To run this notebook the following packages are required, which can be obtained form the website
http://www.risc.jku.at/research/combinat/software/

```
<< Guess.m;
<< HolonomicFunctions.m;
<< Hyper.m;
```

```
Guess  Package by Manuel Kauers — © RISC Linz — V 0.41 2011-11-22
```

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.5.1 (09.08.2011)
⟶ Type ?HolonomicFunctions for help

Theorem 1

```
a[i_Integer, j_Integer] := KroneckerDelta[i, j] + Binomial[i + j - 2 + mu, j]
```

■ Guess the implicit description for $c_{n,j}$

We first compute some data, i.e. the values of $c_{n,j}$ for $1 \leq j \leq 2\, n \leq 30$.

```
Timing[
 dim = 30; data = {};
 Do[
  matrix = Table[a[i, j], {i, n - 1}, {j, n}];
  ns = LinSolveUniv[matrix, mu];
  If[Length[ns] =!= 1, Throw["NullSpace is not of Length 1."]];
  vec = First[ns];
  vec = Together[vec / Last[vec]];
  AppendTo[data, PadRight[vec, dim]];
  , {n, 2, dim, 2}];
]
```

```
{51.8912, Null}
```

```
Dimensions[data]
```

```
{15, 30}
```

```
TableForm[Take[Take[#, 4] & /@ data, 4]]
```

$-\frac{mu}{2}$	1	0	0
$\frac{1}{12}\left(mu + mu^2\right)$	$\frac{1}{12}\left(mu + mu^2\right)$	$\frac{1}{2}\left(-2 - mu\right)$	1
$\frac{-24\,mu-35\,mu^2-12\,mu^3-mu^4}{60\,(9+mu)}$	$\frac{-24\,mu-35\,mu^2-12\,mu^3-mu^4}{60\,(9+mu)}$	$\frac{360+294\,mu+49\,mu^2-6\,mu^3-mu^4}{120\,(9+mu)}$	$\frac{60+53\,mu+14\,mu^2+mu^3}{10\,(9+mu)}$
$\frac{150\,mu+245\,mu^2+113\,mu^3+19\,mu^4+mu^5}{336\,(13+mu)}$	$\frac{150\,mu+245\,mu^2+113\,mu^3+19\,mu^4+mu^5}{336\,(13+mu)}$	$\frac{-1680-1586\,mu-375\,mu^2+25\,mu^3+15\,mu^4+mu^5}{560\,(13+mu)}$	$\frac{-10\,080-10\,266\,mu-3475\,mu^2-41}{1680\,(13+mu)}$

Now we can guess some recurrences for $c_{n,j}$.

```
rec1 = GuessMultRE[data, {f[n, j], f[n, j + 1], f[n + 1, j]},
  {n, j}, 9,  StartPoint → {1, 1}, Constraints → j ≤ 2 n]
```

unlucky prime discarded.

$$
\begin{aligned}
\Big\{ & \frac{1}{1024} (1+j) (-1+j+mu+2n) \\
& \big(2 j^2 + 3 j^3 - 4 j^4 - 3 j^5 + 2 j^6 - 2 j\, mu + 10 j^2 mu + 3 j^3 mu - 11 j^4 mu + 2 j^6 mu + j\, mu^2 + 8 j^2 mu^2 - 3 j^3 mu^2 - \\
& 6 j^4 mu^2 + 3 j^5 mu^2 + 3 j\, mu^3 - 2 j^2 mu^3 - 3 j^3 mu^3 + j^4 mu^3 - j\, mu^4 - 2 j^2 mu^4 - j\, mu^5 - 12 j\, n + 20 j^2 n + \\
& 42 j^3 n - 10 j^4 n - 24 j^5 n + 8 j^6 n - 27 j\, mu\, n + 58 j^2 mu\, n + 59 j^3 mu\, n - 46 j^4 mu\, n - 3 mu^2 n + \\
& 13 j\, mu^2 n - 4 j^2 mu^2 n + 16 j^3 mu^2 n - 20 j^4 mu^2 n - mu^3 n - 6 j\, mu^3 n - 16 j^2 mu^3 n - 9 j^3 mu^3 n + \\
& 3 mu^4 n - 15 j\, mu^4 n - 2 j^2 mu^4 n + mu^5 n - j\, mu^5 n - 84 j\, n^2 + 18 j^2 n^2 + 138 j^3 n^2 + 36 j^4 n^2 - \\
& 48 j^5 n^2 - 6 mu\, n^2 - 55 j\, mu\, n^2 - 141 j^2 mu\, n^2 + 254 j^3 mu\, n^2 - 84 j^4 mu\, n^2 - 11 mu^2 n^2 + 4 j\, mu^2 n^2 - \\
& 150 j^2 mu^2 n^2 + 28 j^3 mu^2 n^2 + 41 mu^3 n^2 - 91 j\, mu^3 n^2 - 3 j^2 mu^3 n^2 + 23 mu^4 n^2 - 14 j\, mu^4 n^2 + mu^5 n^2 - \\
& 132 j\, n^3 - 336 j^2 n^3 + 216 j^3 n^3 + 48 j^4 n^3 - 18 mu\, n^3 + 216 j\, mu\, n^3 - 884 j^2 mu\, n^3 + 312 j^3 mu\, n^3 + \\
& 172 mu^2 n^3 - 128 j\, mu^2 n^3 - 112 j^2 mu^2 n^3 + 186 mu^3 n^3 - 88 j\, mu^3 n^3 + 20 mu^4 n^3 + 408 j\, n^4 - 1080 j^2 n^4 + \\
& 216 j^3 n^4 + 204 mu\, n^4 + 684 j\, mu\, n^4 - 756 j^2 mu\, n^4 + 612 mu^2 n^4 - 144 j\, mu^2 n^4 + 144 mu^3 n^4 + \\
& 1296 j\, n^5 - 864 j^2 n^5 + 648 mu\, n^5 + 432 j\, mu\, n^5 + 432 mu^2 n^5 + 864 j\, n^6 + 432 mu\, n^6\big)\, f[n, j] - \\
& \frac{1}{1024} j (-1+j+mu) (-1+2j+mu) (1+j-2n) (1+mu+4n) \\
& \big(2 j - j^2 - 2 j^3 + j^4 + 4 mu + j\, mu - 5 j^2 mu + 2 j^3 mu + mu^2 - 3 j\, mu^2 + j^2 mu^2 - \\
& mu^3 + 12 n + 6 j\, n - 6 j^2 n + 10 mu\, n - 6 j^2 mu\, n - 3 mu^2 n - 6 j\, mu^2 n - mu^3 n + \\
& 24 n^2 + 12 j\, n^2 - 12 j^2 n^2 - 4 mu\, n^2 - 12 j\, mu\, n^2 - 2 mu^2 n^2\big)\, f[n, 1+j] + \\
& \frac{1}{512} (1+j) (2j+mu) (-2+j-2n) (-1+j-2n) (1+n) (1+2n) (1+j+mu+2n) \\
& (-1+mu+4n) (1+mu+4n)\, f[1+n, j] \Big\}
\end{aligned}
$$

```
rec2 = GuessMultRE[data, {f[n, j], f[n, j + 1], f[n, j + 2]},
  {n, j}, 6,  StartPoint → {1, 1}, Constraints → j ≤ 2 n]
```

unlucky prime discarded.

$$
\begin{aligned}
\Big\{ & -\frac{1}{8} (1+j) (2+j) (j+mu) (2+2j+mu) (j-2n) (-1+j+mu+2n)\, f[n, j] + \\
& \frac{1}{8} (2+j) (-1+j+mu) \big(4 j^2 + 8 j^3 + 4 j^4 + 5 j\, mu + 13 j^2 mu + 8 j^3 mu + mu^2 + 6 j\, mu^2 + 5 j^2 mu^2 + mu^3 + j\, mu^3 + \\
& 4 j\, n + 4 j^2 n - 4 j^2 mu\, n + mu^2 n - 4 j\, mu^2 n - mu^3 n - 8 j\, n^2 - 8 j^2 n^2 - 8 j\, mu\, n^2 - 2 mu^2 n^2\big)\, f[n, 1+j] - \\
& \frac{1}{8} (1+j) (-1+j+mu) (j+mu) (2j+mu) (2+j-2n) (1+j+mu+2n)\, f[n, 2+j] \Big\}
\end{aligned}
$$

Now we transform these recurrences into operators. The Gröbner basis computation serves as a check that the two recurrences are consistent, and to ensure that the generators of the annihilating ideal for $c_{n,j}$ indeed constitute a Gröbner basis.

```
cnj = OreGroebnerBasis[ToOrePolynomial[Join[rec1, rec2], f[n, j]]];
```

```
Support[cnj]
```

$$\{\{S_n, S_j, 1\}, \{S_j^2, S_j, 1\}\}$$

```
LeadingCoefficient /@ Factor[cnj]
```

$$
\begin{aligned}
\{ & 2 (1+j) (2j+mu) (-2+j-2n) (-1+j-2n) (1+n) (1+2n) (1+j+mu+2n) (-1+mu+4n) \\
& (1+mu+4n), -(1+j) (-1+j+mu) (j+mu) (2j+mu) (2+j-2n) (1+j+mu+2n) \}
\end{aligned}
$$

```
UnderTheStaircase[cnj]
```

```
{1, Sj}
```

```
AnnihilatorSingularities[cnj, {1, 1}, Assumptions → mu > 0]
```

```
{{{j → 2 n}, n ≥ 1 && mu > 0}, {{j → 1, n → 1}, mu > 0}, {{j → 2, n → 1}, mu > 0},
 {{j → 1, mu → 1, n → 1}, True}, {{j → 1, mu → 2, n → 1}, True}, {{j → 2, mu → 1, n → 1}, True}}
```

Proof of (1a)

We show that $c_{n,2n} = 1$.

```
Timing[diag = First[FindRelation[cnj, Pattern → ({a_, b_} /; 2 * a == b)]];]
```

```
{13.5128, Null}
```

```
Support[diag]
```

$\{S_n^2 S_j^4, S_n S_j^2, 1\}$

```
lcf = Factor[LeadingCoefficient[diag] /. j → 2 n]
```

$8 (1 + n) (2 + n) (1 + 2 n) (3 + 2 n) (-1 + mu + 2 n) (1 + mu + 2 n) (2 + mu + 2 n) (-1 + mu + 4 n) (mu + 4 n) (1 + mu + 4 n) (2 + mu + 4 n) (3 + mu + 4 n) (4 + mu + 4 n) (5 + mu + 4 n)^2 (6 + mu + 4 n) (7 + mu + 4 n) \big(-3 mu^2 - 4 mu^3 + 2 mu^4 + 4 mu^5 + mu^6 - 6 n - 12 mu n + 3 mu^2 n + 69 mu^3 n + 83 mu^4 n + 23 mu^5 n - 29 n^2 + 184 mu n^2 + 640 mu^2 n^2 + 708 mu^3 n^2 + 225 mu^4 n^2 + 405 n^3 + 2271 mu n^3 + 3055 mu^2 n^3 + 1189 mu^3 n^3 + 2770 n^4 + 6596 mu n^4 + 3554 mu^2 n^4 + 5676 n^5 + 5676 mu n^5 + 3784 n^6\big)$

```
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1, lcf > 0], {n, mu}]
```

```
True
```

```
diag1 = OrePolynomialSubstitute[diag, {S[j] → 1, j → 2 * n}];
```

```
OreReduce[diag1, ToOrePolynomial[{S[n] - 1}]]
```

```
0
```

Proof of (2a)

We show that $\sum_{j=1}^{2n} c_{n,j}\, a_{i,j} = 0$ for all $1 \le i < 2n$.

```
Table[Together[FunctionExpand[Sum[data[[n, j]] * a[i, j], {j, 1, 2 n}]]], {n, 1, 6}, {i, 1, n - 1}]
```

```
{{}, {0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

```
aij0 = Annihilator[Binomial[i + j - 2 + mu, j], {S[i], S[n], S[j]}]
```

```
{(1 + j) Sj + (1 - i - j - mu), Sn - 1, (-1 + i + mu) Si + (1 - i - j - mu)}
```

```
cnj0 = ToOrePolynomial[Append[cnj, S[i] - 1], OreAlgebra[aij0]];
```

```
smnd2 = DFiniteTimes[aij0, cnj0];
```

```
Timing[sum2a = First[FindCreativeTelescoping[smnd2, S[j] - 1]];]
```

```
{542.618, Null}
```

```
Support[sum2a]
```

$\{\{S_i, S_n, 1\}, \{S_n^2, S_n, 1\}\}$

```
(* The Kronecker delta that was left is added here. *)
Timing[sum2 = DFinitePlus[sum2a, DFiniteSubstitute[cnj0, {j → i}]];]
```

{5.96837, Null}

```
ByteCount[sum2]
```

530 040

```
UnderTheStaircase[sum2]
```

$\{1, S_n\}$

```
Support[sum2]
```

$\{\{S_i, S_n, 1\}, \{S_n^2, S_n, 1\}\}$

```
Factor[LeadingCoefficient /@ sum2]
```

$\{i(-1+i+mu)(-1+2i+mu)(1+i-2n)(1+mu+4n)$
$(2i-i^2-2i^3+i^4+4mu+i\,mu-5i^2mu+2i^3mu+mu^2-3i\,mu^2+i^2mu^2-mu^3+12n+6in-6i^2n+$
$10mu\,n-6i^2mu\,n-3mu^2n-6i\,mu^2n-mu^3n+24n^2+12in^2-12i^2n^2-4mu\,n^2-12i\,mu\,n^2-2mu^2n^2),$
$4(-4+i-2n)(-3+i-2n)(2+n)(1+2n)(3+2n)(2+i+mu+2n)(3+i+mu+2n)$
$(-1+mu+4n)(1+mu+4n)^2(3+mu+4n)(5+mu+4n)$
$(2i-i^2-2i^3+i^4+4mu+i\,mu-5i^2mu+2i^3mu+mu^2-3i\,mu^2+i^2mu^2-mu^3+12n+6in-6i^2n+$
$10mu\,n-6i^2mu\,n-3mu^2n-6i\,mu^2n-mu^3n+24n^2+12in^2-12i^2n^2-4mu\,n^2-12i\,mu\,n^2-2mu^2n^2)\}$

```
lcf1 = LeadingCoefficient[sum2[[1]]] /. i → i - 1;
lcf2 = LeadingCoefficient[sum2[[2]]] /. n → n - 2;
```

```
(* This means that we can apply the first recurrence everywhere except i=1. *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1 && 2 ≤ i < n, lcf1 > 0], {i, n, mu}]
```

True

```
(* So we have to check what happens with the
  leading coefficient of the second recurrence for i=1. *)
Factor[lcf2 /. i → 1]
```

$-8(-1+n)n(-3+2n)(-1+2n)^2(-1+mu+2n)(mu+2n)(-9+mu+4n)(-7+mu+4n)^2(-5+mu+4n)$
$(-3+mu+4n)(-72+70mu-9mu^2-mu^3+84n-68mu\,n+mu^2n+mu^3n-24n^2+16mu\,n^2+2mu^2n^2)$

```
(* The linear factors do not introduce singularities,
so we have to look at the irrducible one. *)
irred = Last[%]
```

$-72+70mu-9mu^2-mu^3+84n-68mu\,n+mu^2n+mu^3n-24n^2+16mu\,n^2+2mu^2n^2$

```
CylindricalDecomposition[Implies[mu > 2 && n ≥ 3, irred > 0], {n, mu}]
```

True

Proof of (3a)

We show that $\sum_{j=1}^{2n} c_{n,j}\, a_{2n,j} = \frac{b_{2n}}{b_{2n-1}}$ for all $n \geq 1$.

```
anj0 = Annihilator[Binomial[2 * n + j - 2 + mu, j], {S[n], S[j]}]
```

$$\{(1+j)\, S_j + (1-j-mu-2\,n),\; \left(-mu+mu^2-2\,n+4\,mu\,n+4\,n^2\right) S_n + \left(j-j^2+mu-2\,j\,mu-mu^2+2\,n-4\,j\,n-4\,mu\,n-4\,n^2\right)\}$$

```
smnd3 = DFiniteTimes[anj0, cnj];
```

```
Timing[{sum3a, certificate} = FindCreativeTelescoping[smnd3, S[j] - 1];]
```

```
{164.422, Null}
```

```
(* We cross-check the correctness of the previous output. *)
Timing[OreReduce[sum3a[[1]] + (S[j] - 1) ** certificate[[1, 1]], Together[smnd3]]]
```

```
{19.5212, 0}
```

```
(* The Kronecker delta that was left is added here. *)
sum3 = DFinitePlus[sum3a, DFiniteSubstitute[cnj, {j → 2 n}]];
```

```
Support[sum3]
```

$$\{\{S_n^2,\, S_n,\, 1\}\}$$

```
Factor[sum3]
```

$$\{2\,(3+2\,n)\,(2+mu+2\,n)\,(3+mu+4\,n)\,(5+mu+4\,n)^2\,(7+mu+4\,n)\left(-3\,mu^2-4\,mu^3+2\,mu^4+4\,mu^5+mu^6-6\,n-12\,mu\,n+3\,mu^2\,n+69\,mu^3\,n+83\,mu^4\,n+23\,mu^5\,n-29\,n^2+184\,mu\,n^2+640\,mu^2\,n^2+708\,mu^3\,n^2+225\,mu^4\,n^2+405\,n^3+2271\,mu\,n^3+3055\,mu^2\,n^3+1189\,mu^3\,n^3+2770\,n^4+6596\,mu\,n^4+3554\,mu^2\,n^4+5676\,n^5+5676\,mu\,n^5+3784\,n^6\right) S_n^2 +$$
$$\left(75\,600\,mu+245\,790\,mu^2+256\,569\,mu^3+1127\,mu^4-220\,452\,mu^5-214\,952\,mu^6-105\,930\,mu^7-31\,326\,mu^8-5748\,mu^9-638\,mu^{10}-39\,mu^{11}-mu^{12}+264\,600\,n+1\,679\,580\,mu\,n+2\,127\,777\,mu^2\,n-1\,552\,696\,mu^3\,n-5\,858\,822\,mu^4\,n-5\,981\,562\,mu^5\,n-3\,276\,180\,mu^6\,n-1\,084\,938\,mu^7\,n-222\,786\,mu^8\,n-27\,626\,mu^9\,n-1885\,mu^{10}\,n-54\,mu^{11}\,n+2\,992\,050\,n^2+5\,162\,811\,mu\,n^2-18\,264\,616\,mu^2\,n^2-58\,340\,939\,mu^3\,n^2-68\,211\,289\,mu^4\,n^2-43\,177\,617\,mu^5\,n^2-16\,380\,345\,mu^6\,n^2-3\,818\,217\,mu^7\,n^2-533\,597\,mu^8\,n^2-40\,822\,mu^9\,n^2-1307\,mu^{10}\,n^2+3\,476\,235\,n^3-69\,749\,684\,mu\,n^3-276\,283\,063\,mu^2\,n^3-407\,281\,008\,mu^3\,n^3-313\,447\,113\,mu^4\,n^3-140\,553\,684\,mu^5\,n^3-37\,961\,109\,mu^6\,n^3-6\,061\,144\,mu^7\,n^3-524\,534\,mu^8\,n^3-18\,864\,mu^9\,n^3-86\,899\,955\,n^4-628\,321\,309\,mu\,n^4-1\,342\,293\,424\,mu^2\,n^4-1\,351\,083\,609\,mu^3\,n^4-748\,533\,594\,mu^4\,n^4-241\,122\,375\,mu^5\,n^4-44\,859\,680\,mu^6\,n^4-4\,451\,395\,mu^7\,n^4-181\,475\,mu^8\,n^4-552\,867\,975\,n^5-2\,316\,171\,930\,mu\,n^5-3\,454\,045\,011\,mu^2\,n^5-2\,530\,584\,042\,mu^3\,n^5-1\,014\,047\,637\,mu^4\,n^5-226\,059\,742\,mu^5\,n^5-26\,222\,801\,mu^6\,n^5-1\,228\,510\,mu^7\,n^5-1\,636\,346\,755\,n^6-4\,846\,872\,384\,mu\,n^6-5\,299\,337\,727\,mu^2\,n^6-2\,821\,937\,184\,mu^3\,n^6-785\,417\,129\,mu^4\,n^6-109\,474\,336\,mu^5\,n^6-6\,008\,213\,mu^6\,n^6-2\,879\,771\,580\,n^7-6\,281\,845\,116\,mu\,n^7-5\,008\,600\,008\,mu^2\,n^7-1\,857\,458\,728\,mu^3\,n^7-323\,963\,708\,mu^4\,n^7-21\,404\,572\,mu^5\,n^7-3\,226\,518\,300\,n^8-5\,143\,228\,176\,mu\,n^8-2\,861\,133\,656\,mu^2\,n^8-666\,044\,656\,mu^3\,n^8-55\,152\,716\,mu^4\,n^8-2\,327\,682\,000\,n^9-2\,591\,755\,840\,mu\,n^9-906\,087\,920\,mu^2\,n^9-100\,266\,880\,mu^3\,n^9-1\,048\,351\,120\,n^{10}-734\,110\,592\,mu\,n^{10}-122\,106\,032\,mu^2\,n^{10}-268\,361\,280\,n^{11}-89\,453\,760\,mu\,n^{11}-29\,817\,920\,n^{12}\right) S_n +$$
$$(mu+3\,n)\,(1+mu+3\,n)\,(2+mu+3\,n)\,(-1+mu+6\,n)\,(1+mu+6\,n)\,(3+mu+6\,n)\left(12\,600+14\,715\,mu+7249\,mu^2+1962\,mu^3+310\,mu^4+27\,mu^5+mu^6+63\,315\,n+61\,933\,mu\,n+24\,664\,mu^2\,n+5052\,mu^3\,n+533\,mu^4\,n+23\,mu^5\,n+131\,326\,n^2+103\,333\,mu\,n^2+31\,129\,mu^2\,n^2+4275\,mu^3\,n^2+225\,mu^4\,n^2+143\,925\,n^3+85\,415\,mu\,n^3+17\,271\,mu^2\,n^3+1189\,mu^3\,n^3+87\,910\,n^4+34\,976\,mu\,n^4+3554\,mu^2\,n^4+28\,380\,n^5+5676\,mu\,n^5+3784\,n^6\right)\}$$

We now verify that the given formula satisfies the recurrence.

