac{1}{2}; x^2) \\ H_{n+1}(x) & 2(n+1)x\,{}_1F_1(-\frac{n}{2}; \frac{3}{2}; x^2) \end{pmatrix} \qquad \det \Phi(x) = \frac{2^n n\sqrt{\pi}}{\Gamma(1-\frac{n}{2})} e^{x^2}$$

Generalized Laguerre polynomials

$$\begin{pmatrix} L_n^{(a)}(x) \\ L_{n+1}^{(a)}(x) \end{pmatrix}' = \begin{pmatrix} 1-\frac{n+a+1}{x} & \frac{n+1}{x} \\ -\frac{n+a+1}{x} & \frac{n+1}{x} \end{pmatrix} \begin{pmatrix} L_n^{(a)}(x) \\ L_{n+1}^{(a)}(x) \end{pmatrix}$$

For non-integer n a second solution can be given in terms of the confluent hypergeometric function U.

$$\Phi(x) = \begin{pmatrix} L_n^{(a)}(x) & U(-n,a+1,x) \\ L_{n+1}^{(a)}(x) & \frac{nx}{n+1}U(1-n,a+1,x)+\frac{n+a+1-x}{n+1}U(-n,a+1,x) \end{pmatrix}$$

$$\det\Phi(x) = \frac{\sin(n\pi)\Gamma(n+a+1)}{(n+1)\pi}e^x x^{-a}$$

Gegenbauer polynomials

$$\begin{pmatrix} C_n^{(\lambda)}(x) \\ C_{n+1}^{(\lambda)}(x) \end{pmatrix}' = \begin{pmatrix} \frac{(n+2\lambda)x}{1-x^2} & -\frac{n+1}{1-x^2} \\ \frac{n+2\lambda}{1-x^2} & -\frac{(n+1)x}{1-x^2} \end{pmatrix} \begin{pmatrix} C_n^{(\lambda)}(x) \\ C_{n+1}^{(\lambda)}(x) \end{pmatrix}$$

A second solution can be given in terms of associated Legendre functions of the second kind.

$$\Phi(x) = \begin{pmatrix} C_n^{(\lambda)}(x) & (1-x^2)^{\frac{1-2\lambda}{4}}Q_{n+\lambda-\frac12}^{\lambda-\frac12}(x) \\ C_{n+1}^{(\lambda)}(x) & (1-x^2)^{\frac{1-2\lambda}{4}}Q_{n+\lambda+\frac12}^{\lambda-\frac12}(x) \end{pmatrix}$$

$$\det\Phi(x) = -\frac{2^{\frac12-\lambda}\sqrt{\pi}\sin(\lambda\pi)\Gamma(n+2\lambda)}{\Gamma(n+2)\Gamma(\lambda)}(1-x^2)^{-\lambda+\frac12}$$

For non-integer n a second solution can be given in terms of Gegenbauer functions again.

$$\Phi(x) = \begin{pmatrix} C_n^{(\lambda)}(x) & C_n^{(\lambda)}(-x) \\ C_{n+1}^{(\lambda)}(x) & -C_{n+1}^{(\lambda)}(-x) \end{pmatrix} \qquad \det\Phi(x) = \frac{\sin(n\pi)2^{2-2\lambda}\Gamma(n+2\lambda)}{\Gamma(n+2)\Gamma(\lambda)^2}(1-x^2)^{-\lambda+\frac12}$$

Jacobi polynomials

$$\begin{pmatrix} P_n^{(a,b)}(x) \\ P_{n+1}^{(a,b)}(x) \end{pmatrix}' = \begin{pmatrix} \frac{(a+b+n+1)((\frac{a+b}{2}+n+1)x+\frac{a-b}{2})}{(\frac{a+b}{2}+n+1)(1-x^2)} & -\frac{(n+1)(a+b+n+1)}{(\frac{a+b}{2}+n+1)(1-x^2)} \\ \frac{(a+n+1)(b+n+1)}{(\frac{a+b}{2}+n+1)(1-x^2)} & -\frac{(n+1)((\frac{a+b}{2}+n+1)x-\frac{a-b}{2})}{(\frac{a+b}{2}+n+1)(1-x^2)} \end{pmatrix} \begin{pmatrix} P_n^{(a,b)}(x) \\ P_{n+1}^{(a,b)}(x) \end{pmatrix}$$

