More on Relations 2

Wolfgang Schreiner
Research Institute for Symbolic Computation (RISC-Linz)
Johannes Kepler University, Linz, Austria

Wolfgang.Schreiner@risc.uni-linz.ac.at
http://www.risc.uni-linz.ac.at/people/schreine

Overview

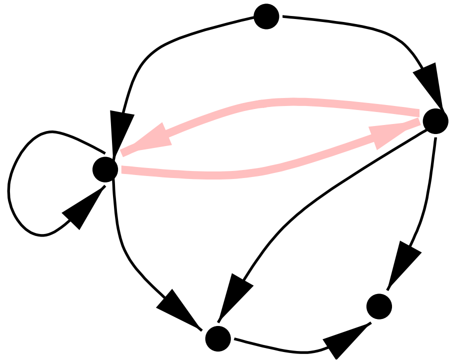
- Partial Order
- Quasi Order
- Total Order
- Hasse Diagrams
- Related Notions

Antisymmetry

Definition: A binary relation is antisymmetric, if different elements are not mutually related.

$$R \text{ is antisymmetric on } S :\Leftrightarrow$$
$$\forall x \in S, y \in S : (\langle x, y \rangle \in R \land \langle y, x \rangle \in R) \Rightarrow x = y.$$

Example: the following relation is not antisymmetric:



Partial Order

Definition: A binary relation on S is a partial order, if it is reflexive, antisymmetric, and transitive:

$$
\begin{aligned}
&R \text{ is partial order on } S :\Leftrightarrow \\
&\quad R \subseteq S \times S \ \wedge \\
&\quad R \text{ is reflexive on } S \ \wedge \\
&\quad R \text{ is antisymmetric on } S \ \wedge \\
&\quad R \text{ is transitive on } S.
\end{aligned}
$$

Example:

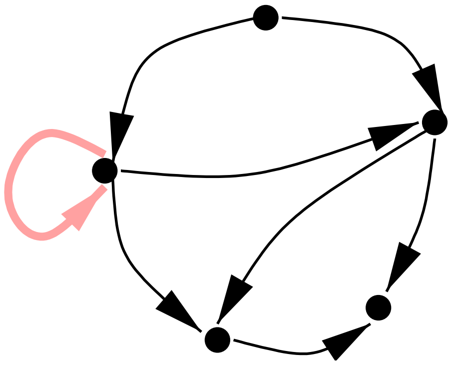
- $\subseteq$ is a partial order on $\mathbb{P}(S)$, for every set S.
- $|$ (divides) is a partial order on $\mathbb{N}$.

Irreflexivity

Definition: A binary relation is irreflexive, if no element is related to itself:

$$\begin{gathered} R \text{ is irreflexive on } S :\Leftrightarrow \\ \forall x \in S : \langle x, x \rangle \notin R. \end{gathered}$$

Example: the following relation is not irreflexive:

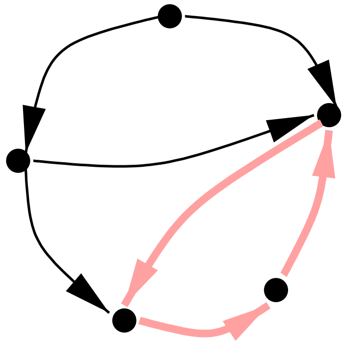


Quasi Order

Definition: A binary relation on S is a quasi order, if it is irreflexive and transitive:

$$
\begin{aligned}
& R \text{ is quasi order on } S :\Leftrightarrow \\
& \quad R \subseteq S \times S \wedge \\
& \quad R \text{ is irreflexive on } S \wedge \\
& \quad R \text{ is transitive on } S.
\end{aligned}
$$

Example: the following relation is not a quasi-order:



Quasi Order from Partial Order

Proposition: Let $\preceq$ be a partial order. Then the relation $\prec$ defined as

$$x \prec y :\Leftrightarrow x \preceq y \land x \neq y.$$

is a quasi-order:

$$\begin{array}{l} \forall S, \preceq : \preceq \text{ is partial order on } S \;\Rightarrow \\ \quad \prec \text{ is quasi order on } S \end{array}$$

Proof: exercise.