```
quo[n_] := (-1)^((n - 1) * (n - 2) / 2) * 2^n * Pochhammer[(mu + 2 n) / 2, Floor[(n + 1) / 2]] *
   Pochhammer[(mu + 4 n + 1) / 2, n - 1] / Pochhammer[n, n] /
    Pochhammer[(-mu - 4 n + 3) / 2, Floor[(n - 1) / 2]];
```

```
(* A numerical test for the first values *)
With[{test = ApplyOreOperator[First[sum3], f[n]]},
 Together[Table[test, {n, 5}] /. f → quo]]
```

{0, 0, 0, 0, 0}

```
(* Symbolic simplification by case distinction n mod 4. *)
test = ApplyOreOperator[First[sum3], f[n]];
Table[FullSimplify[
  test /. n → 4 k + i /. f[kk_] :→ FullSimplify[quo[kk], Element[k, Integers]]], {i, 0, 3}]
```

{0, 0, 0, 0}

We can even find the closed form for the quotient:

```
Hyper[ApplyOreOperator[First[rec], f[n]], f[n], Solutions → All]
```

```
Warning: irreducible factors of degree > 1 in leading
                   coefficient;
some solutions may not be found

Warning: irreducible factors of degree > 1 in trailing
                   coefficient;
some solutions may not be found
```

$$\left\{1, \frac{(mu+3n)(1+mu+3n)(2+mu+3n)(-1+mu+6n)(1+mu+6n)(3+mu+6n)}{2(1+2n)(mu+2n)(-1+mu+4n)(1+mu+4n)^2(3+mu+4n)}\right\}$$

Theorems 2, 3, and 4

```
a[i_Integer, j_Integer, mu_] :=
  Together[-KroneckerDelta[i, j] + FunctionExpand[Binomial[mu + i + j - 2, j]]];
myDet01[0] = 1;
myDet01[n_] := myDet01[n] = Together[Det[Table[-KroneckerDelta[i - 1, j] +
       FunctionExpand[Binomial[(i - 1) + j + mu - 2, j]], {i, n}, {j, n}]]];
myDet10[0] = 1;
myDet10[n_] := myDet10[n] = Together[Det[Table[-KroneckerDelta[i, j - 1] +
       FunctionExpand[Binomial[i + (j - 1) + mu - 2, j - 1]], {i, n}, {j, n}]]];
```

■ $b_{2n-1}(0, 1)$

Here are the first values for $c_{n,j}$:

```
cdata = {{1, 0, 0, 0, 0, 0}, {(mu * (1 + mu)) / 12, (-1 - mu) / 2, 1, 0, 0, 0},
   {- (mu * (1 + mu) * (2 + mu) * (3 + mu)) / (120 * (6 + mu)), - (mu * (1 + mu) * (2 + mu) * (3 + mu)) /
     (120 * (6 + mu)), ((2 + mu) * (3 + mu) * (5 + mu)) / (10 * (6 + mu)), (-3 - mu) / 2, 1, 0}};
TraditionalForm[TableForm[Table[c[n, j] == cdata[[n, j]], {n, 3}, {j, 2 n - 1}]]]
```

$c(1, 1) = 1$

$c(2, 1) = \frac{1}{12} \text{mu} (\text{mu} + 1)$ $\quad c(2, 2) = \frac{1}{2} (-\text{mu} - 1)$ $\quad c(2, 3) = 1$

$c(3, 1) = -\frac{\text{mu} (\text{mu}+1) (\text{mu}+2) (\text{mu}+3)}{120 (\text{mu}+6)}$ $\quad c(3, 2) = -\frac{\text{mu} (\text{mu}+1) (\text{mu}+2) (\text{mu}+3)}{120 (\text{mu}+6)}$ $\quad c(3, 3) = \frac{(\text{mu}+2) (\text{mu}+3) (\text{mu}+5)}{10 (\text{mu}+6)}$ $\quad c(3, 4) = \frac{1}{2} (-\text{mu} - 3)$ $\quad c(3,$

```
(* Test (2a) *)
Together[Table[Sum[cdata[[n, j]] * a[i - 1, j, mu], {j, 1, 2 n - 1}], {n, 1, 3}, {i, 1, 2 n - 2}]]
```

{{}, {0, 0}, {0, 0, 0, 0}}

```
(* Test (3a) *)
Together[Table[Sum[cdata[[n, j]] * a[(2 n - 1) - 1, j, mu], {j, 1, 2 n - 1}] -
   myDet01[2 n - 1] / myDet01[2 n - 2], {n, 1, 3}]]
```

```
{0, 0, 0}
```

We load the guessed recurrences for $c_{n,j}$:

```
cnj = << "gb01odd.m";
```

```
LeadingCoefficient /@ Factor[cnj]
```

```
{2 (1 + 2 j + mu) (3 + j^2 + mu + j mu) (-1 + j - 2 n) (j - 2 n) n (1 + 2 n) (j + mu + 2 n) (-4 + mu + 4 n)
   (-2 + mu + 4 n), (-1 + j + mu) (-1 + 2 j + mu) (3 + j^2 + mu + j mu) (4 + j - 2 n) (j + mu + 2 n)}
```

```
UnderTheStaircase[cnj]
```

```
{1, S_j}
```

```
(* We cannot apply these recurrences if (n,j) = (1,1), (1,2), (2,2) or if j=2n-2. *)
AnnihilatorSingularities[cnj, {1, 1}, Assumptions → mu > 0]
```

```
{{{j → -2 + 2 n}, n ≥ 2 && mu > 0}, {{j → 1, n → 1}, mu > 0},
 {{j → 2, n → 1}, mu > 0}, {{j → 2, n → 2}, mu > 0}, {{j → 1, mu → 2, n → 1}, True},
 {{j → 1, mu → 3, n → 1}, True}, {{j → 1, mu → 4, n → 1}, True}, {{j → 2, mu → 1, n → 1}, True},
 {{j → 2, mu → 1, n → 2}, True}, {{j → 2, mu → 2, n → 1}, True}, {{j → 2, mu → 4, n → 1}, True}}
```

- **Proof of (1a)**

We show that $c_{n,2n-1} = 1$.

```
Timing[diag = First[FindRelation[cnj, Pattern → ({a_, b_} /; 2 * a == b)]];]
```

```
{17.4331, Null}
```

```
Support[diag]
```

```
{S_n^2 S_j^4, S_n S_j^2, 1}
```

```
lcf = Factor[LeadingCoefficient[diag] /. j → 2 n - 2]
```

```
8 n (1 + n) (1 + 2 n) (3 + 2 n) (-3 + mu + 2 n) (-2 + mu + 2 n)
  (-1 + mu + 2 n) (-5 + mu + 4 n) (-4 + mu + 4 n) (-3 + mu + 4 n) (-2 + mu + 4 n)
  (-1 + mu + 4 n) (mu + 4 n) (1 + mu + 4 n) (2 + mu + 4 n)^2 (3 + mu + 4 n) (4 + mu + 4 n)
  (-80 mu + 196 mu^2 - 176 mu^3 + 73 mu^4 - 14 mu^5 + mu^6 - 416 n + 1672 mu n - 2122 mu^2 n + 1143 mu^3 n - 270 mu^4 n +
    23 mu^5 n + 3520 n^2 - 8584 mu n^2 + 6802 mu^2 n^2 - 2120 mu^3 n^2 + 225 mu^4 n^2 - 11 528 n^3 + 18 092 mu n^3 -
    8414 mu^2 n^3 + 1189 mu^3 n^3 + 18 040 n^4 - 16 776 mu n^4 + 3554 mu^2 n^4 - 13 400 n^5 + 5676 mu n^5 + 3784 n^6)
```

```
(* Thus we can use the diagonal recurrence to compute the values for c(n,2n-2). *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1, lcf > 0], {n, mu}]
```

```
True
```

```
diag1 = OrePolynomialSubstitute[diag, {S[j] → 1, j → 2 n - 1}];
```

```
OreReduce[diag1, ToOrePolynomial[{S[n] - 1}]]
```

```
0
```

```
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1, LeadingCoefficient[diag1] > 0], {n, mu}]
```

```
True
```

Proof of (2a)

We show that $\sum_{j=1}^{2n-1} c_{n,j}\, a_{i,j} = 0$ for all $1 \le i < 2n - 1$.

```
aij0 = Annihilator[Binomial[(i - 1) + j - 2 + mu, j], {S[i], S[n], S[j]}]
```

```
{(1 + j) S_j + (2 - i - j - mu), S_n - 1, (-2 + i + mu) S_i + (2 - i - j - mu)}
```

```
cnj0 = ToOrePolynomial[Append[cnj, S[i] - 1], OreAlgebra[aij0]];
```

```
smnd2 = DFiniteTimes[aij0, cnj0];
```

```
Timing[sum2a = First[FindCreativeTelescoping[smnd2, S[j] - 1]];]
```

```
{973.469, Null}
```

```
(* The Kronecker delta that was left is added here. *)
Timing[sum2 = DFinitePlus[sum2a, DFiniteSubstitute[cnj0, {j → i - 1}]];]
```

```
{7.14445, Null}
```

```
ByteCount[sum2]
```

```
551 840
```

```
Support[sum2]
```

```
{{S_i, S_n, 1}, {S_n^2, S_n, 1}}
```

```
Factor[LeadingCoefficient /@ sum2]
```

```
{(-1 + i) (-3 + i + mu) (-2 + i + mu) (-3 + 2 i + mu) (2 + i - 2 n)
   (-2 + mu + 4 n) (-8 + 6 i + 7 i^2 - 6 i^3 + i^4 + 10 mu + i mu - 7 i^2 mu + 2 i^3 mu - 2 mu^2 -
     i mu^2 + i^2 mu^2 + 36 n - 36 i n + 12 i^2 n - 32 mu n + 30 i mu n - 6 i^2 mu n + 9 mu^2 n -
     6 i mu^2 n - mu^3 n - 36 n^2 + 36 i n^2 - 12 i^2 n^2 + 14 mu n^2 - 12 i mu n^2 - 2 mu^2 n^2),
  4 (-4 + i - 2 n) (-3 + i - 2 n) (1 + n) (1 + 2 n) (3 + 2 n) (i + mu + 2 n) (1 + i + mu + 2 n)
   (-4 + mu + 4 n) (-2 + mu + 4 n)^2 (mu + 4 n) (2 + mu + 4 n)
   (-8 + 6 i + 7 i^2 - 6 i^3 + i^4 + 10 mu + i mu - 7 i^2 mu + 2 i^3 mu - 2 mu^2 - i mu^2 +
     i^2 mu^2 + 36 n - 36 i n + 12 i^2 n - 32 mu n + 30 i mu n - 6 i^2 mu n + 9 mu^2 n -
     6 i mu^2 n - mu^3 n - 36 n^2 + 36 i n^2 - 12 i^2 n^2 + 14 mu n^2 - 12 i mu n^2 - 2 mu^2 n^2)}
```

```
lcf1 = LeadingCoefficient[sum2[[1]]] /. i → i - 1;
lcf2 = LeadingCoefficient[sum2[[2]]] /. n → n - 2;
```

```
(* This means that we can apply the first recurrence for i ≥ 3. *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1 && 3 ≤ i ≤ 2 n - 2, lcf1 > 0], {i, n, mu}]
```

```
True
```

```
(* So we have to check what happens with the
  leading coefficient of the second recurrence for i=1,2. *)
Table[CylindricalDecomposition[Implies[mu > 2 && n ≥ 3, (lcf2 /. i → ii) < 0], {n, mu}],
 {ii, 1, 2}]
```

```
{True, True}
```

- **Proof of (3a)**

We show that $\sum_{j=1}^{2n-1} c_{n,j}\, a_{2n-1,j} = \frac{b_{2n-1}(0,1)}{b_{2n-2}(0,1)}$ for all $n \ge 1$.

```
anj0 = Annihilator[Binomial[((2 n - 1) - 1) + j - 2 + mu, j], {S[n], S[j]}]
```

$$\{(1+j)\,S_j + (3-j-mu-2\,n),\ \left(6-5\,mu+mu^2-10\,n+4\,mu\,n+4\,n^2\right)S_n + \left(-6+5\,j-j^2+5\,mu-2\,j\,mu-mu^2+10\,n-4\,j\,n-4\,mu\,n-4\,n^2\right)\}$$

```
smnd3 = DFiniteTimes[anj0, cnj];
```

```
Timing[{sum3a, certificate} = FindCreativeTelescoping[smnd3, S[j] - 1];]
```

```
{341.417, Null}
```

```
(* We cross-check the correctness of the previous output. *)
Timing[OreReduce[sum3a[[1]] + (S[j] - 1) ** certificate[[1, 1]], Together[smnd3]]]
```

```
{30.5939, 0}
```

```
(* The Kronecker delta that was left is added here. *)
sum3 = DFinitePlus[sum3a, DFiniteSubstitute[cnj, {j → 2 n - 2}]];
```

```
Support[sum3]
```

$$\{S_n^2,\ S_n,\ 1\}$$

```
Factor[rec01odd = First[sum3]]
```

$$2\,(3+2\,n)\,(-3+mu+2\,n)\,(-1+mu+2\,n)\,(mu+4\,n)\,(2+mu+4\,n)^2\,(4+mu+4\,n)\,\big(-80\,mu+196\,mu^2-176\,mu^3+73\,mu^4-14\,mu^5+mu^6-416\,n+1672\,mu\,n-2122\,mu^2\,n+1143\,mu^3\,n-270\,mu^4\,n+23\,mu^5\,n+3520\,n^2-8584\,mu\,n^2+6802\,mu^2\,n^2-2120\,mu^3\,n^2+225\,mu^4\,n^2-11\,528\,n^3+18\,092\,mu\,n^3-8414\,mu^2\,n^3+1189\,mu^3\,n^3+18\,040\,n^4-16\,776\,mu\,n^4+3554\,mu^2\,n^4-13\,400\,n^5+5676\,mu\,n^5+3784\,n^6\big)\,S_n^2 -$$
$$(-3+mu+2\,n)\,\big(-7680\,mu^2+768\,mu^3+12\,256\,mu^4-1200\,mu^5-5440\,mu^6+504\,mu^7+918\,mu^8-75\,mu^9-55\,mu^{10}+3\,mu^{11}+mu^{12}-66\,048\,mu\,n-1280\,mu^2\,n+185\,760\,mu^3\,n-6640\,mu^4\,n-126\,720\,mu^5\,n+7152\,mu^6\,n+29\,526\,mu^7\,n-1765\,mu^8\,n-2322\,mu^9\,n+103\,mu^{10}\,n+54\,mu^{11}\,n-132\,096\,n^2-51\,712\,mu\,n^2+1\,051\,328\,mu^2\,n^2+55\,392\,mu^3\,n^2-1\,224\,872\,mu^4\,n^2+18\,468\,mu^5\,n^2+411\,846\,mu^6\,n^2-15\,407\,mu^7\,n^2-43\,550\,mu^8\,n^2+1548\,mu^9\,n^2+1307\,mu^{10}\,n^2-131\,072\,n^3+2\,653\,696\,mu\,n^3+557\,312\,mu^2\,n^3-6\,297\,576\,mu^3\,n^3-235\,332\,mu^4\,n^3+3\,260\,418\,mu^5\,n^3-41\,659\,mu^6\,n^3-479\,224\,mu^7\,n^3+13\,030\,mu^8\,n^3+18\,864\,mu^9\,n^3+2\,522\,624\,n^4+1\,655\,040\,mu\,n^4-18\,172\,496\,mu^2\,n^4-2\,078\,952\,mu^3\,n^4+16\,047\,984\,mu^4\,n^4+268\,486\,mu^5\,n^4-3\,432\,683\,mu^6\,n^4+63\,227\,mu^7\,n^4+181\,475\,mu^8\,n^4+1\,703\,936\,n^5-27\,909\,984\,mu\,n^5-7\,152\,624\,mu^2\,n^5+50\,349\,720\,mu^3\,n^5+2\,811\,980\,mu^4\,n^5-16\,744\,174\,mu^5\,n^5+137\,371\,mu^6\,n^5+1\,228\,510\,mu^7\,n^5-17\,820\,224\,n^6-11\,846\,304\,mu\,n^6+98\,410\,224\,mu^2\,n^6+11\,347\,240\,mu^3\,n^6-56\,369\,828\,mu^4\,n^6-279\,130\,mu^5\,n^6+6\,008\,213\,mu^6\,n^6-7\,864\,320\,n^7+109\,587\,840\,mu\,n^7+24\,889\,472\,mu^2\,n^7-129\,384\,400\,mu^3\,n^7-3\,068\,648\,mu^4\,n^7+21\,404\,572\,mu^5\,n^7+53\,221\,056\,n^8+29\,353\,024\,mu\,n^8-193\,818\,896\,mu^2\,n^8-10\,468\,000\,mu^3\,n^8+55\,152\,716\,mu^4\,n^8+14\,680\,064\,n^9-171\,120\,352\,mu\,n^9-19\,382\,672\,mu^2\,n^9+100\,266\,880\,mu^3\,n^9-67\,609\,280\,n^{10}-19\,497\,120\,mu\,n^{10}+122\,106\,032\,mu^2\,n^{10}-8\,388\,608\,n^{11}+89\,453\,760\,mu\,n^{11}+29\,817\,920\,n^{12}\big)\,S_n +$$
$$(-1+mu+2\,n)\,(-3+mu+3\,n)\,(-2+mu+3\,n)\,(-1+mu+3\,n)\,(-4+mu+6\,n)\,(-2+mu+6\,n)\,(mu+6\,n)\,\big(16\,mu^2+36\,mu^3+28\,mu^4+9\,mu^5+mu^6-96\,n+56\,mu\,n+456\,mu^2\,n+470\,mu^3\,n+180\,mu^4\,n+23\,mu^5\,n-64\,n^2+1796\,mu\,n^2+2884\,mu^2\,n^2+1447\,mu^3\,n^2+225\,mu^4\,n^2+2312\,n^3+7748\,mu\,n^3+5802\,mu^2\,n^3+1189\,mu^3\,n^3+7800\,n^4+11\,604\,mu\,n^4+3554\,mu^2\,n^4+9304\,n^5+5676\,mu\,n^5+3784\,n^6\big)$$

```
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1, LeadingCoefficient[rec01odd] > 0], {n, mu}]
```

```
True
```

We can even find the closed form for the quotient:

```
Hyper[ApplyOreOperator[rec01odd, f[n]], f[n], Solutions → All]
```

```
Warning: irreducible factors of degree > 1 in leading
                   coefficient;
some solutions may not be found

Warning: irreducible factors of degree > 1 in trailing
                   coefficient;
some solutions may not be found
```

$$\left\{\frac{-1+mu+2n}{-3+mu+2n},\ \frac{(-3+mu+3n)(-2+mu+3n)(-1+mu+3n)(-4+mu+6n)(-2+mu+6n)(mu+6n)}{2(1+2n)(-3+mu+2n)(-4+mu+4n)(-2+mu+4n)^2(mu+4n)}\right\}$$

```
basis = RSolve[{y[n + 1] == # * y[n], y[0] == 1}, y[n], n][[1, 1, 2]] & /@ %
```

$$\Big\{\frac{-3+mu+2n}{-3+mu},$$
$$\Big(2^{7-8n}\,729^{-1+n}(-1+mu)\,\text{Pochhammer}\Big[1+\frac{1}{6}(-4+mu),-1+n\Big]\,\text{Pochhammer}\Big[1+\frac{1}{6}(-2+mu),-1+n\Big]$$
$$\text{Pochhammer}\Big[1+\frac{1}{3}(-2+mu),-1+n\Big]\,\text{Pochhammer}\Big[1+\frac{1}{3}(-1+mu),-1+n\Big]$$
$$\text{Pochhammer}\Big[1+\frac{mu}{6},-1+n\Big]\,\text{Pochhammer}\Big[\frac{mu}{3},-1+n\Big]\Big)\Big/$$
$$\Big(\text{Pochhammer}\Big[\frac{3}{2},-1+n\Big]\,\text{Pochhammer}\Big[1+\frac{1}{2}(-3+mu),-1+n\Big]\,\text{Pochhammer}\Big[1+\frac{1}{4}(-2+mu),-1+n\Big]^2$$
$$\text{Pochhammer}\Big[1+\frac{mu}{4},-1+n\Big]\,\text{Pochhammer}\Big[\frac{mu}{4},-1+n\Big]\Big)\Big\}$$

```
inits = Factor[Table[Together[myDet01[2 n - 1] / myDet01[2 n - 2]], {n, 2}]]
```

$$\left\{-1+mu,\ \frac{1}{6}(1+mu)(6+mu)\right\}$$

```
sol01odd = Simplify[
  ({c1, c2} /. First[Solve[Thread[Table[basis.{c1, c2}, {n, 2}] == inits], {c1, c2}]]).basis]
```

$$\Big(2^{8-8n}\,729^{-1+n}(-1+mu)\,\text{Pochhammer}\Big[1+\frac{mu}{6},-1+n\Big]\,\text{Pochhammer}\Big[\frac{mu}{3},-1+n\Big]\,\text{Pochhammer}\Big[\frac{1+mu}{3},-1+n\Big]$$
$$\text{Pochhammer}\Big[\frac{2+mu}{6},-1+n\Big]\,\text{Pochhammer}\Big[\frac{2+mu}{3},-1+n\Big]\,\text{Pochhammer}\Big[\frac{4+mu}{6},-1+n\Big]\Big)\Big/$$
$$\Big(\text{Pochhammer}\Big[\frac{3}{2},-1+n\Big]\,\text{Pochhammer}\Big[1+\frac{mu}{4},-1+n\Big]\,\text{Pochhammer}\Big[\frac{1}{2}(-1+mu),-1+n\Big]$$
$$\text{Pochhammer}\Big[\frac{mu}{4},-1+n\Big]\,\text{Pochhammer}\Big[\frac{2+mu}{4},-1+n\Big]^2\Big)$$

```
FullSimplify[sol01odd]
```

$$\frac{2^{5-mu-4n}\,\pi\,\text{Gamma}\Big[-2+\frac{mu}{2}+3n\Big]\,\text{Gamma}[-3+mu+3n]}{\text{Gamma}\Big[\frac{1}{2}+n\Big]\,\text{Gamma}\Big[\frac{1}{2}(-3+mu)+n\Big]\,\text{Gamma}\Big[-2+\frac{mu}{2}+2n\Big]\,\text{Gamma}\Big[-1+\frac{mu}{2}+2n\Big]}$$

```
Table[Together[FunctionExpand[sol01odd] - myDet01[2 n - 1] / myDet[2 n - 2]], {n, 5}]
```

{0, 0, 0, 0, 0}

$b_{2n-1}(1, 0)$

```
cdata = {{1, 0, 0, 0, 0, 0}, {-mu, -mu / 2, 1, 0, 0, 0},
   {(mu * (2 + mu) * (5 + mu)) / (2 * (6 + mu)), (mu * (2 + mu) * (5 + mu)) / (4 * (6 + mu)),
    ((2 + mu) * (-24 + mu + mu^2)) / (12 * (6 + mu)), (-2 - mu) / 2, 1, 0}};
TraditionalForm[TableForm[Table[c[n, j] == cdata[[n, j]], {n, 3}, {j, 2 n - 1}]]]
```

$c(1, 1) = 1$

$c(2, 1) = -\text{mu}$ $\quad c(2, 2) = -\frac{\text{mu}}{2}$ $\quad c(2, 3) = 1$

$c(3, 1) = \frac{\text{mu}\,(\text{mu}+2)\,(\text{mu}+5)}{2\,(\text{mu}+6)}$ $\quad c(3, 2) = \frac{\text{mu}\,(\text{mu}+2)\,(\text{mu}+5)}{4\,(\text{mu}+6)}$ $\quad c(3, 3) = \frac{(\text{mu}+2)\left(\text{mu}^2+\text{mu}-24\right)}{12\,(\text{mu}+6)}$ $\quad c(3, 4) = \frac{1}{2}(-\text{mu} - 2)$ $\quad c(3, 5) = 1$

```
(* Test (2a) *)
Together[Table[Sum[cdata[[n, j]] * a[i, j - 1, mu], {j, 1, 2 n - 1}], {n, 1, 3}, {i, 1, 2 n - 2}]]
```

```
{{}, {0, 0}, {0, 0, 0, 0}}
```

```
(* Test (3a) *)
Together[Table[Sum[cdata[[n, j]] * a[2 n - 1, j - 1, mu], {j, 1, 2 n - 1}] -
   myDet10[2 n - 1] / myDet10[2 n - 2], {n, 1, 3}]]
```

```
{0, 0, 0}
```

We load the guessed recurrences for $c_{n,j}$:

```
cnj = << "gb10odd.m";
```

```
LeadingCoefficient /@ Factor[cnj]
```

```
{2 (-1 + j - 2 n) (j - 2 n) n (-1 + 2 n) (-2 + j + mu + 2 n) (-4 + mu + 4 n) (-2 + mu + 4 n),
 (-1 + j + mu) (2 + j - 2 n) (-2 + j + mu + 2 n)}
```

```
UnderTheStaircase[cnj]
```

$\{1, S_j\}$

```
(* We cannot apply these recurrences if (n,j) = (1,1), (1,2), (2,2). *)
AnnihilatorSingularities[cnj, {1, 1}, Assumptions → mu > 0]
```

```
{{{j → 1, n → 1}, mu > 0}, {{j → 2, n → 1}, mu > 0}, {{j → 2, n → 2}, mu > 0},
 {{j → 1, mu → 1, n → 1}, True}, {{j → 1, mu → 2, n → 1}, True},
 {{j → 1, mu → 4, n → 1}, True}, {{j → 2, mu → 1, n → 1}, True},
 {{j → 2, mu → 1, n → 2}, True}, {{j → 2, mu → 2, n → 1}, True}, {{j → 2, mu → 4, n → 1}, True}}
```

- **Proof of (1a)**

We show that $c_{n,2n-1} = 1$.