For non-integer n a second solution can be given in terms of Jacobi functions again.

$$\Phi(x) = \begin{pmatrix} P_n^{(a,b)}(x) & P_n^{(b,a)}(-x) \\ P_{n+1}^{(a,b)}(x) & -P_{n+1}^{(b,a)}(-x) \end{pmatrix}$$

$$\det\Phi(x) = \frac{2^{a+b+1}\sin(n\pi)(\frac{a+b}{2}+n+1)\Gamma(a+n+1)\Gamma(b+n+1)}{\pi\,\Gamma(n+2)\Gamma(a+b+n+2)}(1-x)^{-a}(1+x)^{-b}$$

Chebyshev polynomials

The Chebyshev polynomials are elementary functions even for symbolic n and hence the systems given below do not satisfy the criterion given in Theorem 2.56. For their representation as elementary functions see Section A.1. Due to the many relations satisfied by Chebyshev polynomials there are several differential systems having nice fundamental matrix solutions in terms of Chebyshev polynomials.

$A(x)$	$\Phi(x)$	$\det\Phi(x)$
$\begin{pmatrix} \frac{nx}{1-x^2} & -\frac{n}{1-x^2} \\ \frac{n+1}{1-x^2} & -\frac{(n+1)x}{1-x^2} \end{pmatrix}$	$\begin{pmatrix} T_n(x) & \sqrt{1-x^2}U_{n-1}(x) \\ T_{n+1}(x) & \sqrt{1-x^2}U_n(x) \end{pmatrix}$	$\sqrt{1-x^2}$
$\begin{pmatrix} \frac{(n+2)x}{1-x^2} & -\frac{n+1}{1-x^2} \\ \frac{n+2}{1-x^2} & -\frac{(n+1)x}{1-x^2} \end{pmatrix}$	$\begin{pmatrix} U_n(x) & \frac{T_{n+1}(x)}{\sqrt{1-x^2}} \\ U_{n+1}(x) & \frac{T_{n+2}(x)}{\sqrt{1-x^2}} \end{pmatrix}$	$-\frac{1}{\sqrt{1-x^2}}$
$\begin{pmatrix} 0 & n \\ -\frac{n}{1-x^2} & \frac{x}{1-x^2} \end{pmatrix}$	$\begin{pmatrix} T_n(x) & \sqrt{1-x^2}U_{n-1}(x) \\ U_{n-1}(x) & -\frac{T_n(x)}{\sqrt{1-x^2}} \end{pmatrix}$	$-\frac{1}{\sqrt{1-x^2}}$

Fibonacci and Lucas polynomials

The Fibonacci and Lucas polynomials are elementary functions even for symbolic n, see Section A.1. Since Fibonacci and Lucas polynomials can be expressed in terms of each other there are several differential systems with fundamental matrix solutions nicely expressible in terms of $F_n(x)$ and $L_n(x)$, none of those satisfies the criterion given in Theorem 2.56.