```
Timing[diag = First[FindRelation[cnj, Pattern → ({a_, b_} /; 2 * a == b)]];]
```

```
{5.87637, Null}
```

```
Support[diag]
```

$\{S_n^2 S_j^4, S_n S_j^2, 1\}$

```
diag1 = OrePolynomialSubstitute[diag, {S[j] → 1, j → 2 n - 1}];
```

```
OreReduce[diag1, ToOrePolynomial[{S[n] - 1}]]
```

```
0
```

```
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1, LeadingCoefficient[diag1] > 0], {n, mu}]
```

```
True
```

- **Proof of (2a)**

We show that $\sum_{j=1}^{2n-1} c_{n,j}\, a_{i,j} = 0$ for all $1 \le i < 2n-1$.

```
aij0 = Annihilator[Binomial[i + (j - 1) - 2 + mu, j - 1], {S[i], S[n], S[j]}]
```

```
{j Sj + (2 - i - j - mu), Sn - 1, (-1 + i + mu) Si + (2 - i - j - mu)}
```

```
cnj0 = ToOrePolynomial[Append[cnj, S[i] - 1], OreAlgebra[aij0]];
```

```
smnd2 = DFiniteTimes[aij0, cnj0];
```

```
Timing[sum2a = First[FindCreativeTelescoping[smnd2, S[j] - 1]];]
```

```
{257.904, Null}
```

```
(* The Kronecker delta that was left is added here. *)
Timing[sum2 = DFinitePlus[sum2a, DFiniteSubstitute[cnj0, {j → i + 1}]];]
```

```
{3.95625, Null}
```

```
ByteCount[sum2]
```

```
478 584
```

```
Support[sum2]
```

$\{\{S_i, S_n, 1\}, \{S_n^2, S_n, 1\}\}$

```
Factor[LeadingCoefficient /@ sum2]
```

$\{(-1 + i + mu)\,(2 + i - 2n)\,(-2 + mu + 4n)$
$\left(2i - i^2 - 2i^3 + i^4 - 3\,i\,mu - i^2\,mu + 2i^3\,mu + i\,mu^2 + i^2\,mu^2 - 12\,i\,n + 12\,i^2\,n - 2\,mu\,n + 18\,i\,mu\,n - 6\,i^2\,mu\,n + 3\,mu^2\,n - 6\,i\,mu^2\,n - mu^3\,n + 12\,i\,n^2 - 12\,i^2\,n^2 + 2\,mu\,n^2 - 12\,i\,mu\,n^2 - 2\,mu^2\,n^2\right),$
$4\,(-2 + i - 2n)\,(-1 + i - 2n)\,(1 + n)\,(-1 + 2n)\,(1 + 2n)\,(i + mu + 2n)\,(1 + i + mu + 2n)$
$(-4 + mu + 4n)\,(-2 + mu + 4n)^2\,(mu + 4n)\,(2 + mu + 4n)$
$\left(2i - i^2 - 2i^3 + i^4 - 3\,i\,mu - i^2\,mu + 2i^3\,mu + i\,mu^2 + i^2\,mu^2 - 12\,i\,n + 12\,i^2\,n - 2\,mu\,n + 18\,i\,mu\,n - 6\,i^2\,mu\,n + 3\,mu^2\,n - 6\,i\,mu^2\,n - mu^3\,n + 12\,i\,n^2 - 12\,i^2\,n^2 + 2\,mu\,n^2 - 12\,i\,mu\,n^2 - 2\,mu^2\,n^2\right)\}$

```
lcf1 = LeadingCoefficient[sum2[[1]]] /. i → i - 1;
lcf2 = LeadingCoefficient[sum2[[2]]] /. n → n - 2;
```

```
(* This means that we can apply the first recurrence for i ≥ 2. *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1 && 2 ≤ i ≤ 2 n - 2, lcf1 > 0], {i, n, mu}]
```

```
True
```

```
(* So we have to check what happens with the
  leading coefficient of the second recurrence for i=1. *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 3, (lcf2 /. i → 1) < 0], {n, mu}]
```

```
True
```

- **Proof of (3a)**

We show that $\sum_{j=1}^{2n-1} c_{n,j}\, a_{2n-1,j} = \frac{b_{2n-1}(1,0)}{b_{2n-2}(1,0)}$ for all $n \geq 1$.

```
anj0 = Annihilator[Binomial[(2 n - 1) + (j - 1) - 2 + mu, j - 1], {S[n], S[j]}]
```

$$\{j\, S_j + (3 - j - mu - 2\, n),\ (2 - 3\, mu + mu^2 - 6\, n + 4\, mu\, n + 4\, n^2)\, S_n + (-6 + 5\, j - j^2 + 5\, mu - 2\, j\, mu - mu^2 + 10\, n - 4\, j\, n - 4\, mu\, n - 4\, n^2)\}$$

```
smnd3 = DFiniteTimes[anj0, cnj];
```

```
Timing[{sum3a, certificate} = FindCreativeTelescoping[smnd3, S[j] - 1];]
```

```
{4.98031, Null}
```

```
(* We cross-check the correctness of the previous output. *)
Timing[OreReduce[sum3a[[1]] + (S[j] - 1) ** certificate[[1, 1]], Together[smnd3]]]
```

```
{0.684044, 0}
```

```
(* The Kronecker delta that was left is added here. *)
sum3 = DFinitePlus[sum3a, DFiniteSubstitute[cnj, {j → 2 n}]];
```

```
Support[sum3]
```

$$\{\{S_n^2,\ S_n,\ 1\}\}$$

```
Factor[rec10odd = First[sum3]]
```

$$2\ (1+n)\ (-1+2\ n)\ (1+2\ n)\ (mu+2\ n)\ (1+mu+2\ n)$$
$$(-4+mu+4\ n)\ (mu+4\ n)\ (1+mu+4\ n)\ (2+mu+4\ n)^2\ (3+mu+4\ n)\ (4+mu+4\ n)$$
$$\big(24\ mu-92\ mu^2+142\ mu^3-113\ mu^4+49\ mu^5-11\ mu^6+mu^7+96\ n-708\ mu\ n+1624\ mu^2\ n-$$
$$1721\ mu^3\ n+935\ mu^4\ n-253\ mu^5\ n+27\ mu^6\ n-1448\ n^2+6412\ mu\ n^2-10\,072\ mu^2\ n^2+7276\ mu^3\ n^2-$$
$$2465\ mu^4\ n^2+317\ mu^5\ n^2+8584\ n^3-26\,588\ mu\ n^3+28\,690\ mu^2\ n^3-12\,963\ mu^3\ n^3+$$
$$2089\ mu^4\ n^3-26\,480\ n^4+56\,984\ mu\ n^4-38\,626\ mu^2\ n^4+8310\ mu^3\ n^4+45\,400\ n^5-$$
$$61\,604\ mu\ n^5+19\,892\ mu^2\ n^5-41\,000\ n^6+26\,488\ mu\ n^6+15\,136\ n^7\big)\ S_n^2-$$
$$(-1+2\ n)\ (-4+mu+4\ n)\ \big(6912\ mu^3-576\ mu^4-19\,200\ mu^5+1360\ mu^6+19\,184\ mu^7-972\ mu^8-$$
$$8600\ mu^9+155\ mu^{10}+1896\ mu^{11}+47\ mu^{12}-200\ mu^{13}-15\ mu^{14}+8\ mu^{15}+mu^{16}+76\,032\ mu^2\ n+$$
$$5184\ mu^3\ n-351\,744\ mu^4\ n-7920\ mu^5\ n+495\,600\ mu^6\ n+8796\ mu^7\ n-289\,456\ mu^8\ n-10\,505\ mu^9\ n+$$
$$79\,064\ mu^{10}\ n+5496\ mu^{11}\ n-9951\ mu^{12}\ n-1125\ mu^{13}\ n+454\ mu^{14}\ n+74\ mu^{15}\ n+mu^{16}\ n+$$
$$290\,304\ mu\ n^2+105\,984\ mu^2\ n^2-2\,634\,048\ mu^3\ n^2-424\,272\ mu^4\ n^2+5\,497\,840\ mu^5\ n^2+601\,856\ mu^6\ n^2-$$
$$4\,313\,272\ mu^7\ n^2-447\,987\ mu^8\ n^2+1\,488\,990\ mu^9\ n^2+183\,536\ mu^{10}\ n^2-226\,538\ mu^{11}\ n^2-36\,021\ mu^{12}\ n^2+$$
$$11\,856\ mu^{13}\ n^2+2504\ mu^{14}\ n^2+68\ mu^{15}\ n^2+387\,072\ n^3+562\,176\ mu\ n^3-10\,016\,256\ mu^2\ n^3-$$
$$4\,185\,968\ mu^3\ n^3+33\,916\,736\ mu^4\ n^3+8\,250\,328\ mu^5\ n^3-37\,321\,976\ mu^6\ n^3-7\,168\,103\ mu^7\ n^3+$$
$$16\,714\,462\ mu^8\ n^3+3\,178\,360\ mu^9\ n^3-3\,125\,506\ mu^{10}\ n^3-674\,782\ mu^{11}\ n^3+188\,869\ mu^{12}\ n^3+$$
$$51\,717\ mu^{13}\ n^3+2135\ mu^{14}\ n^3+981\,504\ n^4-18\,964\,992\ mu\ n^4-19\,143\,520\ mu^2\ n^4+124\,883\,856\ mu^3\ n^4+$$
$$57\,249\,800\ mu^4\ n^4-206\,192\,524\ mu^5\ n^4-64\,697\,688\ mu^6\ n^4+124\,178\,358\ mu^7\ n^4+34\,329\,035\ mu^8\ n^4-$$
$$29\,152\,486\ mu^9\ n^4-8\,397\,262\ mu^{10}\ n^4+2\,049\,066\ mu^{11}\ n^4+732\,187\ mu^{12}\ n^4+41\,210\ mu^{13}\ n^4-$$
$$13\,957\,632\ n^5-42\,487\,104\ mu\ n^5+271\,765\,024\ mu^2\ n^5+231\,743\,184\ mu^3\ n^5-751\,533\,824\ mu^4\ n^5-$$
$$370\,923\,048\ mu^5\ n^5+640\,314\,190\ mu^6\ n^5+251\,187\,952\ mu^7\ n^5-193\,987\,461\ mu^8\ n^5-74\,091\,192\ mu^9\ n^5+$$
$$15\,990\,620\ mu^{10}\ n^5+7\,550\,240\ mu^{11}\ n^5+548\,443\ mu^{12}\ n^5-36\,740\,480\ n^6+319\,356\,608\ mu\ n^6+$$
$$555\,127\,392\ mu^2\ n^6-1\,797\,806\,352\ mu^3\ n^6-1\,408\,720\,656\ mu^4\ n^6+2\,333\,742\,812\ mu^5\ n^6+$$
$$1\,296\,007\,714\ mu^6\ n^6-946\,288\,252\ mu^7\ n^6-480\,498\,508\ mu^8\ n^6+92\,311\,672\ mu^9\ n^6+58\,739\,306\ mu^{10}\ n^6+$$
$$5\,343\,400\ mu^{11}\ n^6+153\,327\,488\ n^7+730\,527\,040\ mu\ n^7-2\,701\,763\,168\ mu^2\ n^7-3\,548\,287\,248\ mu^3\ n^7+$$
$$5\,994\,766\,376\ mu^4\ n^7+4\,789\,724\,356\ mu^5\ n^7-3\,420\,080\,530\ mu^6\ n^7-2\,334\,248\,801\ mu^7\ n^7+$$
$$398\,626\,329\ mu^8\ n^7+351\,989\,357\ mu^9\ n^7+39\,459\,089\ mu^{10}\ n^7+406\,859\,776\ n^8-2\,288\,708\,608\ mu\ n^8-$$
$$5\,711\,853\,504\ mu^2\ n^8+10\,591\,636\,128\ mu^3\ n^8+12\,642\,493\,760\ mu^4\ n^8-9\,134\,950\,296\ mu^5\ n^8-$$
$$8\,552\,754\,754\ mu^6\ n^8+1\,283\,544\,702\ mu^7\ n^8+1\,642\,855\,242\ mu^8\ n^8+225\,422\,866\ mu^9\ n^8-817\,146\,496\ n^9-$$
$$5\,322\,628\,736\ mu\ n^9+12\,183\,258\,592\ mu^2\ n^9+23\,300\,953\,952\ mu^3\ n^9-17\,745\,668\,504\ mu^4\ n^9-$$
$$23\,548\,634\,612\ mu^5\ n^9+3\,019\,484\,796\ mu^6\ n^9+5\,994\,451\,380\ mu^7\ n^9+1\,007\,198\,460\ mu^8\ n^9-$$
$$2\,181\,559\,040\ n^{10}+8\,133\,863\,488\ mu\ n^{10}+28\,494\,977\,952\ mu^2\ n^{10}-24\,247\,159\,680\ mu^3\ n^{10}-$$
$$47\,990\,978\,664\ mu^4\ n^{10}+4\,931\,222\,616\ mu^5\ n^{10}+17\,047\,152\,264\ mu^6\ n^{10}+3\,532\,097\,448\ mu^7\ n^{10}+$$
$$2\,350\,341\,248\ n^{11}+20\,759\,433\,792\ mu\ n^{11}-21\,901\,792\,736\ mu^2\ n^{11}-70\,212\,849\,200\ mu^3\ n^{11}+$$
$$4\,866\,625\,232\ mu^4\ n^{11}+37\,393\,733\,744\ mu^5\ n^{11}+9\,690\,620\,688\ mu^6\ n^{11}+6\,806\,575\,616\ n^{12}-$$
$$11\,582\,552\,960\ mu\ n^{12}-69\,703\,658\,080\ mu^2\ n^{12}+1\,270\,507\,936\ mu^3\ n^{12}+62\,032\,237\,664\ mu^4\ n^{12}+$$
$$20\,582\,373\,344\ mu^5\ n^{12}-2\,651\,983\,232\ n^{13}-42\,010\,087\,360\ mu\ n^{13}-3\,323\,557\,568\ mu^2\ n^{13}+$$
$$75\,238\,706\,624\ mu^3\ n^{13}+33\,176\,053\,312\ mu^4\ n^{13}-11\,587\,857\,280\ n^{14}-4\,234\,144\,640\ mu\ n^{14}+$$
$$62\,926\,532\,480\ mu^2\ n^{14}+39\,229\,247\,360\ mu^3\ n^{14}-1\,676\,446\,208\ n^{15}+32\,423\,155\,712\ mu\ n^{15}+$$
$$32\,089\,063\,936\ mu^2\ n^{15}+7\,752\,955\,904\ n^{16}+16\,220\,948\,480\ mu\ n^{16}+3\,816\,693\,760\ n^{17}\big)\ S_n+$$
$$n\ (1+2\ n)\ (-2+mu+2\ n)\ (-3+mu+3\ n)\ (-2+mu+3\ n)\ (-1+mu+3\ n)$$
$$(-3+mu+4\ n)$$
$$(-1+mu+4\ n)$$
$$(mu+4\ n)$$
$$(-4+mu+6\ n)$$
$$(-2+mu+6\ n)$$
$$(mu+6\ n)$$
$$\big(288+1008\ mu+1416\ mu^2+1044\ mu^3+446\ mu^4+113\ mu^5+16\ mu^6+mu^7+3984\ n+11\,196\ mu\ n+$$
$$12\,506\ mu^2\ n+7182\ mu^3\ n+2272\ mu^4\ n+381\ mu^5\ n+27\ mu^6\ n+22\,280\ n^2+49\,832\ mu\ n^2+$$
$$43\,162\ mu^2\ n^2+18\,247\ mu^3\ n^2+3802\ mu^4\ n^2+317\ mu^5\ n^2+66\,424\ n^3+115\,068\ mu\ n^3+$$
$$73\,106\ mu^2\ n^3+20\,277\ mu^3\ n^3+2089\ mu^4\ n^3+115\,280\ n^4+146\,284\ mu\ n^4+60\,834\ mu^2\ n^4+$$
$$8310\ mu^3\ n^4+117\,256\ n^5+97\,324\ mu\ n^5+19\,892\ mu^2\ n^5+64\,952\ n^6+26\,488\ mu\ n^6+15\,136\ n^7\big)$$

```
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1, LeadingCoefficient[rec10odd] > 0], {n, mu}]
```

```
True
```

We can even find the closed form for the quotient:

```
sol10odd = (64 / 27) ^ (1 - n) * Pochhammer[(mu + 4 n - 2) / 2, n - 1] *
  Pochhammer[mu / 3, n - 1] * Pochhammer[(mu + 1) / 3, n - 1] *
  Pochhammer[(mu + 2) / 3, n - 1] / (Pochhammer[1 / 2, n - 1] * Pochhammer[(mu + 1) / 2, n - 1] *
     Pochhammer[(mu + 2) / 4, n - 1] * Pochhammer[mu / 4, n - 1])
```

$$\left(\left(\frac{64}{27}\right)^{1-n} \text{Pochhammer}\left[\frac{\text{mu}}{3}, -1+n\right] \text{Pochhammer}\left[\frac{1+\text{mu}}{3}, -1+n\right] \text{Pochhammer}\left[\frac{2+\text{mu}}{3}, -1+n\right] \text{Pochhammer}\left[\frac{1}{2}(-2+\text{mu}+4n), -1+n\right]\right) \Big/ \left(\text{Pochhammer}\left[\frac{1}{2}, -1+n\right] \text{Pochhammer}\left[\frac{\text{mu}}{4}, -1+n\right] \text{Pochhammer}\left[\frac{1+\text{mu}}{2}, -1+n\right] \text{Pochhammer}\left[\frac{2+\text{mu}}{4}, -1+n\right]\right)$$

```
FullSimplify[ApplyOreOperator[rec10odd, sol10odd]]
```

0

```
OreReduce[rec10odd, Annihilator[sol10odd, S[n]]]
```

0

```
Table[Together[FunctionExpand[sol10odd] - myDet10[2 n - 1] / myDet10[2 n - 2]], {n, 5}]
```

{0, 0, 0, 0, 0}

- $b_{2n}(0, 1)$

Here are the first values for $c_{n,j}$:

```
cdata = {{-mu / 2, 1, 0, 0, 0, 0}, {0, ((1 + mu) * (2 + mu)) / 12, (-2 - mu) / 2, 1, 0, 0},
   {0, -((1 + mu) * (2 + mu) * (3 + mu) * (4 + mu)) / (180 * (5 + mu)),
    -((1 + mu) * (2 + mu) * (3 + mu) * (4 + mu)) / (120 * (5 + mu)),
    ((3 + mu) * (4 + mu) * (13 + 3 * mu)) / (30 * (5 + mu)), (-4 - mu) / 2, 1}};
TraditionalForm[TableForm[Table[c[n, j] == cdata[[n, j]], {n, 3}, {j, 2 n}]]]
```

$c(1,\ 1) = -\frac{\text{mu}}{2}$	$c(1,\ 2) = 1$			
$c(2,\ 1) = 0$	$c(2,\ 2) = \frac{1}{12}(\text{mu}+1)(\text{mu}+2)$	$c(2,\ 3) = \frac{1}{2}(-\text{mu}-2)$	$c(2,\ 4) = 1$	
$c(3,\ 1) = 0$	$c(3,\ 2) = -\frac{(\text{mu}+1)(\text{mu}+2)(\text{mu}+3)(\text{mu}+4)}{180(\text{mu}+5)}$	$c(3,\ 3) = -\frac{(\text{mu}+1)(\text{mu}+2)(\text{mu}+3)(\text{mu}+4)}{120(\text{mu}+5)}$	$c(3,\ 4) = \frac{(\text{mu}+3)(\text{mu}+4)(3\,\text{mu}+13)}{30(\text{mu}+5)}$	$c(3,$

```
(* Test (2a) *)
Together[Table[Sum[cdata[[n, j]] * a[i - 1, j, mu], {j, 1, 2 n}], {n, 1, 3}, {i, 1, 2 n - 1}]]
```

{{0}, {0, 0, 0}, {0, 0, 0, 0, 0}}

```
(* Test (3a) *)
Together[Table[
  Sum[cdata[[n, j]] * a[2 n - 1, j, mu], {j, 1, 2 n}] - myDet01[2 n] / myDet01[2 n - 1], {n, 1, 3}]]
```

{0, 0, 0}

We load the guessed recurrences for $c_{n,j}$:

```
cnj = << "gb01even.m";
```

```
LeadingCoefficient /@ Factor[cnj]
```

```
{-2 (1 + j) (-2 + j - 2 n) (-1 + j - 2 n) (1 + n) (1 + 2 n) (1 + mu + 2 n) (1 + j + mu + 2 n)
   (-2 + mu + 4 n) (mu + 4 n) (4 + 2 j^2 - 4 mu + 2 j mu - 2 j^2 mu - 2 j mu^2 - 22 n - 8 j^2 n + 3 mu n -
      8 j mu n + 4 j^2 mu n + 2 mu^2 n + 4 j mu^2 n + mu^3 n + 22 n^2 + 8 j^2 n^2 + 8 mu n^2 + 8 j mu n^2 + 2 mu^2 n^2),
  j (-1 + j + mu) (j + mu) (3 + j - 2 n) (1 + j + mu + 2 n)
   (4 + 2 j^2 - 4 mu + 2 j mu - 2 j^2 mu - 2 j mu^2 - 22 n - 8 j^2 n + 3 mu n - 8 j mu n + 4 j^2 mu n +
      2 mu^2 n + 4 j mu^2 n + mu^3 n + 22 n^2 + 8 j^2 n^2 + 8 mu n^2 + 8 j mu n^2 + 2 mu^2 n^2)}
```

```
UnderTheStaircase[cnj]
```

```
{1, S_j}
```

```
Support[cnj]
```

```
{{S_n, S_j, 1}, {S_j^2, S_j, 1}}
```

```
CylindricalDecomposition[Implies[mu > 0 && n ≥ 2 && 1 ≤ j ≤ 2 n - 2,
  (LeadingCoefficient[cnj[[1]]] /. n → n - 1) < 0], {n, j, mu}]
```

```
True
```

```
CylindricalDecomposition[Implies[mu > 0 && n ≥ 2 && 3 ≤ j ≤ 2 n - 2,
  (LeadingCoefficient[cnj[[2]]] /. j → j - 2) < 0], {n, j, mu}]
```

```
True
```

```
(* We cannot apply the recurrences of cnj if j=2n-1. *)
Expand[{LeadingCoefficient[cnj[[1]]] /. n → n - 1,
   LeadingCoefficient[cnj[[2]]] /. j → j - 2} /. j → 2 n - 1]
```

```
{0, 0}
```

- **Proof of (1a)**

We show that $c_{n,2n-1} = 1$.