$A(x)$	$\Phi(x)$	$\det\Phi(x)$
$\begin{pmatrix} -\frac{(n+1)x}{x^2+4} & \frac{2n}{x^2+4} \\ \frac{2(n+1)}{x^2+4} & \frac{nx}{x^2+4} \end{pmatrix}$	$\begin{pmatrix} F_n(x) & \frac{L_n(x)}{\sqrt{x^2+4}} \\ F_{n+1}(x) & \frac{L_{n+1}(x)}{\sqrt{x^2+4}} \end{pmatrix}$	$-\frac{2\cos(n\pi)}{\sqrt{x^2+4}}$
$\begin{pmatrix} -\frac{nx}{x^2+4} & \frac{2n}{x^2+4} \\ \frac{2(n+1)}{x^2+4} & \frac{(n+1)x}{x^2+4} \end{pmatrix}$	$\begin{pmatrix} L_n(x) & F_n(x)\sqrt{x^2+4} \\ L_{n+1}(x) & F_{n+1}(x)\sqrt{x^2+4} \end{pmatrix}$	$2\cos(n\pi)\sqrt{x^2+4}$
$\begin{pmatrix} -\frac{x}{x^2+4} & \frac{n}{x^2+4} \\ n & 0 \end{pmatrix}$	$\begin{pmatrix} F_n(x) & \frac{L_n(x)}{\sqrt{x^2+4}} \\ L_n(x) & F_n(x)\sqrt{x^2+4} \end{pmatrix}$	$-\frac{4\cos(n\pi)}{\sqrt{x^2+4}}$

A.2.2 Bessel functions and related functions

Bessel functions

The Bessel functions and their shifts in n form a fundamental matrix solution of the following system.

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} \frac{n}{x} & -1 \\ 1 & -\frac{n+1}{x} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

$$\Phi(x) = \begin{pmatrix} J_n(x) & Y_n(x) \\ J_{n+1}(x) & Y_{n+1}(x) \end{pmatrix} \qquad \det\Phi(x) = -\frac{2}{\pi x}$$

Whereas the modified Bessel functions and their shifts are solutions of a related system.

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} \frac{n}{x} & 1 \\ 1 & -\frac{n+1}{x} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

$$\Phi(x) = \begin{pmatrix} I_n(x) & K_n(x) \\ I_{n+1}(x) & -K_{n+1}(x) \end{pmatrix} \qquad \det \Phi(x) = -\frac{1}{x}$$

Anger and Weber functions

The Anger functions $\mathbf{J}_n(x)$ are particular solutions of

$$\begin{pmatrix} \mathbf{J}_n(x) \\ \mathbf{J}_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} \frac{n}{x} & -1 \\ 1 & -\frac{n+1}{x} \end{pmatrix} \begin{pmatrix} \mathbf{J}_n(x) \\ \mathbf{J}_{n+1}(x) \end{pmatrix} + \begin{pmatrix} -\frac{\sin(n\pi)}{\pi x} \\ -\frac{\sin(n\pi)}{\pi x} \end{pmatrix}$$

and can be written in terms of Bessel functions

$$\begin{pmatrix} \mathbf{J}_n(x) \\ \mathbf{J}_{n+1}(x) \end{pmatrix} = \tilde{\lambda}(x) \begin{pmatrix} J_n(x) \\ J_{n+1}(x) \end{pmatrix} - \lambda(x) \begin{pmatrix} Y_n(x) \\ Y_{n+1}(x) \end{pmatrix}$$

with auxiliary functions $\lambda(x)$ and $\tilde{\lambda}(x)$ such that

$$\lambda'(x) = \frac{\sin(n\pi)}{2}(J_{n+1}(x) - J_n(x)) \quad \text{and} \quad \tilde{\lambda}'(x) = \frac{\sin(n\pi)}{2}(Y_{n+1}(x) - Y_n(x)).$$

The Weber functions $\mathbf{E}_n(x)$ are particular solutions of

$$\begin{pmatrix} \mathbf{E}_n(x) \\ \mathbf{E}_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} \frac{n}{x} & -1 \\ 1 & -\frac{n+1}{x} \end{pmatrix} \begin{pmatrix} \mathbf{E}_n(x) \\ \mathbf{E}_{n+1}(x) \end{pmatrix} + \begin{pmatrix} \frac{\cos(n\pi)-1}{\pi x} \\ \frac{\cos(n\pi)+1}{\pi x} \end{pmatrix}$$