```
Timing[diag = First[FindRelation[cnj, Pattern → ({a_, b_} /; 2 * a == b)]];]
```

```
{48.175, Null}
```

```
Support[diag]
```

```
{S_n^2 S_j^4, S_n S_j^2, 1}
```

```
lcf = Factor[LeadingCoefficient[diag] /. j → 2 n - 1]
```

```
8 n (1 + n) (2 + n) (1 + 2 n)^2 (3 + 2 n) (-2 + mu + 2 n) (-1 + mu + 2 n)
 (mu + 2 n)^2 (1 + mu + 2 n) (3 + mu + 2 n) (-3 + mu + 4 n) (-2 + mu + 4 n) (-1 + mu + 4 n)
 (mu + 4 n) (1 + mu + 4 n) (2 + mu + 4 n) (3 + mu + 4 n) (4 + mu + 4 n)^2 (5 + mu + 4 n)
 (6 + mu + 4 n) (-8 mu^2 - 8 mu^3 + 2 mu^4 + 2 mu^5 - 64 mu n - 56 mu^2 n + 52 mu^3 n + 26 mu^4 n -
    5 mu^5 n + mu^6 n - 128 n^2 - 48 mu n^2 + 396 mu^2 n^2 + 78 mu^3 n^2 - 94 mu^4 n^2 + 23 mu^5 n^2 + 160 n^3 +
    1112 mu n^3 - 270 mu^2 n^3 - 671 mu^3 n^3 + 225 mu^4 n^3 + 1024 n^4 - 1704 mu n^4 - 2252 mu^2 n^4 +
    1189 mu^3 n^4 - 2216 n^5 - 3532 mu n^5 + 3554 mu^2 n^5 - 2048 n^6 + 5676 mu n^6 + 3784 n^7)
```

```
(* Thus we can use the diagonal recurrence to compute the values for c(n,2n-2). *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1, lcf > 0], {n, mu}]
```

```
True
```

```
diag1 = OrePolynomialSubstitute[diag, {S[j] → 1, j → 2 n}];

OreReduce[diag1, ToOrePolynomial[{S[n] - 1}]]

0

lcf = Factor[LeadingCoefficient[diag1]]

8 n (1 + n)^2 (2 + n) (1 + 2 n)^2 (3 + 2 n) (-1 + mu + 2 n) (mu + 2 n) (1 + mu + 2 n)
  (2 + mu + 2 n) (3 + mu + 2 n) (-2 + mu + 4 n) (-1 + mu + 4 n) (mu + 4 n) (1 + mu + 4 n)
  (2 + mu + 4 n) (3 + mu + 4 n) (4 + mu + 4 n)^2 (5 + mu + 4 n) (6 + mu + 4 n) (7 + mu + 4 n)
  (16 mu^2 + 24 mu^3 - 10 mu^5 - mu^6 + mu^7 + 128 mu n + 224 mu^2 n - 44 mu^3 n - 190 mu^4 n - 16 mu^5 n +
    25 mu^6 n + 256 n^2 + 576 mu n^2 - 528 mu^2 n^2 - 1394 mu^3 n^2 - 59 mu^4 n^2 + 271 mu^5 n^2 + 320 n^3 -
    1872 mu n^3 - 4804 mu^2 n^3 + 246 mu^3 n^3 + 1639 mu^4 n^3 - 2048 n^4 - 7672 mu n^4 + 2204 mu^2 n^4 +
    5932 mu^3 n^4 - 4432 n^5 + 5224 mu n^5 + 12 784 mu^2 n^5 + 4096 n^6 + 15 136 mu n^6 + 7568 n^7)

CylindricalDecomposition[Implies[mu > 2 && n ≥ 1, lcf > 0], {n, mu}]

True
```

- **Proof of (2a)**

We show that $\sum_{j=1}^{2n} c_{n,j}\, a_{i,j} = 0$ for all $1 \le i < 2n$.

```
aij0 = Annihilator[Binomial[(i - 1) + j - 2 + mu, j], {S[i], S[n], S[j]}]

{(1 + j) S_j + (2 - i - j - mu), S_n - 1, (-2 + i + mu) S_i + (2 - i - j - mu)}

cnj0 = ToOrePolynomial[Append[cnj, S[i] - 1], OreAlgebra[aij0]];

smnd2 = DFiniteTimes[aij0, cnj0];

Timing[sum2a = First[FindCreativeTelescoping[smnd2, S[j] - 1]];]

{4018.02, Null}

(* The Kronecker delta that was left is added here. *)
Timing[sum2 = DFinitePlus[sum2a, DFiniteSubstitute[cnj0, {j → i - 1}]];]

{16.025, Null}

ByteCount[sum2]

1 272 448

Support[sum2]

{{S_i, S_n, 1}, {S_n^2, S_n, 1}}

lcf1 = LeadingCoefficient[sum2[[1]]] /. i → i - 1;
lcf2 = LeadingCoefficient[sum2[[2]]] /. n → n - 2;

(* This means that we can apply the first recurrence for i=2 and i≥4. *)
part2 = Last[Factor[lcf1]];
part1 = Factor[Together[lcf1 / part2]];
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1 && 4 ≤ i < 2 n, part1 < 0], {mu, i, n}] &&
 CylindricalDecomposition[Implies[mu > 60 && n ≥ 1 && 4 ≤ i < 2 n, part2 < 0], {mu, i, n}] &&
 CylindricalDecomposition[Implies[mu > 2 && n ≥ 2, (lcf1 /. i → 2) < 0], {n, mu}]

True
```

```
(* So we have to check what happens with the
  leading coefficient of the second recurrence for i=1,3. *)
Table[CylindricalDecomposition[Implies[mu > 2 && n ≥ 3, (lcf2 /. i → ii) < 0], {n, mu}],
 {ii, 1, 3, 2}]
```

```
{True, True}
```

- **Proof of (3a)**

We show that $\sum_{j=1}^{2n} c_{n,j}\, a_{2n,j} = \frac{b_{2n}(0,1)}{b_{2n-1}(0,1)}$ for all $n \geq 1$.

```
anj0 = Annihilator[Binomial[(2 n - 1) + j - 2 + mu, j], {S[n], S[j]}]
```

$$\{(1+j)\, S_j + (2 - j - mu - 2n),\ (2 - 3\,mu + mu^2 - 6n + 4\,mu\,n + 4n^2)\, S_n + (-2 + 3j - j^2 + 3\,mu - 2j\,mu - mu^2 + 6n - 4jn - 4\,mu\,n - 4n^2)\}$$

```
smnd3 = DFiniteTimes[anj0, cnj];
```

```
Timing[{sum3a, certificate} = FindCreativeTelescoping[smnd3, S[j] - 1];]
```

```
{568.62, Null}
```

```
(* We cross-check the correctness of the previous output. *)
Timing[OreReduce[sum3a[[1]] + (S[j] - 1) ** certificate[[1, 1]], Together[smnd3]]]
```

```
{49.0711, 0}
```

```
(* The Kronecker delta that was left is added here. *)
sum3 = DFinitePlus[sum3a, DFiniteSubstitute[cnj, {j → 2 n - 1}]];
```

```
Support[sum3]
```

$$\{\{S_n^2,\ S_n,\ 1\}\}$$

```
Factor[rec01even = First[sum3]]
```

$$
\begin{aligned}
&2\,(2+n)\,(1+2n)\,(-2+mu+2n)\,(mu+2n)\,(3+mu+2n)\,(2+mu+4n)\,(4+mu+4n)^2\\
&\quad(6+mu+4n)\,\big(-8\,mu^2-8\,mu^3+2\,mu^4+2\,mu^5-64\,mu\,n-56\,mu^2\,n+52\,mu^3\,n+26\,mu^4\,n-\\
&\quad 5\,mu^5\,n+mu^6\,n-128\,n^2-48\,mu\,n^2+396\,mu^2\,n^2+78\,mu^3\,n^2-94\,mu^4\,n^2+23\,mu^5\,n^2+160\,n^3+\\
&\quad 1112\,mu\,n^3-270\,mu^2\,n^3-671\,mu^3\,n^3+225\,mu^4\,n^3+1024\,n^4-1704\,mu\,n^4-2252\,mu^2\,n^4+\\
&\quad 1189\,mu^3\,n^4-2216\,n^5-3532\,mu\,n^5+3554\,mu^2\,n^5-2048\,n^6+5676\,mu\,n^6+3784\,n^7\big)\,S_n^2-\\
&(1+n)\,(-2+mu+2n)\,(2+mu+2n)\,\big(-18\,432\,mu^2-46\,080\,mu^3-39\,040\,mu^4-8960\,mu^5+5824\,mu^6+\\
&\quad 4480\,mu^7+1240\,mu^8+160\,mu^9+8\,mu^{10}-147\,456\,mu\,n-466\,944\,mu^2\,n-452\,864\,mu^3\,n-69\,248\,mu^4\,n+\\
&\quad 142\,448\,mu^5\,n+97\,040\,mu^6\,n+26\,488\,mu^7\,n+3992\,mu^8\,n+683\,mu^9\,n+159\,mu^{10}\,n+21\,mu^{11}\,n+\\
&\quad mu^{12}\,n-294\,912\,n^2-1\,477\,632\,mu\,n^2-1\,589\,760\,mu^2\,n^2+398\,912\,mu^3\,n^2+1\,422\,496\,mu^4\,n^2+\\
&\quad 737\,656\,mu^5\,n^2+129\,364\,mu^6\,n^2+19\,718\,mu^7\,n^2+17\,797\,mu^8\,n^2+6794\,mu^9\,n^2+1021\,mu^{10}\,n^2+\\
&\quad 54\,mu^{11}\,n^2-1\,480\,704\,n^3-1\,290\,496\,mu\,n^3+5\,130\,112\,mu^2\,n^3+6\,964\,560\,mu^3\,n^3+1\,434\,880\,mu^4\,n^3-\\
&\quad 1\,228\,784\,mu^5\,n^3-298\,216\,mu^6\,n^3+256\,207\,mu^7\,n^3+131\,654\,mu^8\,n^3+22\,288\,mu^9\,n^3+1307\,mu^{10}\,n^3+\\
&\quad 1\,464\,832\,n^4+16\,053\,888\,mu\,n^4+16\,373\,984\,mu^2\,n^4-10\,596\,632\,mu^3\,n^4-18\,368\,084\,mu^4\,n^4-\\
&\quad 4\,609\,306\,mu^5\,n^4+2\,448\,523\,mu^6\,n^4+1\,522\,856\,mu^7\,n^4+289\,402\,mu^8\,n^4+18\,864\,mu^9\,n^4+\\
&\quad 16\,229\,120\,n^5+14\,357\,824\,mu\,n^5-69\,122\,672\,mu^2\,n^5-98\,028\,920\,mu^3\,n^5-27\,447\,496\,mu^4\,n^5+\\
&\quad 16\,441\,114\,mu^5\,n^5+11\,620\,523\,mu^6\,n^5+2\,488\,117\,mu^7\,n^5+181\,475\,mu^8\,n^5-1\,184\,896\,n^6-\\
&\quad 153\,822\,240\,mu\,n^6-269\,070\,480\,mu^2\,n^6-86\,289\,464\,mu^3\,n^6+77\,390\,052\,mu^4\,n^6+60\,947\,110\,mu^5\,n^6+\\
&\quad 14\,882\,697\,mu^6\,n^6+1\,228\,510\,mu^7\,n^6-123\,440\,192\,n^7-378\,529\,824\,mu\,n^7-144\,639\,888\,mu^2\,n^7+\\
&\quad 247\,505\,128\,mu^3\,n^7+221\,815\,756\,mu^4\,n^7+63\,202\,646\,mu^5\,n^7+6\,008\,213\,mu^6\,n^7-216\,015\,744\,n^8-\\
&\quad 110\,556\,288\,mu\,n^8+508\,239\,648\,mu^2\,n^8+551\,593\,936\,mu^3\,n^8+190\,506\,752\,mu^4\,n^8+21\,404\,572\,mu^5\,n^8-\\
&\quad 19\,045\,440\,n^9+601\,432\,192\,mu\,n^9+894\,858\,704\,mu^2\,n^9+399\,260\,928\,mu^3\,n^9+55\,152\,716\,mu^4\,n^9+\\
&\quad 310\,932\,608\,n^{10}+853\,760\,352\,mu\,n^{10}+553\,875\,088\,mu^2\,n^{10}+100\,266\,880\,mu^3\,n^{10}+363\,340\,096\,n^{11}+\\
&\quad 457\,589\,600\,mu\,n^{11}+122\,106\,032\,mu^2\,n^{11}+170\,518\,912\,n^{12}+89\,453\,760\,mu\,n^{12}+29\,817\,920\,n^{13}\big)\,S_n+\\
&n\,(mu+2n)\,(2+mu+2n)\,(mu+3n)\,(1+mu+3n)\,(2+mu+3n)\\
&\quad(-4+mu+6n)\\
&\quad(-2+mu+6n)\\
&\quad(mu+6n)\\
&\quad\big(576+1440\,mu+1364\,mu^2+640\,mu^3+159\,mu^4+20\,mu^5+mu^6+7440\,n+12\,756\,mu\,n+\\
&\quad 8688\,mu^2\,n+2951\,mu^3\,n+513\,mu^4\,n+41\,mu^5\,n+mu^6\,n+33\,080\,n^2+42\,884\,mu\,n^2+\\
&\quad 21\,614\,mu^2\,n^2+5199\,mu^3\,n^2+581\,mu^4\,n^2+23\,mu^5\,n^2+73\,576\,n^3+72\,496\,mu\,n^3+\\
&\quad 26\,262\,mu^2\,n^3+4085\,mu^3\,n^3+225\,mu^4\,n^3+91\,664\,n^4+65\,776\,mu\,n^4+15\,518\,mu^2\,n^4+\\
&\quad 1189\,mu^3\,n^4+64\,960\,n^5+30\,524\,mu\,n^5+3554\,mu^2\,n^5+24\,440\,n^6+5676\,mu\,n^6+3784\,n^7\big)
\end{aligned}
$$

```
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1, LeadingCoefficient[rec01even] > 0], {n, mu}]
```

```
True
```

We can even find the closed form for the quotient:

```
Hyper[ApplyOreOperator[rec01even, f[n]], f[n], Solutions → All]
```

```
Warning: irreducible factors of degree > 1 in leading
                   coefficient;
some solutions may not be found

Warning: irreducible factors of degree > 1 in trailing
                   coefficient;
some solutions may not be found
```

$$
\left\{\frac{mu+2n}{-2+mu+2n},\ \frac{n\,(mu+2n)\,(mu+3n)\,(1+mu+3n)\,(2+mu+3n)\,(-4+mu+6n)\,(-2+mu+6n)\,(mu+6n)}{2\,(1+n)\,(-1+2n)\,(-2+mu+2n)\,(1+mu+2n)\,(-2+mu+4n)\,(mu+4n)^2\,(2+mu+4n)}\right\}
$$

```
basis = (RSolve[{y[n + 1] == # * y[n]}, y[n], n][[1, 1, 2]] & /@ %) /. C[_] → 1
```

$$\left\{\frac{-2+\text{mu}+2\,n}{-2+\text{mu}},\ \left(\left(\frac{256}{729}\right)^{1-n}(-2+\text{mu}+2\,n)\,\text{Pochhammer}\left[1+\frac{1}{6}(-4+\text{mu}),-1+n\right]\text{Pochhammer}\left[1+\frac{1}{6}(-2+\text{mu}),-1+n\right]\text{Pochhammer}\left[1+\frac{\text{mu}}{6},-1+n\right]\text{Pochhammer}\left[1+\frac{\text{mu}}{3},-1+n\right]\text{Pochhammer}\left[1+\frac{1+\text{mu}}{3},-1+n\right]\text{Pochhammer}\left[1+\frac{2+\text{mu}}{3},-1+n\right]\right)\Big/\left(\text{mu}\,n\,\text{Pochhammer}\left[\frac{1}{2},-1+n\right]\text{Pochhammer}\left[1+\frac{1}{4}(-2+\text{mu}),-1+n\right]\text{Pochhammer}\left[1+\frac{\text{mu}}{4},-1+n\right]^2\text{Pochhammer}\left[1+\frac{1+\text{mu}}{2},-1+n\right]\text{Pochhammer}\left[1+\frac{2+\text{mu}}{4},-1+n\right]\right)\right\}$$

```
inits = Factor[Table[Together[myDet01[2 n] / myDet01[2 n - 1]], {n, 2}]]
```

$$\left\{\text{mu},\ \frac{1}{4}(2+\text{mu})(5+\text{mu})\right\}$$

```
sol01even = Simplify[
  ({c1, c2} /. First[Solve[Thread[Table[basis.{c1, c2}, {n, 2}] == inits], {c1, c2}]]).basis]
```

$$\left(\left(\frac{256}{729}\right)^{1-n}(-2+\text{mu}+2\,n)\,\text{Pochhammer}\left[1+\frac{\text{mu}}{6},-1+n\right]\text{Pochhammer}\left[1+\frac{\text{mu}}{3},-1+n\right]\text{Pochhammer}\left[\frac{2+\text{mu}}{6},-1+n\right]\text{Pochhammer}\left[\frac{4+\text{mu}}{6},-1+n\right]\text{Pochhammer}\left[\frac{4+\text{mu}}{3},-1+n\right]\text{Pochhammer}\left[\frac{5+\text{mu}}{3},-1+n\right]\right)\Big/\left(n\,\text{Pochhammer}\left[\frac{1}{2},-1+n\right]\text{Pochhammer}\left[1+\frac{\text{mu}}{4},-1+n\right]^2\text{Pochhammer}\left[\frac{2+\text{mu}}{4},-1+n\right]\text{Pochhammer}\left[\frac{3+\text{mu}}{2},-1+n\right]\text{Pochhammer}\left[\frac{6+\text{mu}}{4},-1+n\right]\right)$$

```
FullSimplify[sol01even]
```

$$\frac{2^{2-\text{mu}-4\,n}(-2+\text{mu}+2\,n)\,\pi\,\text{Gamma}\left[-2+\frac{\text{mu}}{2}+3\,n\right]\text{Gamma}[\text{mu}+3\,n]}{n\,\text{Gamma}\left[-\frac{1}{2}+n\right]\text{Gamma}\left[\frac{1+\text{mu}}{2}+n\right]\text{Gamma}\left[-1+\frac{\text{mu}}{2}+2\,n\right]\text{Gamma}\left[\frac{\text{mu}}{2}+2\,n\right]}$$

```
Table[Together[FunctionExpand[sol01even] - myDet01[2 n] / myDet01[2 n - 1]], {n, 5}]
```

{0, 0, 0, 0, 0}

■ $b_{2n}(1, 0)$

```
cdata =
  {{1 - mu, 1, 0, 0, 0, 0}, {((-1 + mu) * (1 + mu) * (6 + mu)) / 12, (-1 - mu) / 2, (-1 - mu) / 2, 1, 0, 0},
   {-((-1 + mu) * (1 + mu) * (3 + mu) * (10 + mu) * (12 + mu)) / 720, ((1 + mu) * (3 + mu)) / 6,
    ((1 + mu) * (3 + mu)) / 6, ((-2 + mu) * (3 + mu)) / 12, (-3 - mu) / 2, 1}};
TraditionalForm[TableForm[Table[c[n, j] == cdata[[n, j]], {n, 3}, {j, 2 n}]]]
```

$c(1,\ 1) = 1 - \text{mu}$ $\quad c(1,\ 2) = 1$

$c(2,\ 1) = \frac{1}{12}(\text{mu}-1)(\text{mu}+1)(\text{mu}+6)$ $\quad c(2,\ 2) = \frac{1}{2}(-\text{mu}-1)$ $\quad c(2,\ 3) = \frac{1}{2}(-\text{mu}-1)$

$c(3,\ 1) = -\frac{1}{720}(\text{mu}-1)(\text{mu}+1)(\text{mu}+3)(\text{mu}+10)(\text{mu}+12)$ $\quad c(3,\ 2) = \frac{1}{6}(\text{mu}+1)(\text{mu}+3)$ $\quad c(3,\ 3) = \frac{1}{6}(\text{mu}+1)(\text{mu}+3)$

```
(* Test (2a) *)
Together[Table[Sum[cdata[[n, j]] * a[i, j - 1, mu], {j, 1, 2 n}], {n, 1, 3}, {i, 1, 2 n - 1}]]
```

```
{{0}, {0, 0, 0}, {0, 0, 0, 0, 0}}
```

```
(* Test (3a) *)
Together[Table[
  Sum[cdata[[n, j]] * a[2 n, j - 1, mu], {j, 1, 2 n}] - myDet10[2 n] / myDet10[2 n - 1], {n, 1, 3}]]
```

```
{0, 0, 0}
```

We load the guessed recurrences for $c_{n,j}$:

```
cnj = << "gb10even.m";
```

```
{lc1, lc2, lc3} = LeadingCoefficient /@ Factor[cnj]
```

```
{(-2 + j + mu) (j + mu) (-3 + 2 j + mu) (1 + j - 2 n) (mu + 4 n)
  (-4 j + 10 j^2 - 8 j^3 + 2 j^4 + 6 j mu - 9 j^2 mu + 3 j^3 mu - 2 j mu^2 + j^2 mu^2 + 6 j^2 n - 6 j^3 n - 4 mu n +
    16 j mu n - 12 j^2 mu n + 4 mu^2 n - 4 j mu^2 n + 12 j^2 n^2 - 8 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2),
 2 (-1 + j) (-2 + j + mu) (-3 + 2 j + mu) (-1 + 2 n) (j + mu + 2 n) (-2 + mu + 4 n) (mu + 4 n)
  (-4 j + 10 j^2 - 8 j^3 + 2 j^4 + 6 j mu - 9 j^2 mu + 3 j^3 mu - 2 j mu^2 + j^2 mu^2 + 6 j^2 n - 6 j^3 n - 4 mu n +
    16 j mu n - 12 j^2 mu n + 4 mu^2 n - 4 j mu^2 n + 12 j^2 n^2 - 8 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2),
 8 (-4 + j - 2 n) (-3 + j - 2 n) (1 + n) (-1 + 2 n) (1 + 2 n) (3 + 2 n) (j + mu + 2 n)
  (1 + j + mu + 2 n) (-2 + mu + 4 n) (mu + 4 n)^2 (2 + mu + 4 n) (4 + mu + 4 n)
  (-4 j + 10 j^2 - 8 j^3 + 2 j^4 + 6 j mu - 9 j^2 mu + 3 j^3 mu - 2 j mu^2 + j^2 mu^2 + 6 j^2 n - 6 j^3 n - 4 mu n +
    16 j mu n - 12 j^2 mu n + 4 mu^2 n - 4 j mu^2 n + 12 j^2 n^2 - 8 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2)}
```

```
LeadingPowerProduct /@ cnj
```

$\{S_j^2,\ S_n S_j,\ S_n^2\}$

```
{lc1, lc2, lc3} = Factor[{lc1 /. j → j - 2, lc2 /. {j → j - 1, n → n - 1}, lc3 /. n → n - 2}];
```

```
(* So the first recurrence can be applied for j≥3. *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1 && 3 ≤ j ≤ 2 n, lc1 < 0], {mu, n, j}]
```

```
True
```

```
(* For j=1 the second recurrence may be used. *)
CylindricalDecomposition[Implies[mu > 3 && n ≥ 2, (lc2 /. j → 1) < 0], {mu, n}]
```

```
True
```

```
(* For j=2 the third recurrence may be used. *)
CylindricalDecomposition[Implies[mu > 3 && n ≥ 3, (lc3 /. j → 2) > 0], {mu, n}]
```

```
True
```

- **Proof of (1a)**

We show that $c_{n,2n} = 1$.

```
Timing[diag = First[FindRelation[cnj, Pattern → ({a_, b_} /; 2 * a == b)]];]
```

```
{2159.99, Null}
```

```
Support[diag]
```

$\{S_n^3 S_j^6, S_n^2 S_j^4, S_n S_j^2, 1\}$

```
diag1 = OrePolynomialSubstitute[diag, {S[j] → 1, j → 2 n}];
```

```
OreReduce[diag1, ToOrePolynomial[{S[n] - 1}]]
```

```
0
```

```
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1, LeadingCoefficient[diag1] > 0], {n, mu}]
```

```
True
```

- **Proof of (2a)**

We show that $\sum_{j=1}^{2n} c_{n,j}\, a_{i,j} = 0$ for all $1 \le i < 2n$.

```
aij0 = Annihilator[Binomial[i + (j - 1) - 2 + mu, j - 1], {S[i], S[n], S[j]}]
```

$\{j\, S_j + (2 - i - j - mu), S_n - 1, (-1 + i + mu)\, S_i + (2 - i - j - mu)\}$

```
cnj0 = ToOrePolynomial[Append[cnj, S[i] - 1], OreAlgebra[aij0]];
```

```
smnd2 = DFiniteTimes[aij0, cnj0];
```

```
Timing[sum2a = First[FindCreativeTelescoping[smnd2, S[j] - 1]];]
```

```
{8059.34, Null}
```

```
(* The Kronecker delta that was left is added here. *)
Timing[sum2 = DFinitePlus[sum2a, DFiniteSubstitute[cnj0, {j → i + 1}]];]
```

```
{7.78449, Null}
```

```
ByteCount[sum2]
```

```
946976
```

```
Support[sum2]
```

$\{\{S_n^2, S_i, S_n, 1\}, \{S_i S_n, S_i, S_n, 1\}, \{S_i^2, S_i, S_n, 1\}\}$

```
Factor[LeadingCoefficient /@ sum2]
```

```
{8 (-3 + i - 2 n) (-2 + i - 2 n) (1 + n) (-1 + 2 n) (1 + 2 n) (3 + 2 n)
  (1 + i + mu + 2 n) (2 + i + mu + 2 n) (-2 + mu + 4 n) (mu + 4 n)^2 (2 + mu + 4 n) (4 + mu + 4 n)
  (-2 i^2 + 2 i^4 - 3 i mu + 3 i^3 mu - mu^2 + i^2 mu^2 - 6 i n - 12 i^2 n - 6 i^3 n - 8 i mu n -
    12 i^2 mu n - 4 i mu^2 n + 12 n^2 + 24 i n^2 + 12 i^2 n^2 + 16 mu n^2 + 24 i mu n^2 + 8 mu^2 n^2),
 2 i (-1 + i + mu) (-1 + 2 i + mu) (-1 + 2 n) (1 + i + mu + 2 n) (-2 + mu + 4 n) (mu + 4 n)
  (-2 i^2 + 2 i^4 - 3 i mu + 3 i^3 mu - mu^2 + i^2 mu^2 - 6 i n - 12 i^2 n - 6 i^3 n - 8 i mu n -
    12 i^2 mu n - 4 i mu^2 n + 12 n^2 + 24 i n^2 + 12 i^2 n^2 + 16 mu n^2 + 24 i mu n^2 + 8 mu^2 n^2),
 (-1 + i + mu) (1 + i + mu) (-1 + 2 i + mu) (2 + i - 2 n) (mu + 4 n)
  (-2 i^2 + 2 i^4 - 3 i mu + 3 i^3 mu - mu^2 + i^2 mu^2 - 6 i n - 12 i^2 n - 6 i^3 n - 8 i mu n -
    12 i^2 mu n - 4 i mu^2 n + 12 n^2 + 24 i n^2 + 12 i^2 n^2 + 16 mu n^2 + 24 i mu n^2 + 8 mu^2 n^2)}
```

```
lcf1 = LeadingCoefficient[sum2[[1]]] /. n → n - 2;
lcf2 = LeadingCoefficient[sum2[[2]]] /. {i → i - 1, n → n - 1};
lcf3 = LeadingCoefficient[sum2[[3]]] /. i → i - 2;
```

```
(* This means that we can apply the first recurrence for all i≤2n-3. *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 3 && 1 ≤ i ≤ 2 n - 3, lcf1 > 0], {i, n, mu}]
```

```
True
```

```
(* So we have to check what happens with the
  leading coefficient of the second recurrence for i=2n-2,2n-1. *)
Table[CylindricalDecomposition[Implies[mu > 2 && n ≥ 3, (lcf2 /. i → 2 n - ii) > 0], {n, mu}],
 {ii, 1, 2}]
```

```
{True, True}
```

- **Proof of (3a)**

We show that $\sum_{j=1}^{2n} c_{n,j}\, a_{2n,j} = \frac{b_{2n}(1,0)}{b_{2n-1}(1,0)}$ for all $n \geq 1$.

```
anj0 = Annihilator[Binomial[2 n + (j - 1) - 2 + mu, j - 1], {S[n], S[j]}]
```

```
{j S_j + (2 - j - mu - 2 n),
 (-mu + mu^2 - 2 n + 4 mu n + 4 n^2) S_n + (-2 + 3 j - j^2 + 3 mu - 2 j mu - mu^2 + 6 n - 4 j n - 4 mu n - 4 n^2)}
```

```
smnd3 = DFiniteTimes[anj0, cnj];
```

```
Timing[{sum3, certificate} = FindCreativeTelescoping[smnd3, S[j] - 1];]
```

```
{416.274, Null}
```

```
(* We cross-check the correctness of the previous output. *)
Timing[OreReduce[sum3[[1]] + (S[j] - 1) ** certificate[[1, 1]], Together[smnd3]]]
```

```
{32.9261, 0}
```

```
Support[sum3]
```

```
{{S_n^2, S_n, 1}}
```

```
Factor[rec10even = First[sum3]]
```

$$-8 (-1+2 n) (1+2 n) (3+2 n) (3+mu+2 n) (-2+mu+4 n) (mu+4 n) (2+mu+4 n) (4+mu+4 n)^2 (6+mu+4 n) \left(2 mu+5 mu^2+4 mu^3+mu^4+4 n+36 mu n+ 46 mu^2 n+16 mu^3 n+62 n^2+166 mu n^2+88 mu^2 n^2+194 n^3+204 mu n^3+172 n^4\right) S_n^2 - 2 (-1+2 n) (-2+mu+4 n) (mu+4 n) (2+mu+6 n) (4+mu+6 n) (6+mu+6 n) \left(-1584 mu-3612 mu^2-2816 mu^3-791 mu^4+64 mu^5+82 mu^6+16 mu^7+mu^8-4896 n-25776 mu n- 34492 mu^2 n-17184 mu^3 n-2068 mu^4 n+912 mu^5 n+296 mu^6 n+24 mu^7 n-44760 n^2-132868 mu n^2- 116520 mu^2 n^2-34488 mu^3 n^2+1060 mu^4 n^2+2016 mu^5 n^2+232 mu^6 n^2-163972 n^3-320112 mu n^3- 181032 mu^2 n^3-24912 mu^3 n^3+5424 mu^4 n^3+1152 mu^5 n^3-311612 n^4-404212 mu n^4- 134740 mu^2 n^4-1824 mu^3 n^4+2976 mu^4 n^4-331564 n^5-271224 mu n^5-43008 mu^2 n^5+ 2880 mu^3 n^5-197660 n^6-89584 mu n^6-3568 mu^2 n^6-61168 n^7-11040 mu n^7-7568 n^8\right) S_n + (-1+mu+2 n) (mu+3 n) (1+mu+3 n) (2+mu+3 n) (-4+mu+6 n) (-2+mu+6 n) (mu+6 n) (2+mu+6 n) (4+mu+6 n) (6+mu+6 n) \left(432+408 mu+139 mu^2+20 mu^3+mu^4+1398 n+980 mu n+222 mu^2 n+ 16 mu^3 n+1676 n^2+778 mu n^2+88 mu^2 n^2+882 n^3+204 mu n^3+172 n^4\right)$$

```
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1, LeadingCoefficient[rec10even] < 0], {n, mu}]
```

```
True
```

We can even find the closed form for the quotient:

```
Hyper[ApplyOreOperator[rec10even, f[n]], f[n], Solutions → All]
```

```
Warning: irreducible factors of degree > 1 in leading
                   coefficient;
some solutions may not be found

Warning: irreducible factors of degree > 1 in trailing
                   coefficient;
some solutions may not be found
```

$$\left\{-\frac{(-1+mu+2 n) (-4+mu+6 n) (-2+mu+6 n) (mu+6 n)}{4 (-1+2 n) (1+2 n) (-2+mu+4 n) (mu+4 n)}, \frac{(mu+3 n) (1+mu+3 n) (2+mu+3 n) (-4+mu+6 n) (-2+mu+6 n) (mu+6 n)}{2 (-1+2 n) (1+mu+2 n) (-2+mu+4 n) (mu+4 n)^2 (2+mu+4 n)}\right\}$$

```
basis = (RSolve[{y[n + 1] == # * y[n]}, y[n], n][[1, 1, 2]] & /@ %) /. C[_] → 1
```

$$\left\{\left(\left(-\frac{27}{16}\right)^{-1+n} \text{Pochhammer}\left[1+\frac{1}{6}(-4+\text{mu}), -1+n\right] \text{Pochhammer}\left[1+\frac{1}{6}(-2+\text{mu}), -1+n\right] \text{Pochhammer}\left[1+\frac{1}{2}(-1+\text{mu}), -1+n\right] \text{Pochhammer}\left[1+\frac{\text{mu}}{6}, -1+n\right]\right) \Big/ \left(\text{Pochhammer}\left[\frac{1}{2}, -1+n\right] \text{Pochhammer}\left[\frac{3}{2}, -1+n\right] \text{Pochhammer}\left[1+\frac{1}{4}(-2+\text{mu}), -1+n\right] \text{Pochhammer}\left[1+\frac{\text{mu}}{4}, -1+n\right]\right), \left(\left(\frac{256}{729}\right)^{1-n} \text{Pochhammer}\left[1+\frac{1}{6}(-4+\text{mu}), -1+n\right] \text{Pochhammer}\left[1+\frac{1}{6}(-2+\text{mu}), -1+n\right] \text{Pochhammer}\left[1+\frac{\text{mu}}{6}, -1+n\right] \text{Pochhammer}\left[1+\frac{\text{mu}}{3}, -1+n\right] \text{Pochhammer}\left[1+\frac{1+\text{mu}}{3}, -1+n\right] \text{Pochhammer}\left[1+\frac{2+\text{mu}}{3}, -1+n\right]\right) \Big/ \left(\text{Pochhammer}\left[\frac{1}{2}, -1+n\right] \text{Pochhammer}\left[1+\frac{1}{4}(-2+\text{mu}), -1+n\right] \text{Pochhammer}\left[1+\frac{\text{mu}}{4}, -1+n\right]^2 \text{Pochhammer}\left[1+\frac{1+\text{mu}}{2}, -1+n\right] \text{Pochhammer}\left[1+\frac{2+\text{mu}}{4}, -1+n\right]\right)\right\}$$

```
inits = Factor[Table[Together[myDet10[2 n] / myDet10[2 n - 1]], {n, 2}]]
```

{2, 5 + mu}

```
sol10even = Simplify[
  ({c1, c2} /. First[Solve[Thread[Table[basis.{c1, c2}, {n, 2}] == inits], {c1, c2}]]).basis]
```

$$\left(2^{9-8n}\, 729^{-1+n} \text{Pochhammer}\left[1+\frac{\text{mu}}{6}, -1+n\right] \text{Pochhammer}\left[1+\frac{\text{mu}}{3}, -1+n\right] \text{Pochhammer}\left[\frac{2+\text{mu}}{6}, -1+n\right] \text{Pochhammer}\left[\frac{4+\text{mu}}{6}, -1+n\right] \text{Pochhammer}\left[\frac{4+\text{mu}}{3}, -1+n\right] \text{Pochhammer}\left[\frac{5+\text{mu}}{3}, -1+n\right]\right) \Big/ \left(\text{Pochhammer}\left[\frac{1}{2}, -1+n\right] \text{Pochhammer}\left[1+\frac{\text{mu}}{4}, -1+n\right]^2 \text{Pochhammer}\left[\frac{2+\text{mu}}{4}, -1+n\right] \text{Pochhammer}\left[\frac{3+\text{mu}}{2}, -1+n\right] \text{Pochhammer}\left[\frac{6+\text{mu}}{4}, -1+n\right]\right)$$

```
FullSimplify[sol10even]
```

$$\frac{2^{3-\text{mu}-4n}\, \pi\, \text{Gamma}\left[-2+\frac{\text{mu}}{2}+3n\right] \text{Gamma}[\text{mu}+3n]}{\text{Gamma}\left[-\frac{1}{2}+n\right] \text{Gamma}\left[\frac{1+\text{mu}}{2}+n\right] \text{Gamma}\left[-1+\frac{\text{mu}}{2}+2n\right] \text{Gamma}\left[\frac{\text{mu}}{2}+2n\right]}$$

```
Table[Together[FunctionExpand[sol10even] - myDet10[2 n] / myDet10[2 n - 1]], {n, 5}]
```

{0, 0, 0, 0, 0}

∎

- $b_{2n}(0, 0) = -b_{2n-1}(1, 1)$

Here are the first values for $c_{n,j}$:

$c(1,\ 0) = 1 \qquad c(1,\ 1) = \frac{1}{1-\text{mu}}$

$c(2,\ 0) = 1 \qquad c(2,\ 1) = -\frac{6}{\text{mu}^2+5\,\text{mu}-6} \qquad c(2,\ 2) = -\frac{6}{\text{mu}^2+5\,\text{mu}-6} \qquad c(2,\ 3) = \frac{12}{\text{mu}^3+6\,\text{mu}^2-\text{mu}-6}$

$c(3,\ 0) = 1 \qquad c(3,\ 1) = -\frac{120}{\text{mu}^3+21\,\text{mu}^2+98\,\text{mu}-120} \qquad c(3,\ 2) = -\frac{120}{\text{mu}^3+21\,\text{mu}^2+98\,\text{mu}-120} \qquad c(3,\ 3) = -\frac{60\,(\text{mu}-2)}{\text{mu}^4+22\,\text{mu}^3+119\,\text{mu}^2-22\,\text{mu}-120} \qquad c(3,\ 4)$

We load the guessed recurrences for $c_{n,j}$:

```
cnj = << "gb0011.m";
```

```
LeadingCoefficient /@ Factor[cnj]
```

```
{(-1 + j + mu) (1 + j + mu) (-1 + 2 j + mu) (2 + j - 2 n) (mu + 4 n)
   (-2 j^2 + 2 j^4 - 3 j mu + 3 j^3 mu - mu^2 + j^2 mu^2 - 6 j n - 12 j^2 n - 6 j^3 n - 8 j mu n -
     12 j^2 mu n - 4 j mu^2 n + 12 n^2 + 24 j n^2 + 12 j^2 n^2 + 16 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2),
 j (-1 + j + mu) (-1 + 2 j + mu) (-1 + mu + 2 n) (1 + j + mu + 2 n)
   (-2 j^2 + 2 j^4 - 3 j mu + 3 j^3 mu - mu^2 + j^2 mu^2 - 6 j n - 12 j^2 n - 6 j^3 n - 8 j mu n -
     12 j^2 mu n - 4 j mu^2 n + 12 n^2 + 24 j n^2 + 12 j^2 n^2 + 16 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2),
 -(-3 + j - 2 n) (-2 + j - 2 n) (1 + n) (-1 + mu + 2 n) (1 + mu + 2 n) (1 + j + mu + 2 n)
   (2 + j + mu + 2 n) (mu + 4 n) (2 + mu + 6 n) (4 + mu + 6 n) (6 + mu + 6 n)
   (-2 j^2 + 2 j^4 - 3 j mu + 3 j^3 mu - mu^2 + j^2 mu^2 - 6 j n - 12 j^2 n - 6 j^3 n - 8 j mu n -
     12 j^2 mu n - 4 j mu^2 n + 12 n^2 + 24 j n^2 + 12 j^2 n^2 + 16 mu n^2 + 24 j mu n^2 + 8 mu^2 n^2)}
```

```
UnderTheStaircase[cnj]
```

```
{1, S_j, S_n}
```

```
Support[cnj]
```

```
{{S_j^2, S_n, S_j, 1}, {S_n S_j, S_n, S_j, 1}, {S_n^2, S_n, S_j, 1}}
```

```
lcf1 = Factor[LeadingCoefficient[cnj[[1]]] /. j → j - 2];
lcf2 = Factor[LeadingCoefficient[cnj[[2]]] /. {j → j - 1, n → n - 1}];
lcf3 = Factor[LeadingCoefficient[cnj[[3]]] /. n → n - 2];
```

```
(* The first recurrence can be applied j≥3. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1 && 3 ≤ j ≤ 2 n - 1, lcf1 < 0], {n, j, mu}]
```

```
True
```

```
(* The second recurrence can be applied for j≥2. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 2 && 2 ≤ j ≤ 2 n - 1, lcf2 > 0], {n, j, mu}]
```

```
True
```

```
(* The third recurrence can be applied for j≤2n-3. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 3 && 0 ≤ j ≤ 2 n - 3, lcf3 < 0], {n, j, mu}]
```

```
True
```

The union of all these areas is the full area where we want to apply $c_{n,j}$.

- **Proof of (1c)**

We show that $c_{n,0} = 1$.

```
Timing[rec = First[DFiniteSubstitute[cnj, {j → 0}]];]
```

```
{1.03606, Null}
```

```
Support[rec]
```

$\{S_n^2, S_n, 1\}$

```
OreReduce[rec, ToOrePolynomial[{S[n] - 1}]]
```

0

```
lcf = Factor[LeadingCoefficient[rec]]
```

$(1+n)^2 (3+2n) (1+mu+2n)^2 (2+mu+2n) (mu+4n) (2+mu+6n) (4+mu+6n) (6+mu+6n) \left(-mu^2+12n^2+16\,mu\,n^2+8\,mu^2 n^2\right)$

Clearly, for mu > 0 and $n \geq 1$, this leading coefficient never becomes 0.

- **Proof of (2c)**

We show that $\sum_{j=0}^{2n-1} c_{n,j}\, a_{i,j} = 0$ for all $0 < i \leq 2n - 1$.

```
aij0 = Annihilator[Binomial[i + j - 2 + mu, j], {S[i], S[n], S[j]}]
```

$\{(1+j) S_j + (1-i-j-mu),\ S_n - 1,\ (-1+i+mu) S_i + (1-i-j-mu)\}$

```
cnj0 = ToOrePolynomial[Append[cnj, S[i] - 1], OreAlgebra[aij0]];

smnd2 = DFiniteTimes[aij0, cnj0];

Timing[sum2a = First[FindCreativeTelescoping[smnd2, S[j] - 1]];]
```

{6006.41, Null}

```
(* The Kronecker delta that was left is added here. *)
Timing[sum2 = DFinitePlus[sum2a, DFiniteSubstitute[cnj0, {j → i}]];]
```

{5.45634, Null}

```
ByteCount[sum2]
```

677 968

```
Support[sum2]
```

$\{\{S_n^2, S_i, S_n, 1\}, \{S_i S_n, S_i, S_n, 1\}, \{S_i^2, S_i, S_n, 1\}\}$

```
lcf1 = LeadingCoefficient[sum2[[1]]] /. n → n - 2;
lcf2 = LeadingCoefficient[sum2[[2]]] /. {i → i - 1, n → n - 1};
lcf3 = LeadingCoefficient[sum2[[3]]] /. i → i - 2;
```

```
(* The first recurrence can be applied for i≤2n-3. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 3 && 1 ≤ i ≤ 2 n - 3, lcf1 > 0], {mu, i, n}]
```

True

```
(* The second recurrence can be applied for 2≤i≤2n-2. *)
CylindricalDecomposition[Implies[mu > 1 && n ≥ 2 && 2 ≤ i ≤ 2 n - 2, lcf2 > 0], {mu, i, n}]
```

True

```
(* The third recurrence can be applied i≥3. *)
CylindricalDecomposition[Implies[mu > 1 && n ≥ 1 && 3 ≤ i ≤ 2 n - 1, lcf3 < 0], {mu, i, n}]
```

True

Proof of (3c)

We show that $\sum_{j=0}^{2n-1} c_{n,j}\, a_{0,j} = \frac{b_{2n}(0,0)}{b_{2n-1}(1,1)}$ for all $n \geq 1$.