and can be written in terms of Bessel functions

$$\begin{pmatrix} \mathbf{E}_n(x) \\ \mathbf{E}_{n+1}(x) \end{pmatrix} = \tilde{\lambda}(x) \begin{pmatrix} J_n(x) \\ J_{n+1}(x) \end{pmatrix} - \lambda(x) \begin{pmatrix} Y_n(x) \\ Y_{n+1}(x) \end{pmatrix}$$

with auxiliary functions $\lambda(x)$ and $\tilde{\lambda}(x)$ such that

$$\begin{aligned} \lambda'(x) &= \frac{1+\cos(n\pi)}{2} J_n(x) + \frac{1-\cos(n\pi)}{2} J_{n+1}(x) \\ \tilde{\lambda}'(x) &= \frac{1+\cos(n\pi)}{2} J_n(x) + \frac{1-\cos(n\pi)}{2} J_{n+1}(x). \end{aligned}$$

Struve functions

The Struve functions $\mathbf{H}_n(x)$ are particular solutions of

$$\begin{pmatrix} \mathbf{H}_n(x) \\ \mathbf{H}_{n+1}(x) \end{pmatrix}' = \begin{pmatrix} \frac{n}{x} & -1 \\ 1 & -\frac{n+1}{x} \end{pmatrix} \begin{pmatrix} \mathbf{H}_n(x) \\ \mathbf{H}_{n+1}(x) \end{pmatrix} + \begin{pmatrix} \frac{x^n}{2^n\sqrt{\pi}\Gamma(n+\frac{3}{2})} \\ 0 \end{pmatrix}$$

and can be written in terms of Bessel functions

$$\begin{pmatrix}\mathbf{H}_n(x)\\ \mathbf{H}_{n+1}(x)\end{pmatrix} = \tilde{\lambda}(x)\begin{pmatrix}J_n(x)\\ J_{n+1}(x)\end{pmatrix} - \lambda(x)\begin{pmatrix}Y_n(x)\\ Y_{n+1}(x)\end{pmatrix}$$

with auxiliary functions $\lambda(x)$ and $\tilde{\lambda}(x)$ such that

$$\lambda'(x) = -\frac{\sqrt{\pi}}{2^{n+1}\Gamma(n+\frac{3}{2})}x^{n+1}J_{n+1}(x) \quad \text{and} \quad \tilde{\lambda}'(x) = -\frac{\sqrt{\pi}}{2^{n+1}\Gamma(n+\frac{3}{2})}x^{n+1}Y_{n+1}(x).$$

The modified Struve functions $\mathbf{L}_n(x)$ are particular solutions of

$$\begin{pmatrix}\mathbf{L}_n(x)\\ \mathbf{L}_{n+1}(x)\end{pmatrix}' = \begin{pmatrix}\frac{n}{x} & 1\\ 1 & -\frac{n+1}{x}\end{pmatrix}\begin{pmatrix}\mathbf{L}_n(x)\\ \mathbf{L}_{n+1}(x)\end{pmatrix} + \begin{pmatrix}\frac{x^n}{2^n\sqrt{\pi}\Gamma(n+\frac{3}{2})}\\ 0\end{pmatrix}$$

and can be written in terms of Bessel functions

$$\begin{pmatrix}\mathbf{L}_n(x)\\ \mathbf{L}_{n+1}(x)\end{pmatrix} = \tilde{\lambda}(x)\begin{pmatrix}I_n(x)\\ I_{n+1}(x)\end{pmatrix} - \lambda(x)\begin{pmatrix}K_n(x)\\ -K_{n+1}(x)\end{pmatrix}$$

with auxiliary functions $\lambda(x)$ and $\tilde{\lambda}(x)$ such that

$$\lambda'(x) = -\frac{1}{2^n\sqrt{\pi}\Gamma(n+\frac{3}{2})}x^{n+1}I_{n+1}(x) \quad \text{and} \quad \tilde{\lambda}'(x) = \frac{1}{2^n\sqrt{\pi}\Gamma(n+\frac{3}{2})}x^{n+1}K_{n+1}(x).$$

Lommel functions

The Lommel functions $s_{m,n}(x)$ are particular solutions of

$$\begin{pmatrix}s_{m,n}(x)\\ (n-m+1)s_{m-1,n+1}(x)\end{pmatrix}' = \begin{pmatrix}\frac{n}{x} & -1\\ 1 & -\frac{n+1}{x}\end{pmatrix}\begin{pmatrix}s_{m,n}(x)\\ (n-m+1)s_{m-1,n+1}(x)\end{pmatrix} + \begin{pmatrix}0\\ -x^{m-1}\end{pmatrix}$$

and can be written in terms of Bessel functions

$$\begin{pmatrix}s_{m,n}(x)\\ (n-m+1)s_{m-1,n+1}(x)\end{pmatrix} = \tilde{\lambda}(x)\begin{pmatrix}J_n(x)\\ J_{n+1}(x)\end{pmatrix} - \lambda(x)\begin{pmatrix}Y_n(x)\\ Y_{n+1}(x)\end{pmatrix}$$

with auxiliary functions $\lambda(x)$ and $\tilde{\lambda}(x)$ such that

$$\lambda'(x) = -\frac{\pi}{2}x^mJ_n(x) \quad \text{and} \quad \tilde{\lambda}'(x) = -\frac{\pi}{2}x^mY_n(x).$$

For avoiding the shift in m we might also use the system

$$\begin{pmatrix}s_{m,n}(x)\\ \frac{n-m+1}{n+m+1}s_{m,n+2}(x)\end{pmatrix}' = \begin{pmatrix}\frac{n}{x}-\frac{x}{2(n+1)} & -\frac{x}{2(n+1)}\\ \frac{x}{2(n+1)} & \frac{x}{2(n+1)}-\frac{n+2}{x}\end{pmatrix}\begin{pmatrix}s_{m,n}(x)\\ \frac{n-m+1}{n+m+1}s_{m,n+2}(x)\end{pmatrix} + \begin{pmatrix}\frac{x^m}{n+m+1}\\ -\frac{x^m}{n+m+1}\end{pmatrix},$$

which gives rise to the representation

$$\begin{pmatrix}s_{m,n}(x)\\ \frac{n-m+1}{n+m+1}s_{m,n+2}(x)\end{pmatrix} = \tilde{\lambda}(x)\begin{pmatrix}J_n(x)\\ J_{n+2}(x)\end{pmatrix} - \lambda(x)\begin{pmatrix}Y_n(x)\\ Y_{n+2}(x)\end{pmatrix}$$

with auxiliary functions $\lambda(x)$ and $\tilde{\lambda}(x)$ such that

$$\lambda'(x) = -\frac{\pi}{2(n+m+1)}x^{m+1}J_{n+1}(x) \quad \text{and} \quad \tilde{\lambda}'(x) = -\frac{\pi}{2(n+m+1)}x^{m+1}Y_{n+1}(x).$$

Note that $J_{n+2}(x) = \frac{2(n+1)}{x}J_{n+1}(x) - J_n(x)$ and $Y_{n+2}(x) = \frac{2(n+1)}{x}Y_{n+1}(x) - Y_n(x)$. For the Lommel functions $S_{m,n}(x)$ the same holds.

Airy functions

The Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ form a fundamental system of the Airy equation.

$$\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}$$

$$\Phi(x) = \begin{pmatrix} \mathrm{Ai}(x) & \mathrm{Bi}(x) \\ \mathrm{Ai}'(x) & \mathrm{Bi}'(x) \end{pmatrix} \qquad \det \Phi(x) = \frac{1}{\pi}$$