```
a0j = Annihilator[Binomial[j - 2 + mu, j], {S[n], S[j]}]
```

```
{(1 + j) Sj + (1 - j - mu), Sn - 1}
```

```
smnd3 = DFiniteTimes[a0j, cnj];
```

```
Timing[{sum3a, certificate} = FindCreativeTelescoping[smnd3, S[j] - 1];]
```

```
{11.5487, Null}
```

```
(* We cross-check the correctness of the previous output. *)
Timing[OreReduce[sum3a[[1]] + (S[j] - 1) ** certificate[[1, 1]], Together[smnd3]]]
```

```
{1.21608, 0}
```

```
(* The Kronecker delta that was left is added here,
but we showed before that c_{n,0}=1. *)
sum3 = DFinitePlus[sum3a, Annihilator[1, S[n]]];
```

```
sum3
```

```
{Sn - 1}
```

This means that the sum evaluates to a constant, which is:

```
Together[With[{n = 1}, Sum[c[n, j] * a[0, j, mu], {j, 0, 2 n - 1}]] /. c[1, 1] → 1 / (1 - mu)]
```

```
-1
```

- $b_{2n-1}(0, 0) = 0$

We load the guessed recurrences for $c_{n,j}$:

```
cnj = << "gb00.m";
```

```
LeadingCoefficient /@ Factor[cnj]
```

```
{2 (j - 2 n) (1 + j - 2 n) n (-1 + 2 n) (-1 + j + mu + 2 n) (-4 + mu + 4 n) (-2 + mu + 4 n),
 - (j + mu) (3 + j - 2 n) (-1 + j + mu + 2 n)}
```

```
UnderTheStaircase[cnj]
```

```
{1, Sj}
```

```
(* We cannot apply these recurrences if (n,j) = (1,1), (1,2), (2,1), (2,2). *)
AnnihilatorSingularities[cnj, {1, 1}, Assumptions → mu > 0]
```

```
{{{j → 1, n → 1}, mu > 0}, {{j → 1, n → 2}, mu > 0}, {{j → 2, n → 1}, mu > 0}, {{j → 2, n → 2}, mu > 0},
 {{j → 1, mu → 1, n → 1}, True}, {{j → 1, mu → 1, n → 2}, True}, {{j → 1, mu → 2, n → 1}, True},
 {{j → 1, mu → 4, n → 1}, True}, {{j → 2, mu → 2, n → 1}, True}, {{j → 2, mu → 4, n → 1}, True}}
```

```
(* We cannot apply these recurrences if (n,j) = (1,0), (1,1), (2,1). *)
AnnihilatorSingularities[cnj, {1, 0}, Assumptions → mu > 0]
```

```
{{{j → 0, n → 1}, mu > 0}, {{j → 1, n → 1}, mu > 0}, {{j → 1, n → 2}, mu > 0},
 {{j → 0, mu → 1, n → 1}, True}, {{j → 0, mu → 2, n → 1}, True},
 {{j → 0, mu → 4, n → 1}, True}, {{j → 1, mu → 1, n → 1}, True},
 {{j → 1, mu → 1, n → 2}, True}, {{j → 1, mu → 2, n → 1}, True}, {{j → 1, mu → 4, n → 1}, True}}
```

- **Numerical check / initial values**

```
eqns = ApplyOreOperator[cnj, c[n, j]];
eqns = Flatten[Table[eqns, {n, 5}, {j, 0, 2 n - 2}]];
csol = {c[1, 0] → 1, c[1, 1] → 0, c[2, 1] → -mu / 2};
eqns = eqns /. csol;
csol = Join[csol, First[Solve[Thread[eqns == 0], Union[Cases[eqns, c[__], Infinity]]]]];
csol = Together[csol];
TableForm[Together[Table[c[n, j], {n, 3}, {j, 0, 2 n - 2}] /. csol]]
```

```
Solve::svars : Equations may not give solutions for all "solve" variables. >>
```

1				
$-\text{mu}$	$-\frac{\text{mu}}{2}$	1		
$\frac{10\,\text{mu}+7\,\text{mu}^2+\text{mu}^3}{2\,(6+\text{mu})}$	$\frac{(5+\text{mu})\,(2\,\text{mu}+\text{mu}^2)}{4\,(6+\text{mu})}$	$\frac{-48-22\,\text{mu}+3\,\text{mu}^2+\text{mu}^3}{12\,(6+\text{mu})}$	$\frac{1}{2}\,(-2-\text{mu})$	1

```
Table[c[n, 2 n - 2], {n, 5}] /. csol
```

```
{1, 1, 1, 1, 1}
```

```
Table[Together[
  (Sum[c[n, j] * FunctionExpand[Binomial[i + j - 2 + mu, j]], {j, 0, 2 n - 2}] - c[n, i]) /. csol], {n,
  5}, {i, 0, 2 n - 2}]
```

```
{{0}, {0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

- **Show that $c_{n,j}$ is nontrivial**

We show that $c_{n,2n-2} = 1$.

```
Timing[diag = First[FindRelation[cnj, Pattern → ({a_, b_} /; 2 * a == b)]];]
```

```
{6.3404, Null}
```

```
Support[diag]
```

$\{S_n^2 S_j^4,\ S_n S_j^2,\ 1\}$

```
diag1 = OrePolynomialSubstitute[diag, {S[j] → 1, j → 2 n - 2}];
```

```
OreReduce[diag1, ToOrePolynomial[{S[n] - 1}]]
```

```
0
```

```
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1, LeadingCoefficient[diag1] > 0], {mu, n}]
```

```
True
```

- **Proof of the linear combination**

We show that $\sum_{j=0}^{2n-2} c_{n,j}\, a_{i,j} = 0$ for all $0 \le i \le 2n-2$.

```
aij0 = Annihilator[Binomial[i + j - 2 + mu, j], {S[i], S[n], S[j]}]
```

$\{(1+j)\, S_j + (1-i-j-mu),\ S_n - 1,\ (-1+i+mu)\, S_i + (1-i-j-mu)\}$

```
cnj0 = ToOrePolynomial[Append[cnj, S[i] - 1], OreAlgebra[aij0]];

smnd2 = DFiniteTimes[aij0, cnj0];

Timing[sum2a = First[FindCreativeTelescoping[smnd2, S[j] - 1]];]
```

{304.615, Null}

```
(* The Kronecker delta that was left is added here. *)
Timing[sum2 = DFinitePlus[sum2a, DFiniteSubstitute[cnj0, {j → i}]];]
```

{3.93625, Null}

```
ByteCount[sum2]
```

478 584

```
Support[sum2]
```

$\{\{S_i, S_n, 1\}, \{S_n^2, S_n, 1\}\}$

```
Factor[LeadingCoefficient /@ sum2]
```

$\{(-1+i+mu)\,(2+i-2n)\,(-2+mu+4n)\,(2i - i^2 - 2i^3 + i^4 - 3\,i\,mu - i^2\,mu + 2\,i^3\,mu + i\,mu^2 + i^2\,mu^2 - 12\,i\,n + 12\,i^2\,n - 2\,mu\,n + 18\,i\,mu\,n - 6\,i^2\,mu\,n + 3\,mu^2\,n - 6\,i\,mu^2\,n - mu^3\,n + 12\,i\,n^2 - 12\,i^2\,n^2 + 2\,mu\,n^2 - 12\,i\,mu\,n^2 - 2\,mu^2\,n^2),$
$4\,(-2+i-2n)\,(-1+i-2n)\,(1+n)\,(-1+2n)\,(1+2n)\,(i+mu+2n)\,(1+i+mu+2n)\,(-4+mu+4n)\,(-2+mu+4n)^2\,(mu+4n)\,(2+mu+4n)\,(2i - i^2 - 2i^3 + i^4 - 3\,i\,mu - i^2\,mu + 2\,i^3\,mu + i\,mu^2 + i^2\,mu^2 - 12\,i\,n + 12\,i^2\,n - 2\,mu\,n + 18\,i\,mu\,n - 6\,i^2\,mu\,n + 3\,mu^2\,n - 6\,i\,mu^2\,n - mu^3\,n + 12\,i\,n^2 - 12\,i^2\,n^2 + 2\,mu\,n^2 - 12\,i\,mu\,n^2 - 2\,mu^2\,n^2)\}$

```
lcf1 = LeadingCoefficient[sum2[[1]]] /. i → i - 1;
lcf2 = LeadingCoefficient[sum2[[2]]] /. n → n - 2;

(* This means that we can apply the first recurrence for i≥2. *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 1 && 2 ≤ i < n, lcf1 > 0], {mu, i, n}]
```

True

```
(* So we have to check what happens with the
  leading coefficient of the second recurrence for i=1. *)
CylindricalDecomposition[Implies[mu > 2 && n ≥ 3, (lcf2 /. i → 1) < 0], {n, mu}]
```

True

Conclusion

n even

```
(* the formula for even n *)
evaleven = (-1)^(n / 2) * 2^(n * (n + 2) / 4) *
   Pochhammer[mu / 2, n / 2] / (n / 2)! * Prod[(i!)^2 / ((2 i)!)^2, {i, 0, (n - 2) / 2}] *
   Prod[Pochhammer[(mu + 6 i - 1) / 2, (n - 4 i + 2) / 2]^2 *
     Pochhammer[(-mu - 3 n + 6 i) / 2, (n - 4 i) / 2]^2, {i, 1, Floor[n / 4]}];
```

We simplify the quotient by hand, since *Mathematica* cannot do this automatically:

```
{quo0, quo2} = (evaleven / (evaleven /. n → n - 2)) /. {{n → 4 k}, {n → 4 k + 2}};

{quo0, quo2} = ExpandAll[{quo0, quo2} /. a_Floor :→ FullSimplify[a, Element[k, Integers]]]
```

$$\left\{-\left(2^{4k}(-1+2k)!\,\text{Pochhammer}\left[\frac{\text{mu}}{2},2k\right]\text{Prod}\left[\frac{(i!)^2}{((2i)!)^2},\{i,0,-1+2k\}\right]\text{Prod}\left[\text{Pochhammer}\left[3i-6k-\frac{\text{mu}}{2},-2i+2k\right]^2\text{Pochhammer}\left[-\frac{1}{2}+3i+\frac{\text{mu}}{2},1-2i+2k\right]^2,\{i,1,k\}\right]\right)\Big/\left((2k)!\,\text{Pochhammer}\left[\frac{\text{mu}}{2},-1+2k\right]\text{Prod}\left[\frac{(i!)^2}{((2i)!)^2},\{i,0,-2+2k\}\right]\text{Prod}\left[\text{Pochhammer}\left[3+3i-6k-\frac{\text{mu}}{2},-1-2i+2k\right]^2\text{Pochhammer}\left[-\frac{1}{2}+3i+\frac{\text{mu}}{2},-2i+2k\right]^2,\{i,1,-1+k\}\right]\right),\right.$$

$$\left.-\left(2^{2+4k}(2k)!\,\text{Pochhammer}\left[\frac{\text{mu}}{2},1+2k\right]\text{Prod}\left[\frac{(i!)^2}{((2i)!)^2},\{i,0,2k\}\right]\text{Prod}\left[\text{Pochhammer}\left[-3+3i-6k-\frac{\text{mu}}{2},1-2i+2k\right]^2\text{Pochhammer}\left[-\frac{1}{2}+3i+\frac{\text{mu}}{2},2-2i+2k\right]^2,\{i,1,k\}\right]\right)\Big/\left((1+2k)!\,\text{Pochhammer}\left[\frac{\text{mu}}{2},2k\right]\text{Prod}\left[\frac{(i!)^2}{((2i)!)^2},\{i,0,-1+2k\}\right]\text{Prod}\left[\text{Pochhammer}\left[3i-6k-\frac{\text{mu}}{2},-2i+2k\right]^2\text{Pochhammer}\left[-\frac{1}{2}+3i+\frac{\text{mu}}{2},1-2i+2k\right]^2,\{i,1,k\}\right]\right)\right\}$$

```
quo0 = quo0 /. Pochhammer[-1 / 2 + 3 i + mu / 2, 1 - 2 i + 2 k] →
      Pochhammer[-1 / 2 + 3 i + mu / 2, -2 i + 2 k] * (-1 / 2 + i + mu / 2 + 2 k) /. Pochhammer[
      3 i - 6 k - mu / 2, -2 i + 2 k] :→ Pochhammer[3 + 3 i - 6 k - mu / 2, -1 - 2 i + 2 k] * FunctionExpand[
       Pochhammer[3 i - 6 k - mu / 2, -2 i + 2 k] / Pochhammer[3 + 3 i - 6 k - mu / 2, -1 - 2 i + 2 k]];
quo2 = quo2 /. Pochhammer[-1 / 2 + 3 i + mu / 2, 2 - 2 i + 2 k] →
      Pochhammer[-1 / 2 + 3 i + mu / 2, 1 - 2 i + 2 k] * (1 / 2 + i + 2 k + mu / 2) /. Pochhammer[
      3 i - 6 k - mu / 2, -2 i + 2 k] :→ Pochhammer[-3 + 3 i - 6 k - mu / 2, 1 - 2 i + 2 k] * FunctionExpand[
       Pochhammer[3 i - 6 k - mu / 2, -2 i + 2 k] / Pochhammer[-3 + 3 i - 6 k - mu / 2, 1 - 2 i + 2 k]];

{quo0, quo2} = {quo0, quo2} //. Prod[a1_ * a2_, a3_] → Prod[a1, a3] * Prod[a2, a3];

{quo0, quo2} = {quo0, quo2} //.
   Prod[a1_, {i, a2_, a3_}] / Prod[a1_, {i, a2_, a4_}] /; Expand[a3 - a4] == 1 :→ (a1 /. i → a3);
```

```
{quo0, quo2} = {quo0, quo2} /. Prod → Product
```

$$\left\{-\left(2^{4k}\,9^{3k}\,((-1+2k)!)^3\,\text{Pochhammer}\left[\frac{1}{6}(10-12k-mu),k\right]^2\text{Pochhammer}\left[1-2k-\frac{mu}{6},k\right]^2\text{Pochhammer}\left[\frac{4}{3}-2k-\frac{mu}{6},k\right]^2\text{Pochhammer}\left[\frac{mu}{2},2k\right]\text{Pochhammer}\left[\frac{1}{2}(1+4k+mu),k\right]^2\right)\Big/\left(\left(2-3k-\frac{mu}{2}\right)^2(2k)!\,((2(-1+2k))!)^2\,\text{Pochhammer}\left[\frac{1}{2}(2-8k-mu),k\right]^2\text{Pochhammer}\left[\frac{1}{2}(4-8k-mu),k\right]^2\text{Pochhammer}\left[\frac{mu}{2},-1+2k\right]\right),\right.$$

$$\left.-\left(2^{2+4k}\,9^{3k}\,((2k)!)^3\,\text{Pochhammer}\left[\frac{1}{6}(-12k-mu),k\right]^2\text{Pochhammer}\left[\frac{1}{3}-2k-\frac{mu}{6},k\right]^2\text{Pochhammer}\left[\frac{2}{3}-2k-\frac{mu}{6},k\right]^2\text{Pochhammer}\left[\frac{mu}{2},1+2k\right]\text{Pochhammer}\left[\frac{1}{2}(3+4k+mu),k\right]^2\right)\Big/\left(((4k)!)^2(1+2k)!\,\text{Pochhammer}\left[-1-4k-\frac{mu}{2},k\right]^2\text{Pochhammer}\left[-4k-\frac{mu}{2},k\right]^2\text{Pochhammer}\left[\frac{mu}{2},2k\right]\right)\right\}$$

```
FullSimplify[quo0 / ((-sol01even * sol10even) /. n → 2 k), Element[k, Integers]]
```

1

```
FullSimplify[quo2 / ((-sol01even * sol10even) /. n → 2 k + 1), Element[k, Integers]]
```

1

- ***n* odd**

```
(* the formula for odd n *)
evalodd = (-1) ^ ((n - 1) / 2) * 2^ ((n + 3) * (n + 1) / 4) *
   Pochhammer[(mu - 1) / 2, (n + 1) / 2] * Prod[i ! * (i + 1) ! / (2 i) ! / (2 i + 2) !, {i, 0, (n - 1) / 2}] *
   Prod[Pochhammer[(mu + 6 i - 1) / 2, (n - 4 i + 1) / 2] ^2 *
     Pochhammer[(-mu - 3 n + 6 i - 3) / 2, (n - 4 i + 3) / 2] ^2, {i, 1, Floor[(n + 1) / 4]}];
```

We simplify the quotient by hand, since *Mathematica* cannot do this automatically:

```
{quo1, quo3} = (evalodd / (evalodd /. n → n - 2)) /. {{n → 4 k + 1}, {n → 4 k + 3}};
```

```
{quo1, quo3} = ExpandAll[{quo1, quo3} /. a_Floor :> FullSimplify[a, Element[k, Integers]]]
```

$$\left\{-\left(2^{2+4k}\,\text{Pochhammer}\left[-\frac{1}{2}+\frac{mu}{2},1+2k\right]\text{Prod}\left[\frac{i!\,(1+i)!}{(2i)!\,(2+2i)!},\{i,0,2k\}\right]\text{Prod}\left[\text{Pochhammer}\left[-3+3i-6k-\frac{mu}{2},2-2i+2k\right]^2\text{Pochhammer}\left[-\frac{1}{2}+3i+\frac{mu}{2},1-2i+2k\right]^2,\{i,1,k\}\right]\right)\Big/\left(\text{Pochhammer}\left[-\frac{1}{2}+\frac{mu}{2},2k\right]\text{Prod}\left[\frac{i!\,(1+i)!}{(2i)!\,(2+2i)!},\{i,0,-1+2k\}\right]\text{Prod}\left[\text{Pochhammer}\left[3i-6k-\frac{mu}{2},1-2i+2k\right]^2\text{Pochhammer}\left[-\frac{1}{2}+3i+\frac{mu}{2},-2i+2k\right]^2,\{i,1,k\}\right]\right),$$

$$-\left(2^{4+4k}\,\text{Pochhammer}\left[-\frac{1}{2}+\frac{mu}{2},2+2k\right]\text{Prod}\left[\frac{i!\,(1+i)!}{(2i)!\,(2+2i)!},\{i,0,1+2k\}\right]\text{Prod}\left[\text{Pochhammer}\left[-6+3i-6k-\frac{mu}{2},3-2i+2k\right]^2\text{Pochhammer}\left[-\frac{1}{2}+3i+\frac{mu}{2},2-2i+2k\right]^2,\{i,1,1+k\}\right]\right)\Big/\left(\text{Pochhammer}\left[-\frac{1}{2}+\frac{mu}{2},1+2k\right]\text{Prod}\left[\frac{i!\,(1+i)!}{(2i)!\,(2+2i)!},\{i,0,2k\}\right]\text{Prod}\left[\text{Pochhammer}\left[-3+3i-6k-\frac{mu}{2},2-2i+2k\right]^2\text{Pochhammer}\left[-\frac{1}{2}+3i+\frac{mu}{2},1-2i+2k\right]^2,\{i,1,k\}\right]\right)\right\}$$

```
quo1 = quo1 /. Pochhammer[-1 / 2 + 3 i + mu / 2, 1 - 2 i + 2 k] →
      Pochhammer[-1 / 2 + 3 i + mu / 2, -2 i + 2 k] * (-1 / 2 + i + mu / 2 + 2 k) /. Pochhammer[
      3 i - 6 k - mu / 2, 1 - 2 i + 2 k] :> Pochhammer[-3 + 3 i - 6 k - mu / 2, 2 - 2 i + 2 k] * FunctionExpand[
       Pochhammer[3 i - 6 k - mu / 2, 1 - 2 i + 2 k] / Pochhammer[-3 + 3 i - 6 k - mu / 2, 2 - 2 i + 2 k]];
quo3 = quo3 /. Pochhammer[-1 / 2 + 3 i + mu / 2, 2 - 2 i + 2 k] →
      Pochhammer[-1 / 2 + 3 i + mu / 2, 1 - 2 i + 2 k] * (1 / 2 + i + 2 k + mu / 2) /.
    Pochhammer[-3 + 3 i - 6 k - mu / 2, 2 - 2 i + 2 k] :>
     Pochhammer[-6 + 3 i - 6 k - mu / 2, 3 - 2 i + 2 k] * FunctionExpand[
       Pochhammer[-3 + 3 i - 6 k - mu / 2, 2 - 2 i + 2 k] / Pochhammer[-6 + 3 i - 6 k - mu / 2, 3 - 2 i + 2 k]];

{quo1, quo3} = {quo1, quo3} //. Prod[a1_ * a2_, a3_] → Prod[a1, a3] * Prod[a2, a3];

{quo1, quo3} = {quo1, quo3} //.
    Prod[a1_, {i, a2_, a3_}] / Prod[a1_, {i, a2_, a4_}] /; Expand[a3 - a4] == 1 :> (a1 /. i → a3);
```

```
{quo1, quo3} = {quo1, quo3} /. Prod → Product
```

$$\Big\{-\Big(2^{2+4k}\,9^{3k}\,(2k)!\,(1+2k)!\,\text{Pochhammer}\Big[\frac{1}{6}(-12k-\text{mu}),k\Big]^2\,\text{Pochhammer}\Big[\frac{1}{3}-2k-\frac{\text{mu}}{6},k\Big]^2\,\text{Pochhammer}\Big[\frac{2}{3}-2k-\frac{\text{mu}}{6},k\Big]^2\,\text{Pochhammer}\Big[-\frac{1}{2}+\frac{\text{mu}}{2},1+2k\Big]\,\text{Pochhammer}\Big[\frac{1}{2}(1+4k+\text{mu}),k\Big]^2\Big)\Big/\Big((4k)!\,(2+4k)!\,\text{Pochhammer}\Big[\frac{1}{2}(2-8k-\text{mu}),k\Big]^2\,\text{Pochhammer}\Big[-4k-\frac{\text{mu}}{2},k\Big]^2\,\text{Pochhammer}\Big[-\frac{1}{2}+\frac{\text{mu}}{2},2k\Big]\Big),$$

$$-\Big(2^{4+4k}\,9^{3k}\,(1+2k)!\,(2+2k)!\,\text{Pochhammer}\Big[-6-6k+3(1+k)-\frac{\text{mu}}{2},3+2k-2(1+k)\Big]^2\,\text{Pochhammer}\Big[-1-2k-\frac{\text{mu}}{6},k\Big]^2\,\text{Pochhammer}\Big[-\frac{2}{3}-2k-\frac{\text{mu}}{6},k\Big]^2\,\text{Pochhammer}\Big[-\frac{1}{3}-2k-\frac{\text{mu}}{6},k\Big]^2\,\text{Pochhammer}\Big[-\frac{1}{2}+\frac{\text{mu}}{2},2+2k\Big]\,\text{Pochhammer}\Big[-\frac{1}{2}+3(1+k)+\frac{\text{mu}}{2},1+2k-2(1+k)\Big]^2\,\text{Pochhammer}\Big[\frac{1}{2}(3+4k+\text{mu}),1+k\Big]^2\Big)\Big/\Big((2(1+2k))!\,(2+2(1+2k))!\,\text{Pochhammer}\Big[-2-4k-\frac{\text{mu}}{2},k\Big]^2\,\text{Pochhammer}\Big[-1-4k-\frac{\text{mu}}{2},k\Big]^2\,\text{Pochhammer}\Big[-\frac{1}{2}+\frac{\text{mu}}{2},1+2k\Big]\Big)\Big\}$$

```
FullSimplify[quo1 / ((-sol01odd * sol10odd) /. n → 2 k + 1), Element[k, Integers]]
```

1

```
FullSimplify[quo3 / ((-sol01odd * sol10odd) /. n → 2 k + 2), Element[k, Integers]]
```

1

- **closed forms for the solutions and for the determinants $b_n(0, 1)$ and $b_n(1, 0)$**

First we simplify the expression a little bit:

```
sols = {sol01odd /. n → n + 1, sol01even, sol10odd /. n → n + 1, sol10even};
sols = Simplify[FullSimplify[sols] /. Gamma[a1_] /; Not[IntegerQ[a1 /. {mu → 0, n → 0}]] :>
      Sqrt[Pi] * 2^(1 - 2 * a1) * Gamma[2 * a1] / Gamma[a1 + 1 / 2]];
sols = sols //. {Gamma[a1_] / Gamma[a2_] /; FullSimplify[Element[a1 - a2, Integers] &&
        (a1 - a2 ≥ 0), Element[n, Integers] && n ≥ 1] :> Pochhammer[a2, Expand[a1 - a2]],
    Gamma[a1_] / Gamma[a2_] /; FullSimplify[Element[a2 - a1, Integers] && (a2 - a1 ≥ 0),
       Element[n, Integers] && n ≥ 1] :> 1 / Pochhammer[a1, Expand[a2 - a1]]}
```

$$\left\{\frac{2\,\text{Pochhammer}\left[\frac{\text{mu}}{2}+2\,n,\,1+n\right]\,\text{Pochhammer}[-1+\text{mu}+2\,n,\,1+n]}{\text{Pochhammer}[2+n,\,1+n]\,\text{Pochhammer}\left[\frac{\text{mu}}{2}+n,\,1+n\right]},\right.$$
$$\frac{(-2+\text{mu}+2\,n)\,\text{Pochhammer}\left[-1+\frac{\text{mu}}{2}+2\,n,\,-1+n\right]\,\text{Pochhammer}[1+\text{mu}+2\,n,\,-1+n]}{n\,\text{Pochhammer}[n,\,-1+n]\,\text{Pochhammer}\left[1+\frac{\text{mu}}{2}+n,\,-1+n\right]},$$
$$\frac{2\,\text{Pochhammer}\left[\frac{\text{mu}}{2}+2\,n,\,1+n\right]\,\text{Pochhammer}[1+\text{mu}+2\,n,\,-1+n]}{\text{Pochhammer}[1+n,\,n]\,\text{Pochhammer}\left[1+\frac{\text{mu}}{2}+n,\,n\right]},$$
$$\left.\frac{2\,\text{Pochhammer}\left[-1+\frac{\text{mu}}{2}+2\,n,\,-1+n\right]\,\text{Pochhammer}[1+\text{mu}+2\,n,\,-1+n]}{\text{Pochhammer}[n,\,-1+n]\,\text{Pochhammer}\left[1+\frac{\text{mu}}{2}+n,\,-1+n\right]}\right\}$$

```
TraditionalForm[sols]
```

$$\left\{\frac{2\left(\frac{\text{mu}}{2}+2\,n\right)_{n+1}(\text{mu}+2\,n-1)_{n+1}}{(n+2)_{n+1}\left(\frac{\text{mu}}{2}+n\right)_{n+1}},\ \frac{(\text{mu}+2\,n-2)\left(\frac{\text{mu}}{2}+2\,n-1\right)_{n-1}(\text{mu}+2\,n+1)_{n-1}}{n\,(n)_{n-1}\left(\frac{\text{mu}}{2}+n+1\right)_{n-1}},\right.$$
$$\left.\frac{2\left(\frac{\text{mu}}{2}+2\,n\right)_{n+1}(\text{mu}+2\,n+1)_{n-1}}{(n+1)_{n}\left(\frac{\text{mu}}{2}+n+1\right)_{n}},\ \frac{2\left(\frac{\text{mu}}{2}+2\,n-1\right)_{n-1}(\text{mu}+2\,n+1)_{n-1}}{(n)_{n-1}\left(\frac{\text{mu}}{2}+n+1\right)_{n-1}}\right\}$$