Scorer functions

The Scorer functions $\mathrm{Gi}(x)$ and $\mathrm{Hi}(x)$ are particular solutions of the systems

$$\begin{pmatrix} \mathrm{Gi}(x) \\ \mathrm{Gi}'(x) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Gi}(x) \\ \mathrm{Gi}'(x) \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{\pi} \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{Hi}(x) \\ \mathrm{Hi}'(x) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Hi}(x) \\ \mathrm{Hi}'(x) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\pi} \end{pmatrix}$$

and can be written in terms of the Airy functions as

$$\begin{aligned} \mathrm{Gi}(x) &= \mathrm{Ai}(x) \int_0^x \mathrm{Bi}(t)\, dt - \mathrm{Bi}(x) \left(\int_0^x \mathrm{Ai}(t)\, dt - \frac{1}{3} \right) \\ \mathrm{Hi}(x) &= -\mathrm{Ai}(x) \int_0^x \mathrm{Bi}(t)\, dt + \mathrm{Bi}(x) \left(\int_0^x \mathrm{Ai}(t)\, dt + \frac{2}{3} \right) \end{aligned}$$

Kelvin functions

Kelvin functions can be represented in terms of Bessel functions.

$$\begin{aligned} \mathrm{ber}_n(x) &= \frac{J_n(\frac{-1+i}{\sqrt{2}}x) + J_n(\frac{-1-i}{\sqrt{2}}x)}{2} \\ \mathrm{bei}_n(x) &= \frac{J_n(\frac{-1+i}{\sqrt{2}}x) - J_n(\frac{-1-i}{\sqrt{2}}x)}{2i} \\ \mathrm{ker}_n(x) &= \frac{(-i)^n K_n(\frac{1+i}{\sqrt{2}}x) + i^n K_n(\frac{1-i}{\sqrt{2}}x)}{2} \\ \mathrm{kei}_n(x) &= \frac{(-i)^n K_n(\frac{1+i}{\sqrt{2}}x) - i^n K_n(\frac{1-i}{\sqrt{2}}x)}{2i} \end{aligned}$$

A.2.3 Other functions

Complete elliptic integrals

The following definition of the complete elliptic integrals of first and second kind is used in the computer algebra system Mathematica.

$$\begin{aligned} K(x) &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - x\sin(t)^2}}\,dt \\ E(x) &= \int_0^{\frac{\pi}{2}} \sqrt{1 - x\sin(t)^2}\,dt \end{aligned}$$

They are related by the system

$$\begin{pmatrix} K(x) \\ E(x) \end{pmatrix}' = \begin{pmatrix} -\frac{1}{2x} & \frac{1}{2x(1-x)} \\ -\frac{1}{2x} & \frac{1}{2x} \end{pmatrix} \begin{pmatrix} K(x) \\ E(x) \end{pmatrix}.$$

In fact all the solutions of this system can be written in terms of K and E as it has a fundamental matrix solution given by

$$\Phi(x) = \begin{pmatrix} K(x) & K(1-x) \\ E(x) & K(1-x) - E(1-x) \end{pmatrix} \quad \text{with} \quad \det\Phi(x) = -\frac{\pi}{2}.$$

Note that a different definition of K and E is more common: $K(k) := F(\frac{\pi}{2}, k)$ and $E(k) := E(\frac{\pi}{2}, k)$ are given by $K(x)$ and $E(x)$ above with $x = k^2$.

Kummer functions

The system

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} 0 & \frac{a}{b} \\ \frac{b}{x} & 1 - \frac{b}{x} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

has a fundamental matrix solution in terms of confluent hypergeometric functions.

$$\Phi(x) = \begin{pmatrix} {}_1F_1(a; b; x) & U(a, b, x) \\ {}_1F_1(a+1; b+1; x) & -bU(a+1, b+1, x) \end{pmatrix} \qquad \det\Phi(x) = -\frac{\Gamma(b+1)}{\Gamma(a+1)} e^x x^{-b}$$

Whittaker functions

The system

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} \frac{1}{2} - \frac{k}{x} & \frac{k+m+\frac{1}{2}}{x} \\ \frac{m-k-\frac{1}{2}}{x} & \frac{k+1}{x} - \frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

has a fundamental matrix solution in terms of Whittaker functions and their shifts in k.

$$\Phi(x) = \begin{pmatrix} M_{k,m}(x) & W_{k,m}(x) \\ M_{k+1,m}(x) & -\frac{1}{k+m+\frac{1}{2}} W_{k+1,m}(x) \end{pmatrix} \qquad \det\Phi(x) = -\frac{2m\Gamma(2m)x}{(k+m+\frac{1}{2})\Gamma(m-k+\frac{1}{2})}$$

Whereas the system

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} \frac{(m+\frac{1}{2})^2-\frac{k}{2}x}{(m+\frac{1}{2})x} & \frac{(m+\frac{1}{2})^2-k^2}{8(m+1)(m+\frac{1}{2})^2} \\ 2(m+1) & -\frac{(m+\frac{1}{2})^2-\frac{k}{2}x}{(m+\frac{1}{2})x} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

has a fundamental matrix solution in terms of Whittaker functions and their shifts in m.

$$\Phi(x) = \begin{pmatrix} M_{k,m}(x) & W_{k,m}(x) \\ M_{k,m+1}(x) & -\frac{4(m+1)(m+\frac{1}{2})}{k+m+\frac{1}{2}} W_{k,m+1}(x) \end{pmatrix} \qquad \det \Phi(x) = -\frac{2(m+\frac{1}{2})\Gamma(2m+3)}{(k+m+\frac{1}{2})\Gamma(m-k+\frac{3}{2})}$$

Hypergeometric functions

For $d := a+b-c+1$ the system

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} 0 & \frac{ab}{c} \\ \frac{c}{x(1-x)} & \frac{d}{1-x} - \frac{c}{x} \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

has a fundamental matrix solution in terms of hypergeometric functions. If none of a, b, c, d is a non-positive integer then we may use the following fundamental matrix.

$$\Phi(x) = \begin{pmatrix} {}_2F_1(a, b; c; x) & {}_2F_1(a, b; d; 1-x) \\ {}_2F_1(a+1, b+1; c+1; x) & -\frac{c}{d} {}_2F_1(a+1, b+1; d+1; 1-x) \end{pmatrix}$$

$$\det \Phi(x) = -\frac{\Gamma(c+1)\Gamma(d)}{\Gamma(a+1)\Gamma(b+1)} x^{-c}(1-x)^{-d}$$

Mathieu functions

The Mathieu cosine $C(a, q, x)$ and Mathieu sine $S(a, q, x)$, which are denoted by $w_{\mathrm{I}}(x; a, q)$ and $w_{\mathrm{II}}(x; a, q)$ in [DLMF], form a fundamental system of Mathieu's equation.

$$\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2q\cos(2x) - a & 0 \end{pmatrix} \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}$$

$$\Phi(x) = \begin{pmatrix} C(a, q, x) & S(a, q, x) \\ C'(a, q, x) & S'(a, q, x) \end{pmatrix} \qquad \det \Phi(x) = 1$$

A.3 Other special functions

A.3.1 Elliptic functions and related functions

The elliptic functions of Jacobi and Weierstraß are inverses of Liouvillian functions, so by an appropriate change of variable an integrand involving such functions could become better accessible to our algorithms as discussed in Section 2.6.3. However, radicals are involved, for which we refer to Section 3.5.

Jacobian elliptic functions and related functions

For example, based on the elliptic integral of the first kind we can make the change of variable

$$x = F(\arcsin(u), k) = \int_0^u \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt$$

and we get the following correspondences, where $\mathcal{E}(x,k) = E(\mathrm{am}(x,k),k)$ is Jacobi's Epsilon function.

$$\mathrm{sn}(x,k) = u \qquad \mathrm{cn}(x,k) = \sqrt{1-u^2} \qquad \mathrm{dn}(x,k) = \sqrt{1-k^2u^2}$$

$$\mathrm{am}(x,k) = \int_0^u \frac{1}{\sqrt{1-t^2}}\,dt \qquad \mathcal{E}(x,k) = \int_0^u \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt$$