We numerically check whether they agree with the quotients of determinants:

```
Table[Together[sols[[1]] - myDet01[2 n + 1] / myDet01[2 n]], {n, 1, 4}]
```

```
{0, 0, 0, 0}
```

```
Table[Together[sols[[2]] - myDet01[2 n] / myDet01[2 n - 1]], {n, 1, 4}]
```

```
{0, 0, 0, 0}
```

```
Table[Together[sols[[3]] - myDet10[2 n + 1] / myDet10[2 n]], {n, 1, 4}]
```

```
{0, 0, 0, 0}
```

```
Table[Together[sols[[4]] - myDet10[2 n] / myDet10[2 n - 1]], {n, 1, 4}]
```

```
{0, 0, 0, 0}
```

In the following we check the closed forms numerically:

```
Do[Q[i][n_] := Evaluate[sols[[i]]], {i, 4}]
```

```
Table[Together[(mu - 1) * Product[Q[1][k] * Q[2][k], {k, 1, (n - 1) / 2}] - myDet01[n]], {n, 1, 9, 2}]
```

```
{0, 0, 0, 0, 0}
```

```
Table[Together[Product[Q[1][k], {k, 0, n / 2 - 1}] * Product[Q[2][k], {k, 1, n / 2}] - myDet01[n]],
 {n, 2, 10, 2}]
```

{0, 0, 0, 0, 0}

```
Table[Together[Product[Q[3][k] * Q[4][k], {k, 1, (n - 1) / 2}] - myDet10[n]], {n, 1, 9, 2}]
```

{0, 0, 0, 0, 0}

```
Table[Together[Product[Q[3][k], {k, 0, n / 2 - 1}] * Product[Q[4][k], {k, 1, n / 2}] - myDet10[n]],
 {n, 2, 10, 2}]
```

{0, 0, 0, 0, 0}

Theorem 5

```
a[i_Integer, j_Integer, mu_] :=
  Together[-KroneckerDelta[i, j] + FunctionExpand[Binomial[mu + i + j - 2, j]]];
```

Here are the first values for $c_{n,j}$:

```
cdata = {{1, (-2 * (1 + mu)) / (mu * (3 + mu)), 0, 0, 0, 0},
   {1, ((1 + mu) * (-24 - mu + mu^2)) / (3 * mu * (3 + mu) * (5 + mu)), (-2 * (2 + mu)) / (mu * (5 + mu)),
    4 / (mu * (5 + mu)), 0, 0}, {1, (2 * (1 + mu) * (-1620 - 381 * mu + 61 * mu^2 + 19 * mu^3 + mu^4)) /
      (5 * mu * (3 + mu) * (5 + mu) * (7 + mu) * (12 + mu)),
     ((2 + mu) * (-1620 - 301 * mu + mu^3)) / (10 * mu * (5 + mu) * (7 + mu) * (12 + mu)),
     (-6 * (-20 + 79 * mu + 20 * mu^2 + mu^3)) / (5 * mu * (5 + mu) * (7 + mu) * (12 + mu)),
     (6 * (4 + mu) * (15 + mu)) / (mu * (3 + mu) * (7 + mu) * (12 + mu)),
     (-12 * (15 + mu)) / (mu * (3 + mu) * (7 + mu) * (12 + mu))}};
TraditionalForm[TableForm[Table[c[n, j] == cdata[[n, j]], {n, 3}, {j, 2 n}]]]
```

$c(1,\,1)=1$	$c(1,\,2)=-\frac{2\,(\text{mu}+1)}{\text{mu}\,(\text{mu}+3)}$		
$c(2,\,1)=1$	$c(2,\,2)=\frac{(\text{mu}+1)\left(\text{mu}^2-\text{mu}-24\right)}{3\,\text{mu}\,(\text{mu}+3)\,(\text{mu}+5)}$	$c(2,\,3)=-\frac{2\,(\text{mu}+2)}{\text{mu}\,(\text{mu}+5)}$	$c(2,\,4)=\frac{4}{\text{mu}\,(\text{mu}+5)}$
$c(3,\,1)=1$	$c(3,\,2)=\frac{2\,(\text{mu}+1)\left(\text{mu}^4+19\,\text{mu}^3+61\,\text{mu}^2-381\,\text{mu}-1620\right)}{5\,\text{mu}\,(\text{mu}+3)\,(\text{mu}+5)\,(\text{mu}+7)\,(\text{mu}+12)}$	$c(3,\,3)=\frac{(\text{mu}+2)\left(\text{mu}^3-301\,\text{mu}-1620\right)}{10\,\text{mu}\,(\text{mu}+5)\,(\text{mu}+7)\,(\text{mu}+12)}$	$c(3,\,4)=-\frac{6\left(\text{mu}^3+20\,\text{mu}^2+79\,\text{mu}-20\right)}{5\,\text{mu}\,(\text{mu}+5)\,(\text{mu}+7)\,(\text{mu}+12)}$

In this case we have to prove slight variations of formulas 1c, 2c, and 3c:

```
Table[Together[Sum[cdata[[n, j]] * a[i, j, mu], {j, 1, 2 n}]], {n, 3}, {i, 2, 2 n}]
```

{{0}, {0, 0, 0}, {0, 0, 0, 0, 0}}

```
Table[Together[Sum[cdata[[n, j]] * a[1, j, mu], {j, 1, 2 n}]], {n, 3}]
```

$\left\{-\frac{4}{3+\text{mu}},\ -\frac{4}{3+\text{mu}},\ -\frac{4}{3+\text{mu}}\right\}$

```
Table[Together[Det[Table[a[i, j, mu], {i, 1, 2 n}, {j, 1, 2 n}]] /
   Det[Table[a[i, j, mu], {i, 2, 2 n}, {j, 2, 2 n}]]], {n, 3}]
```

$\left\{-\frac{4}{3+\text{mu}},\ -\frac{4}{3+\text{mu}},\ -\frac{4}{3+\text{mu}}\right\}$

We load the guessed recurrences for $c_{n,j}$:

```
cnj = << "gb1122.m";
```

```
LeadingCoefficient /@ Factor[cnj]
```

```
{-2 (1 + j) (-4 + j - 2 n) (-3 + j - 2 n) (-2 + j - 2 n) (-1 + j - 2 n) (1 + j - 2 n) (1 + n)
  (3 + 2 n) (2 + mu + 2 n) (3 + mu + 2 n) (2 + j + mu + 2 n) (3 + j + mu + 2 n) (3 + mu + 3 n)
  (4 + mu + 3 n) (5 + mu + 3 n) (2 + mu + 4 n) (2 + mu + 6 n) (4 + mu + 6 n) (6 + mu + 6 n)
  (4 + 2 j^2 + 7 mu + 2 j mu + 2 j^2 mu + 4 mu^2 + 2 j mu^2 + mu^3 + 22 n + 8 j^2 n + 19 mu n + 8 j mu n +
    4 j^2 mu n + 6 mu^2 n + 4 j mu^2 n + mu^3 n + 22 n^2 + 8 j^2 n^2 + 8 mu n^2 + 8 j mu n^2 + 2 mu^2 n^2),
 (1 + j) (-1 + j + mu) (j + mu) (1 + j + mu) (3 + j + mu) (j - 2 n) (1 + j - 2 n) (2 + j - 2 n)
  (2 + j + mu + 2 n) (2 + mu + 4 n) (4 + 2 j^2 + 7 mu + 2 j mu + 2 j^2 mu + 4 mu^2 + 2 j mu^2 + mu^3 + 22 n + 8 j^2 n +
    19 mu n + 8 j mu n + 4 j^2 mu n + 6 mu^2 n + 4 j mu^2 n + mu^3 n + 22 n^2 + 8 j^2 n^2 + 8 mu n^2 + 8 j mu n^2 + 2 mu^2 n^2),
 -j (-1 + j + mu) (j + mu) (1 + j - 2 n) (3 + j + mu + 2 n)
  (4 + 2 j^2 + 7 mu + 2 j mu + 2 j^2 mu + 4 mu^2 + 2 j mu^2 + mu^3 + 22 n + 8 j^2 n + 19 mu n + 8 j mu n +
    4 j^2 mu n + 6 mu^2 n + 4 j mu^2 n + mu^3 n + 22 n^2 + 8 j^2 n^2 + 8 mu n^2 + 8 j mu n^2 + 2 mu^2 n^2)}
```

```
UnderTheStaircase[cnj]
```

$\{1, S_j, S_n, S_j^2, S_n S_j\}$

```
Support[cnj]
```

$\{\{S_n^2, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n S_j^2, S_n S_j, S_j^2, S_n, S_j, 1\}\}$

```
lcf1 = Factor[LeadingCoefficient[cnj[[1]]] /. n → n - 2];
lcf2 = Factor[LeadingCoefficient[cnj[[2]]] /. j → j - 3];
lcf3 = Factor[LeadingCoefficient[cnj[[3]]] /. {j → j - 2, n → n - 1}];
```

```
(* The second recurrence can be applied for j≥4. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1 && 4 ≤ j ≤ 2 n, lcf2 < 0], {n, j, mu}]
```

```
True
```

```
(* The first recurrence can be applied for j≤2n-6. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1 && 1 ≤ j ≤ 2 n - 6, lcf1 > 0], {n, j, mu}]
```

```
True
```

```
(* The third recurrence can be applied for 3≤j≤2n-2. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1 && 3 ≤ j ≤ 2 n - 2, lcf3 > 0], {n, j, mu}]
```

```
True
```

The union of all these areas is the full area where we want to apply $c_{n,j}$.

- **Proof of (1c')**

We show that $c_{n,1} = 1$.

```
Timing[rec = First[DFiniteSubstitute[cnj, {j → 1}]];]
```

```
{5.47634, Null}
```

```
Support[rec]
```

$\{S_n^2, S_n, 1\}$

```
OreReduce[rec, ToOrePolynomial[{S[n] - 1}]]
```

```
0
```

```
lcf = Factor[LeadingCoefficient[rec]]
```

```
4 (3 + 2 n)^2 (4 + mu + 2 n) (3 + mu + 3 n) (4 + mu + 3 n) (5 + mu + 3 n)
 (2 + mu + 4 n) (2 + 3 mu + mu^2 + 10 n + 7 mu n + mu^2 n + 10 n^2 + 2 mu n^2)
```

Clearly, for mu > 0 and $n \geq 1$, this leading coefficient never becomes 0.

- **Proof of (2c')**

We show that $\sum_{j=1}^{2n} c_{n,j}\, a_{i,j} = 0$ for all $1 < i \leq 2\,n$.

```
aij0 = Annihilator[Binomial[i + j - 2 + mu, j], {S[n], S[j], S[i]}];
```

```
cnj0 = ToOrePolynomial[Prepend[cnj, S[i] - 1], OreAlgebra[aij0]];
```

```
smnd2 = Together[DFiniteTimes[aij0, cnj0]];
```

```
{sum2a, delta2a} = << "conj36_2c_ct.m";
```

```
MapThread[Timing[OreReduce[#1 + (S[j] - 1) ** #2, smnd2]] &, {sum2a, delta2a}]
```

```
{{642.532, 0}, {49.5391, 0}, {7.34046, 0}}
```

```
(* The Kronecker delta that was left is added here. *)
Timing[sum2 = DFinitePlus[Expand[sum2a], DFiniteSubstitute[cnj0, {j → i}]];]
```

```
{38.3784, Null}
```

```
Support[sum2]
```

$\{\{S_n^2, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_i^3, S_n S_i, S_i^2, S_n, S_i, 1\}, \{S_n S_i^2, S_n S_i, S_i^2, S_n, S_i, 1\}\}$

```
lcf1 = LeadingCoefficient[sum2[[1]]] /. n → n - 2;
lcf2 = LeadingCoefficient[sum2[[2]]] /. i → i - 3;
lcf3 = LeadingCoefficient[sum2[[3]]] /. {i → i - 2, n → n - 1};
```

```
(* This means that we can apply the first recurrence for i ≤ 2n-6. *)
CylindricalDecomposition[Implies[mu > 50 && n ≥ 3 && 1 ≤ i ≤ 2 n - 6, lcf1 < 0], {mu, i, n}]
```

```
True
```

```
(* This means that we can apply the second recurrence for i ≥ 4. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1 && 4 ≤ i ≤ 2 n, lcf2 < 0], {mu, i, n}]
```

```
True
```

```
(* This means that we can apply the second recurrence for 3 ≤ i ≤ 2n-2. *)
CylindricalDecomposition[Implies[mu > 0 && n ≥ 1 && 3 ≤ i ≤ 2 n - 2, lcf3 < 0], {mu, i, n}]
```

```
True
```

- **Proof of (3c')**

We show that $\sum_{j=1}^{2n} c_{n,j}\, a_{1,j} = \frac{b_{2n}(1,1,\text{mu}-2)}{b_{2n-1}(2,2,\text{mu}-2)}$ for all $n \geq 1$.

```
a1j = Annihilator[Binomial[1 + j - 2 + mu, j], {S[n], S[j]}]
```

$\{(1 + j)\, S_j + (-j - \text{mu}),\ S_n - 1\}$

```
smnd3 = DFiniteTimes[a1j, cnj];
```

```
Timing[{sum3a, certificate} = FindCreativeTelescoping[smnd3, S[j] - 1];]
```

{13.5848, Null}

```
(* We cross-check the correctness of the previous output. *)
Timing[OreReduce[sum3a[[1]] + (S[j] - 1) ** certificate[[1, 1]], Together[smnd3]]]
```

{1.99613, 0}

```
(* The Kronecker delta that was left is added here,
but we showed before that c_{n,1}=1. *)
sum3 = DFinitePlus[sum3a, Annihilator[1, S[n]]];

(* This means that sum3 evaluates to a constant. *)
sum3
```

$\{S_n - 1\}$

```
(* the formula for b_{2n}(1,1,mu), copied from Thm 2. *)
eval11 = (-1) ^ (n / 2) * 2^ (n * (n + 2) / 4) *
   Pochhammer[mu / 2, n / 2] / (n / 2) ! * Prod[(i !) ^2 / ((2 i) !) ^2, {i, 0, (n - 2) / 2}] *
   Prod[Pochhammer[(mu + 6 i - 1) / 2, (n - 4 i + 2) / 2] ^2 *
     Pochhammer[(-mu - 3 n + 6 i) / 2, (n - 4 i) / 2] ^2, {i, 1, Floor[n / 4]}];

(* the formula for b_{2n-1}(2,2,mu-2) *)
eval22 = (-1) ^ ((n - 1) / 2) * 2^ ((n - 1) (n + 5) / 4) * (mu + 1) * Pochhammer[(mu - 2) / 2, (n + 1) / 2] /
     ((n + 1) / 2) ! * Prod[(i !) ^2 / ((2 i) !) ^2, {i, 0, (n - 1) / 2}] *
   Prod[Pochhammer[(mu + 6 i - 3) / 2, (n - 4 i + 3) / 2] ^2, {i, 1, Floor[(n + 3) / 4]}] *
   Prod[Pochhammer[(-mu - 3 n + 6 i - 1) / 2, (n - 4 i + 1) / 2] ^2, {i, 1, Floor[(n + 1) / 4]}];

(* In order to evaluate the quotient symbolically, we make a case distinction *)
quo = (eval11 /. n → 2 n) / (eval22 /. mu → mu + 2 /. n → 2 n - 1);
{quo0, quo1} =
  ExpandAll[quo /. {{n → 2 k}, {n → 2 k - 1}} /. a_Floor :→ FullSimplify[a, Element[k, Integers]]];

quo0 = quo0 /. With[{a1 = Pochhammer[3 + 3 i - 6 k - mu / 2, -2 - 2 i + 2 k],
      a2 = Pochhammer[3 i - 6 k - mu / 2, -2 i + 2 k]}, a1 :→ a2 * FunctionExpand[a1 / a2]] /.
   With[{a1 = Pochhammer[-1 / 2 + 3 i + mu / 2, 1 - 2 i + 2 k],
     a2 = Pochhammer[5 / 2 + 3 i + mu / 2, -1 - 2 i + 2 k]}, a1 :→ a2 * FunctionExpand[a1 / a2]];

quo1 = quo1 /. With[{a1 = Pochhammer[3 + 3 i - 6 k - mu / 2, -1 - 2 i + 2 k],
      a2 = Pochhammer[6 + 3 i - 6 k - mu / 2, -3 - 2 i + 2 k]}, a1 :→ a2 * FunctionExpand[a1 / a2]] /.
   With[{a1 = Pochhammer[-1 / 2 + 3 i + mu / 2, -2 i + 2 k],
     a2 = Pochhammer[5 / 2 + 3 i + mu / 2, -2 - 2 i + 2 k]}, a1 :→ a2 * FunctionExpand[a1 / a2]];

{quo0, quo1} = Factor[{quo0, quo1}] //. Prod[a1_ * a2_, a3_] → Prod[a1, a3] * Prod[a2, a3] //.
  Prod[a1_, {i, 1, k - 1}] / Prod[a1_, {i, 1, k}] :→ (1 / a1 /. i → k)
```

$$\left\{-\frac{4}{3+\text{mu}},\ -\frac{4\left(\frac{1}{2}+3k+\frac{\text{mu}}{2}\right)^2\left(\frac{3}{2}+3k+\frac{\text{mu}}{2}\right)^2\,\text{Prod}\left[\frac{1}{256},\{i,1,-1+k\}\right]}{(3+\text{mu})\,(1+6k+\text{mu})^2\,(3+6k+\text{mu})^2\,\text{Prod}\left[\frac{1}{16},\{i,1,-1+k\}\right]\,\text{Prod}\left[\frac{1}{16},\{i,1,k\}\right]}\right\}$$

```
{quo0, quo1} = Factor[{quo0, quo1} /. Prod → Product]
```

$$\left\{-\frac{4}{3+\text{mu}},\ -\frac{4}{3+\text{mu}}\right\}$$

```
(* some initial values *)
Factor[
 Table[Det[Table[-KroneckerDelta[i, j] + FunctionExpand[Binomial[mu + i + j - 2, j]], {i, n},
     {j, n}]], {n, 2, 8, 2}] / Table[Det[Table[-KroneckerDelta[i, j] +
      FunctionExpand[Binomial[(mu + 2) + i + j - 2, j + 1]], {i, n}, {j, n}]], {n, 1, 7, 2}]]
```

$$\left\{-\frac{4}{3+\text{mu}},\ -\frac{4}{3+\text{mu}},\ -\frac{4}{3+\text{mu}},\ -\frac{4}{3+\text{mu}}\right\}$$

Conjecture 6

```
Clear[myC, myE, myF, myT, myS1, myS2, myP1, myP2, myG, D34];
myC[n_Integer?Positive] := ((-1)^n + 3) / 2 * Product[Floor[i / 2]! / i!, {i, 1, n}];
myE[n_Integer?Positive, mu_] := Pochhammer[mu + 1, n] *
   Product[(mu + 2 i + 6)^(2 * Floor[(i + 2) / 3]), {i, 1, Floor[3 / 2 * Floor[(n - 1) / 2] - 2]}] *
   Product[(mu + 2 i + 2 * Floor[3 / 2 * Floor[n / 2 + 1]] - 1)^
     (2 * Floor[Floor[n / 2] / 2 - (i - 1) / 3] - 1), {i, 1, Floor[3 / 2 * Floor[n / 2] - 2]}];
myF[m : (0 | 1), n_Integer?Positive, mu_] :=
  Product[(mu + 2 i + n + m)^(1 - 2 i - m), {i, 1, Floor[(n - 1) / 4]}] *
   Product[(mu - 2 i + 2 n - 2 m + 1)^(1 - 2 i - m), {i, 1, Floor[n / 4 - 1]}];
myF[n_Integer?Positive, mu_] := Which[
   EvenQ[n], myE[n, mu] * myF[0, n, mu],
   OddQ[n], myE[n, mu] * myF[1, n, mu] * Product[mu + 2 n + 2 i - 1, {i, 1, (n - 5) / 2}]];
myT[k_, mu_] :=
  -12 + 84 * k + 288 * k^2 - 5856 * k^3 + 20352 * k^4 - 41472 * k^5 + 55296 * k^6 + 10 * mu + 76 * k * mu -
   2176 * k^2 * mu + 9888 * k^3 * mu - 25344 * k^4 * mu + 41472 * k^5 * mu + 10 * mu^2 - 261 * k * mu^2 +
   1676 * k^2 * mu^2 - 5472 * k^3 * mu^2 + 11520 * k^4 * mu^2 - 10 * mu^3 + 115 * k * mu^3 -
   488 * k^2 * mu^3 + 1440 * k^3 * mu^3 + 2 * mu^4 - 15 * k * mu^4 + 76 * k^2 * mu^4 + k * mu^5;
myS1[n_Integer?Positive, mu_] := Sum[myT[k, mu] * 2^(6 k) * (mu + 8 k - 1) *
    Pochhammer[1 / 2, 2 k - 1]^2 * Pochhammer[(mu + 4 k + 2) / 2, 2 n - 2 k - 2] *
    Pochhammer[(mu + 5) / 2, 2 k - 3] * Pochhammer[(mu + 4 k + 2) / 2, k - 2] /
     ((2 k)! * Pochhammer[(mu + 6 k - 3) / 2, 3 k + 4]), {k, 1, n - 1}];
myS2[n_Integer?Positive, mu_] :=
  Sum[myT[k + 1 / 2, mu] * 2^(6 k) * (mu + 8 k + 3) *
    Pochhammer[1 / 2, 2 k]^2 * Pochhammer[(mu + 4 k + 4) / 2, 2 n - 2 k - 2] *
    Pochhammer[(mu + 5) / 2, 2 k - 2] * Pochhammer[(mu + 4 k + 4) / 2, k - 2] /
     ((2 k + 1)! * Pochhammer[(mu + 6 k + 1) / 2, 3 k + 5]), {k, 1, n - 1}];
myP1[n_Integer?Positive, mu_] := Together[
   2^(3 n - 1) * Pochhammer[(mu + 6 n - 3) / 2, 3 n - 2] / Pochhammer[(mu + 5) / 2, 2 n - 3] *
    (2^(-13) * mu * (mu - 1) * myS1[n, mu] + 1 / (mu + 3)^2 * Pochhammer[(mu + 2) / 2, 2 n - 2])];
myP2[n_Integer?Positive, mu_] := Together[
   2^(3 n - 1) * Pochhammer[(mu + 6 n + 1) / 2, 3 n - 1] / Pochhammer[(mu + 5) / 2, 2 n - 2] *
    (2^(-9) mu * (mu - 1) * myS2[n, mu] +
      (mu + 14) / ((mu + 7) * (mu + 9)) * Pochhammer[(mu + 4) / 2, 2 n - 2])];
myG[n_Integer?Positive, mu_] := If[EvenQ[n], myP2[n / 2, mu], myP1[(n + 1) / 2, mu]];
D34[n_Integer?Positive, mu_] := myC[n] * myF[n, mu] * myG[Floor[(n + 1) / 2], mu];

Table[
 Expand[D34[n, mu] - Det[Table[FunctionExpand[KroneckerDelta[i, j] + Binomial[i + j - 2 + mu, j]],
     {i, n}, {j, n}]]], {n, 10}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Here is a faster variant to compute D34:

```
Clear[myGfast, D34fast, myP1fast, myP2fast];
myP1fast[1, mu_] = 1;
myP1fast[2, mu_] = 3432 + 722 * mu + 45 * mu^2 + mu^3;
myP1fast[n_Integer /; n ≥ 3, mu_] := myP1fast[n, mu] = Together[
```

```
(-8 * (-10 + mu + 4 * n) * (-8 + mu + 4 * n) * (-14 + mu + 6 * n) * (-12 + mu + 6 * n) * (-10 + mu + 6 * n) *
    (-9 + mu + 8 * n) * (-31 + mu + 12 * n) * (-29 + mu + 12 * n) * (-27 + mu + 12 * n) *
    (-25 + mu + 12 * n) * (-23 + mu + 12 * n) * (-21 + mu + 12 * n) * (81 - 72 * n + 16 * n^2) *
    (49 - 56 * n + 16 * n^2) * (123 168 - 78 946 * mu + 18 939 * mu^2 - 2053 * mu^3 + 93 * mu^4 -
      mu^5 + (-9 + mu) * (70 956 - 30 208 * mu + 3989 * mu^2 - 158 * mu^3 + mu^4) * n +
      4 * (346 032 - 149 656 * mu + 21 803 * mu^2 - 1202 * mu^3 + 19 * mu^4) * n^2 +
      96 * (-9 + mu) * (1861 - 402 * mu + 15 * mu^2) * n^3 + 384 * (2753 - 606 * mu + 30 * mu^2) *
       n^4 + 41 472 * (-9 + mu) * n^5 + 55 296 * n^6) * myP1fast[-2 + n, mu] +
   (-13 + mu + 8 * n) * (-6 * (-15 + mu) * (-11 + mu) * (-9 + mu) * (77 272 834 343 040 -
         90 508 623 095 808 * mu + 46 786 094 223 720 * mu^2 - 14 041 912 717 156 * mu^3 +
         2 707 887 452 266 * mu^4 - 350 541 498 059 * mu^5 + 30 888 280 625 * mu^6 - 1 838 952 303 *
          mu^7 + 72 032 193 * mu^8 - 1 778 033 * mu^9 + 26 555 * mu^10 - 241 * mu^11 + mu^12) +
      (-13 + mu) * (615 591 764 176 296 960 - 787 691 318 438 414 592 * mu +
         453 271 146 257 615 040 * mu^2 - 154 970 921 382 725 880 * mu^3 +
         35 030 740 197 791 460 * mu^4 - 5 511 255 715 119 386 * mu^5 + 618 465 455 797 003 * mu^6 -
         49 890 145 667 170 * mu^7 + 2 877 469 024 970 * mu^8 - 116 576 723 262 * mu^9 +
         3 218 550 024 * mu^10 - 58 094 110 * mu^11 + 655 730 * mu^12 - 4400 * mu^13 + 13 * mu^14) *
       n + (43 790 163 197 061 415 680 - 55 769 554 581 921 674 496 * mu +
         32 100 807 569 482 408 752 * mu^2 - 11 046 065 343 390 418 896 * mu^3 +
         2 532 539 665 806 086 200 * mu^4 - 408 068 212 472 225 048 * mu^5 +
         47 486 735 062 736 003 * mu^6 - 4 036 853 597 489 641 * mu^7 + 250 606 824 181 572 * mu^8 -
         11 237 476 473 228 * mu^9 + 356 071 800 098 * mu^10 - 7 704 642 502 * mu^11 +
         108 621 484 * mu^12 - 941 780 * mu^13 + 4611 * mu^14 - 9 * mu^15) * n^2 + 2 * (-13 + mu) *
       (5 765 368 315 087 296 000 - 6 423 796 647 403 130 880 * mu + 3 186 986 194 272 026 736 * mu^2 -
         928 737 086 880 929 008 * mu^3 + 176 577 512 806 080 224 * mu^4 -
         23 002 876 518 214 396 * mu^5 + 2 097 912 117 891 133 * mu^6 - 134 465 197 774 532 * mu^7 +
         5 992 468 266 728 * mu^8 - 181 075 265 324 * mu^9 + 3 560 096 842 * mu^10 -
         42 928 700 * mu^11 + 293 696 * mu^12 - 1000 * mu^13 + mu^14) * n^3 +
      8 * (44 967 647 815 472 773 440 - 49 875 119 477 893 931 904 * mu + 24 771 543 294 236 452 512 *
          mu^2 - 7 277 588 373 063 623 552 * mu^3 + 1 407 087 781 066 080 464 * mu^4 -
         188 436 568 279 081 716 * mu^5 + 17 910 169 812 661 579 * mu^6 - 1 217 322 600 443 922 *
          mu^7 + 58 827 888 448 174 * mu^8 - 1 983 671 151 898 * mu^9 + 45 113 742 796 * mu^10 -
         655 655 046 * mu^11 + 5 605 730 * mu^12 - 24 666 * mu^13 + 41 * mu^14) * n^4 +
      32 * (-13 + mu) * (1 545 137 447 830 050 528 - 1 468 846 207 754 989 056 * mu +
         613 359 955 784 013 384 * mu^2 - 148 046 294 338 567 160 * mu^3 +
         22 867 645 137 091 796 * mu^4 - 2 363 768 523 778 396 * mu^5 + 166 104 951 524 749 * mu^6 -
         7 900 529 853 234 * mu^7 + 248 588 564 859 * mu^8 - 4 947 975 304 * mu^9 +
         57 722 923 * mu^10 - 345 266 * mu^11 + 785 * mu^12) * n^5 + 128 *
       (6 923 436 910 786 740 816 - 6 551 979 917 272 781 760 * mu + 2 741 775 205 145 125 620 * mu^2 -
         668 624 737 408 815 316 * mu^3 + 105 402 483 452 844 020 * mu^4 - 11 258 804 752 461 004 *
          mu^5 + 830 334 150 499 955 * mu^6 - 42 256 983 681 030 * mu^7 + 1 457 399 275 653 * mu^8 -
         32 763 679 904 * mu^9 + 447 520 681 * mu^10 - 3 258 554 * mu^11 + 9319 * mu^12) * n^6 +
      1024 * (-13 + mu) * (72 414 477 952 775 604 - 57 105 723 925 009 800 * mu +
         19 399 742 350 341 207 * mu^2 - 3 719 307 354 992 416 * mu^3 + 442 850 412 559 382 * mu^4 -
         33 955 375 237 500 * mu^5 + 1 681 820 711 178 * mu^6 - 52 507 834 704 * mu^7 +
         974 233 650 * mu^8 - 9 518 828 * mu^9 + 36 355 * mu^10) * n^7 +
      4096 * (204 759 442 490 425 380 - 160 746 724 570 083 012 * mu + 54 801 297 077 548 677 * mu^2 -
         10 648 677 530 738 482 * mu^3 + 1 300 829 127 395 384 * mu^4 -
         103 865 351 431 818 * mu^5 + 5 455 145 057 379 * mu^6 - 184 594 947 228 * mu^7 +
         3 811 103 508 * mu^8 - 42 749 540 * mu^9 + 194 248 * mu^10) * n^8 +
      49 152 * (-13 + mu) * (920 215 916 156 142 - 577 914 239 846 832 * mu +
         151 701 784 373 213 * mu^2 - 21 614 250 577 806 * mu^3 + 1 815 722 558 519 * mu^4 -
         91 353 917 016 * mu^5 + 2 663 224 490 * mu^6 - 40 669 644 * mu^7 + 245 586 * mu^8) * n^9 +
      196 608 * (1 693 595 159 851 230 - 1 058 822 980 698 432 * mu + 279 542 833 819 585 * mu^2 -
         40 572 445 515 984 * mu^3 + 3 526 446 267 001 * mu^4 - 187 021 320 840 * mu^5 +
```

```
          5 872 755 784 * mu^6 - 99 020 958 * mu^7 + 679 074 * mu^8) * n^10 +
       21 233 664 * (-13 + mu) * (550 446 775 412 - 258 091 315 032 * mu + 47 985 773 125 * mu^2 -
          4 496 668 860 * mu^3 + 222 288 724 * mu^4 - 5 456 352 * mu^5 + 51 547 * mu^6) * n^11 +
       28 311 552 * (1 958 821 138 060 - 914 306 594 496 * mu + 171 668 385 371 * mu^2 -
          16 540 689 390 * mu^3 + 859 090 262 * mu^4 - 22 689 546 * mu^5 + 236 549 * mu^6) * n^12 +
       21 403 533 312 * (-13 + mu) * (57 395 792 - 17 859 456 * mu + 1 964 631 * mu^2 -
          89 610 * mu^3 + 1425 * mu^4) * n^13 + 12 230 590 464 *
        (290 157 464 - 89 880 912 * mu + 10 081 119 * mu^2 - 483 594 * mu^3 + 8337 * mu^4) * n^14 +
       3 522 410 053 632 * (-13 + mu) * (12 823 - 1986 * mu + 69 * mu^2) * n^15 +
       3 522 410 053 632 * (19 340 - 2982 * mu + 111 * mu^2) * n^16 +
       380 420 285 792 256 * (-13 + mu) * n^17 +
       169 075 682 574 336 * n^18) * myP1fast[-1 + n, mu]) /
  ((-1 + n) * (-3 + 2 * n) * (-6 + mu + 4 * n) * (-5 + mu + 4 * n) * (-4 + mu + 4 * n) *
    (-3 + mu + 4 * n) *
    (-9 + mu + 6 * n) *
    (-7 + mu + 6 * n) *
    (-5 + mu + 6 * n) *
    (-17 + mu + 8 * n) *
    (-2 * (-2 619 750 + 910 279 * mu - 117 666 * mu^2 + 6856 * mu^3 - 168 * mu^4 + mu^5) +
      (-17 + mu) * (862 188 - 199 648 * mu + 14 213 * mu^2 - 302 * mu^3 + mu^4) * n +
      4 * (4 278 168 - 996 880 * mu + 77 747 * mu^2 - 2282 * mu^3 + 19 * mu^4) * n^2 +
      96 * (-17 + mu) * (6541 - 762 * mu + 15 * mu^2) * n^3 +
      384 * (9773 - 1146 * mu + 30 * mu^2) * n^4 + 41 472 * (-17 + mu) * n^5 + 55 296 * n^6))];
myP2fast[1, mu_] = mu + 14;
myP2fast[2, mu_] = 201 552 + 38 364 * mu + 2552 * mu^2 + 81 * mu^3 + mu^4;
myP2fast[n_Integer /; n ≥ 3, mu_] := myP2fast[n, mu] =
   Together[(-8 * (-8 + mu + 4 * n) * (-6 + mu + 4 * n) * (-12 + mu + 6 * n) * (-10 + mu + 6 * n) *
         (-8 + mu + 6 * n) * (-5 + mu + 8 * n) * (-25 + mu + 12 * n) * (-23 + mu + 12 * n) * (-21 + mu + 12 * n) *
         (-19 + mu + 12 * n) * (-17 + mu + 12 * n) * (-15 + mu + 12 * n) * (49 - 56 * n + 16 * n^2) *
         (25 - 40 * n + 16 * n^2) * (-((-3 + mu) * (2788 - 2196 * mu + 577 * mu^2 - 54 * mu^3 + mu^4)) +
           2 * (-5 + mu) * (7620 - 5536 * mu + 1253 * mu^2 - 86 * mu^3 + mu^4) * n +
           8 * (35 820 - 26 716 * mu + 6791 * mu^2 - 662 * mu^3 + 19 * mu^4) * n^2 +
           192 * (-5 + mu) * (601 - 222 * mu + 15 * mu^2) * n^3 + 768 * (863 - 336 * mu + 30 * mu^2) * n^4 +
           82 944 * (-5 + mu) * n^5 + 110 592 * n^6) * myP2fast[-2 + n, mu] +
       (-9 + mu + 8 * n) * (-((-11 + mu) * (-7 + mu) * (-5 + mu) * (-3 + mu) *
              (-941 137 562 880 + 1 369 543 037 568 * mu - 856 059 425 680 * mu^2 + 301 467 356 208 *
                 mu^3 - 65 925 560 840 * mu^4 + 9 300 152 544 * mu^5 - 851 420 265 * mu^6 + 49 707 939 *
                 mu^7 - 1 788 230 * mu^8 + 38 538 * mu^9 - 505 * mu^10 + 3 * mu^11)) +
            (-9 + mu) * (2 174 231 624 313 600 - 4 271 307 638 939 136 * mu + 3 746 500 640 981 808 * mu^2 -
               1 938 172 937 860 384 * mu^3 + 658 024 132 807 528 * mu^4 -
               154 336 161 708 664 * mu^5 + 25 631 896 940 311 * mu^6 - 3 038 647 883 536 * mu^7 +
               255 911 958 856 * mu^8 - 15 059 474 264 * mu^9 + 601 933 862 * mu^10 -
               15 728 672 * mu^11 + 258 008 * mu^12 - 2528 * mu^13 + 11 * mu^14) * n -
           4 * (-41 108 205 131 322 624 + 79 558 217 840 920 896 * mu - 69 190 984 849 287 408 * mu^2 +
               35 769 692 404 688 632 * mu^3 - 12 252 335 726 377 252 * mu^4 +
               2 933 722 316 842 738 * mu^5 - 504 752 475 079 572 * mu^6 + 63 145 862 893 203 * mu^7 -
               5 745 369 671 196 * mu^8 + 376 356 342 416 * mu^9 - 17 384 266 580 * mu^10 +
               548 066 954 * mu^11 - 11 274 720 * mu^12 + 143 142 * mu^13 - 1032 * mu^14 + 3 * mu^15) *
             n^2 + 4 * (-9 + mu) * (23 845 345 590 072 960 - 40 269 695 568 954 624 * mu +
               30 117 142 128 190 992 * mu^2 - 13 158 415 762 916 400 * mu^3 +
               3 730 778 330 679 232 * mu^4 - 721 067 843 021 868 * mu^5 + 97 108 500 711 985 * mu^6 -
               9 153 045 269 192 * mu^7 + 597 928 404 668 * mu^8 - 26 432 573 136 * mu^9 +
               759 984 806 * mu^10 - 13 416 552 * mu^11 + 134 684 * mu^12 - 676 * mu^13 + mu^14) * n^3 +
           16 * (195 243 602 402 676 096 - 325 694 901 477 820 032 * mu + 242 221 596 032 134 128 * mu^2 -
```

```
106 098 978 486 724 128 * mu^3 + 30 459 989 915 673 992 * mu^4 - 6 033 975 669 037 412 *
  mu^5 + 845 417 566 861 997 * mu^6 - 84 452 424 919 988 * mu^7 + 5 983 741 160 080 * mu^8 -
 295 319 349 276 * mu^9 + 9 821 158 066 * mu^10 - 208 695 676 * mu^11 +
 2 610 160 * mu^12 - 16 816 * mu^13 + 41 * mu^14) * n^4 + 64 * (-9 + mu) *
(14 622 810 947 299 008 - 20 883 005 872 697 088 * mu + 13 042 640 269 010 160 * mu^2 -
 4 687 978 533 249 048 * mu^3 + 1 073 821 472 622 084 * mu^4 - 163 965 505 744 412 * mu^5 +
 16 961 587 465 549 * mu^6 - 1 184 203 363 074 * mu^7 + 54 575 767 659 * mu^8 -
 1 588 856 808 * mu^9 + 27 087 171 * mu^10 - 236 578 * mu^11 + 785 * mu^12) * n^5 +
256 * (68 159 047 060 299 744 - 96 275 531 839 385 520 * mu + 59 912 949 582 646 848 * mu^2 -
 21 651 596 638 546 640 * mu^3 + 5 041 402 403 618 604 * mu^4 - 793 018 597 591 700 * mu^5 +
 85 896 040 596 299 * mu^6 - 6 405 365 947 182 * mu^7 + 323 083 532 589 * mu^8 -
 10 605 978 520 * mu^9 + 211 271 829 * mu^10 - 2 240 614 * mu^11 + 9319 * mu^12) * n^6 +
2048 * (-9 + mu) * (1 541 341 241 341 668 - 1 813 373 921 002 968 * mu +
 915 446 118 884 163 * mu^2 - 259 814 716 685 092 * mu^3 + 45 629 241 741 242 * mu^4 -
 5 143 009 129 752 * mu^5 + 373 337 413 062 * mu^6 - 17 038 328 436 * mu^7 +
 461 072 406 * mu^8 - 6 556 280 * mu^9 + 36 355 * mu^10) * n^7 +
8192 * (4 500 207 031 276 008 - 5 239 264 901 634 576 * mu + 2 640 189 965 261 667 * mu^2 -
 755 987 780 804 488 * mu^3 + 135 697 154 047 598 * mu^4 -
 15 878 627 119 200 * mu^5 + 1 219 280 284 095 * mu^6 - 60 190 646 760 * mu^7 +
 1 809 241 320 * mu^8 - 29 487 896 * mu^9 + 194 248 * mu^10) * n^8 +
98 304 * (-9 + mu) * (43 421 763 841 182 - 40 431 075 715 248 * mu + 15 676 365 711 905 * mu^2 -
 3 287 037 266 982 * mu^3 + 404 944 404 503 * mu^4 - 29 779 385 976 * mu^5 +
 1 265 065 310 * mu^6 - 28 070 508 * mu^7 + 245 586 * mu^8) * n^9 +
393 216 * (81 960 492 523 446 - 75 530 247 171 240 * mu + 29 304 166 747 543 * mu^2 -
 6 232 480 720 254 * mu^3 + 791 707 261 321 * mu^4 - 61 212 289 536 * mu^5 +
 2 795 677 186 * mu^6 - 68 402 040 * mu^7 + 679 074 * mu^8) * n^10 +
42 467 328 * (-9 + mu) * (56 814 548 324 - 39 259 013 448 * mu + 10 716 187 369 * mu^2 -
 1 468 655 040 * mu^3 + 105 783 400 * mu^4 - 3 770 148 * mu^5 + 51 547 * mu^6) * n^11 +
56 623 104 * (206 001 269 260 - 140 889 461 280 * mu + 38 650 036 817 * mu^2 -
 5 426 533 428 * mu^3 + 409 650 908 * mu^4 - 15 687 096 * mu^5 + 236 549 * mu^6) * n^12 +
42 807 066 624 * (-9 + mu) * (12 790 352 - 5 830 560 * mu + 935 607 * mu^2 -
 61 962 * mu^3 + 1425 * mu^4) * n^13 + 24 461 180 928 *
(65 459 144 - 29 536 992 * mu + 4 812 879 * mu^2 - 334 554 * mu^3 + 8337 * mu^4) * n^14 +
7 044 820 107 264 * (-9 + mu) * (6091 - 1374 * mu + 69 * mu^2) * n^15 +
7 044 820 107 264 * (9242 - 2064 * mu + 111 * mu^2) * n^16 +
760 840 571 584 512 * (-9 + mu) * n^17 + 338 151 365 148 672 * n^18) * myP2fast[-1 + n, mu]) /
((-1 + n) * (-1 + 2 * n) * (-4 + mu + 4 * n) * (-3 + mu + 4 * n) * (-2 + mu + 4 * n) *
 (-1 + mu + 4 * n) *
 (-5 + mu + 6 * n) * (-3 + mu + 6 * n) *
 (-1 + mu + 6 * n) * (-13 + mu + 8 * n) *
 (2 136 180 - 963 208 * mu + 161 921 * mu^2 - 12 281 * mu^3 + 391 * mu^4 - 3 * mu^5 +
  2 * (-13 + mu) * (298 788 - 89 728 * mu + 8309 * mu^2 - 230 * mu^3 + mu^4) * n +
  8 * (1 475 028 - 447 124 * mu + 45 455 * mu^2 - 1742 * mu^3 + 19 * mu^4) * n^2 +
  192 * (-13 + mu) * (3841 - 582 * mu + 15 * mu^2) * n^3 +
  768 * (5723 - 876 * mu + 30 * mu^2) * n^4 + 82 944 * (-13 + mu) * n^5 + 110 592 * n^6))];
myGfast[n_Integer?Positive, mu_] := If[EvenQ[n], myP2fast[n / 2, mu], myP1fast[(n + 1) / 2, mu]];
D34fast[n_Integer?Positive, mu_] := myC[n] * myF[n, mu] * myGfast[Floor[(n + 1) / 2], mu];

First /@ {Timing[Table[D34[n, mu], {n, 100}];], Timing[Table[D34fast[n, mu], {n, 100}];]}

{54.8674, 0.716046}
```

Conjecture 37 of (Krattenthaler 2005)

```
det37[n_, r_] := Together[Det[Table[-KroneckerDelta[i, j + r - 1] +
      FunctionExpand[Binomial[mu + i + j - 2, j + r - 1]], {i, n}, {j, n}]]];
eval37[n_, r_] := (-1)^((n - r) / 2) * 2^((n^2 + 6 n - 2 n r + r^2 - 4 r + 2) / 4) *
  Product[i!, {i, 0, r - 2}] *
  Product[((n - 2 i - 2)!)^2 / (((n - 2 i - 3) / 2)!)^2 / (n + 2 i)! / (n + 2 i + 2)!, {i, 0, (r - 3) / 2}] *
  (mu - r) * Pochhammer[(mu + 1) / 2, (n - r) / 2] *
  Product[Pochhammer[mu - r + i, n + r - 2 i + 1], {i, 1, r - 1}] *
  Product[i! * (i + 1)! / (2 i)! / (2 i + 2)!, {i, 0, (n - 1) / 2}] *
  Product[Pochhammer[mu / 2 + 3 i + r + 3 / 2, (n - 4 i - r - 2) / 2]^2 *
    Pochhammer[-mu / 2 - 3 n / 2 + r / 2 + 3 i + 1, (n - 4 i - r) / 2]^2, {i, 0, Floor[(n - r - 2) / 4]}]
```

```
Table[Together[det37[n, r] - eval37[n, r]], {n, 1, 7, 2}, {r, 1, n, 2}]
```

{{0}, {0, 0}, {0, 0, 0}, {0, 0, 0, 0}}

```
FullSimplify[Together[det37[3, 2] - eval37[3, 2]]]
```

$$\frac{1}{48} (-1 + mu)\, mu\, (1 + mu)\, (2 + mu) \left((1 + mu)\, (4 + mu) - 8\, i\, 2^{1/4}\, (-2 + mu)\, \text{Pochhammer}\left[\frac{1 + mu}{2}, \frac{1}{2}\right]\right)$$

```
Together[det37[3, 5] - eval37[3, 5]]
```

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. >>

Indeterminate