Note that the right hand sides are all Liouvillian functions. Alternatively, we could also utilize $x = \int_u^1 \frac{1}{\sqrt{(1-t^2)(1-k^2+k^2t^2)}}\,dt$ or $x = \int_u^1 \frac{1}{\sqrt{(1-t^2)(k^2-1+t^2)}}\,dt$ with inverses $\mathrm{cn}(x,k) = u$ or $\mathrm{dn}(x,k) = u$ respectively, which give expressions for the other functions similar to above. A slightly modified version of the change of variable above is

$$x = F(u,k) = \int_0^u \frac{1}{\sqrt{1-k^2\sin(t)^2}}\,dt,$$

which gives rise to the following correspondences.

$$\mathrm{sn}(x,k) = \sin(u) \qquad \mathrm{cn}(x,k) = \cos(u) \qquad \mathrm{dn}(x,k) = \sqrt{1-k^2\sin(u)^2}$$

$$\mathrm{am}(x,k) = u \qquad \mathcal{E}(x,k) = \int_0^u \sqrt{1-k^2\sin(t)^2}\,dt$$

Weierstraß elliptic functions and related functions

Let g_2 and g_3 be fixed, in addition to the elliptic function $\wp(x)$ we also consider the Weierstraß Zeta and Sigma functions. The change of variable

$$x = w(u) := \int_u^\infty \frac{1}{\sqrt{4t^3-g_2t-g_3}}\,dt$$

yields the following correspondences, the right hand sides all being Liouvillian functions.

$$\begin{aligned}
\wp(x) &= u \\
\wp'(x) &= -\sqrt{4u^3-g_2u-g_3} \\
\zeta(x) &= z(u) := \frac{1}{w(u)} - \int_u^\infty \frac{t-\frac{1}{w(t)^2}}{\sqrt{4t^3-g_2t-g_3}}\,dt \\
\sigma(x) &= s(u) := w(u)\exp\left(\int_u^\infty \frac{z(t)-\frac{1}{w(t)}}{\sqrt{4t^3-g_2t-g_3}}\,dt\right)
\end{aligned}$$

Note that we have $z'(u) = \frac{u}{\sqrt{4u^3-g_2u-g_3}}$ and $\frac{s'(u)}{s(u)} = -\frac{z(u)}{\sqrt{4u^3-g_2u-g_3}}$ explicitly.

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Curriculum Vitae

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Education

Jun 2001	Matura (secondary school graduation diploma) with distinction at Europagymnasium Auhof, Linz, Austria
Oct 2001–Sep 2002, Oct 2003–Oct 2008	Studies in Technical Mathematics (branch: mathematics in natural sciences) at Johannes Kepler University Linz, Austria
Oct 2008	Diplom-Ingenieur (diploma degree) with distinction at Johannes Kepler University Linz, Austria
Nov 2008–present	Doctoral studies in the Doctoral Program "Computational Mathematics" at Johannes Kepler University Linz, Austria Advisor: Prof. Dr. Peter Paule

Research visits

Feb–Apr 2011	North Carolina State University, Raleigh, USA host: Prof. Dr. Michael F. Singer
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Awards

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Publications

1. Elena Kartashova, Clemens G. Raab, Christian Feurer, Günther Mayrhofer, Wolfgang Schreiner. *Symbolic Computations for Nonlinear Wave Resonances.* In: Extreme Ocean Waves, E. Pelinovsky, Ch. Kharif (eds.), pp. 97–128, Springer, 2008.

2. Clemens G. Raab. *Räume homogenen Typs und Diffusion Wavelets.* Diploma Thesis, Johannes Kepler Universität Linz, Austria, 2008. (in German)

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6. Moulay A. Barkatou, Clemens G. Raab, *Solving Linear Ordinary Differential Systems in Hyperexponential Extensions*, Proceedings of ISSAC'12, pp. 51–58, 2012.

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