Quasi Orders and Anti-Symmetry

Proposition: Every quasi order is antisymmetric:

$$\forall S, \prec : \prec \text{ is quasi order on } S \; \Rightarrow \\ \prec \text{ is antisymmetric on } S.$$

Proof: Take arbitrary S and quasi order $\prec$ on S. Assume there exist $x \in S$ and $y \in S$ with $x \neq y$ such that $x \prec y$ and $y \prec x$. By transitivity, we have $x \prec x$ which contradicts the irreflexivity of $\prec$.

Only difference between partial orders and quasi orders is reflexivity versus irreflexivity.

Example

- $\leq$ is a partial order on $\mathbb{R}$.
- $<$ is a quasi-order on $\mathbb{R}$.
- $\subset$ (proper subset) is a quasi-order on $\mathbb{P}(S)$, for every set S.

Comparability

Definition: Two elements x and y are incomparable with respect to a relation $\preceq$ if neither $x \preceq y$ nor $y \preceq x$:

$$x \text{ and } y \text{ are incomparable w.r.t. } \preceq \; :\Leftrightarrow$$
$$x \not\preceq y \land y \not\preceq x.$$

Example: $\{0, 1\}$ and $\{1, 2\}$ are incomparable w.r.t. $\subseteq$.

Total Order

Definition: A partial order is a total order or linear order or chain if no elements are incomparable with respect to this order:

$$
\begin{aligned}
&R \text{ is total order on } S \;:\Leftrightarrow \\
&\quad R \text{ is partial order on } S \,\wedge \\
&\quad \neg\exists x \in S, y \in S : x \text{ and } y \text{ are incomparable w.r.t. } R.
\end{aligned}
$$

Example:

- $\subseteq$ is not a total order on $\mathbb{P}(\{0,1,2\})$.
- $\leq$ is a total order on $\mathbb{R}$.

Example

The lexicographic order on $\mathbb{N} \times \mathbb{N}$ is a total order:

$$x \preceq y :\Leftrightarrow x_0 < y_0 \vee (x_0 = y_0 \wedge x_1 \leq y_1)$$

$$\langle 0, 0\rangle \prec \langle 0, 1\rangle \prec \langle 0, 2\rangle \prec \ldots \prec \langle 1, 0\rangle \prec \langle 1, 1\rangle \prec \langle 1, 2\rangle \prec \ldots$$

Comparison first by first component, then by second component.

Example

Let $\preceq$ be a total order on an alphabet A, let $W_n := \mathbb{N}_n \to A$ be the set of all words with n letters, and let

$$\overline{w} := \textbf{such } u : \exists n \in \mathbb{N} : w \in W_n \wedge u \in W_{n-1} \wedge \forall i \in \mathbb{N}_{n-1} : u_i = w_{i+1}$$

The lexicographic order $\preceq_n \subseteq W_n \times W_n$ defined as

$$w \preceq_n u :\Leftrightarrow n = 0 \vee w_0 \prec u_0 \vee (w_0 = u_0 \wedge \overline{w} \preceq_{n-1} \overline{u})$$

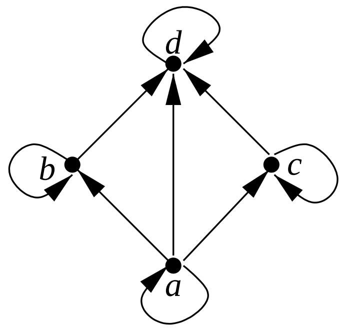
is a total order.

$$\text{“back”} \prec_4 \text{“bare”} \prec_4 \text{“base”} \prec_4 \text{“bear”} \prec_4 \text{“bend”} \prec_4 \text{“care”}$$

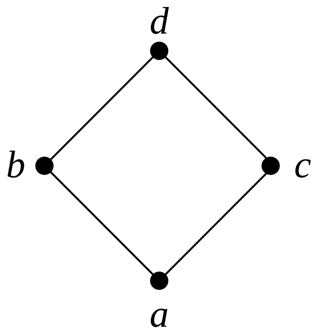
Compare letters in the order of their positions.

Hasse Diagrams

Directed Graph:

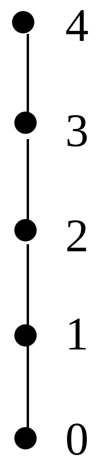


Hasse Diagram:



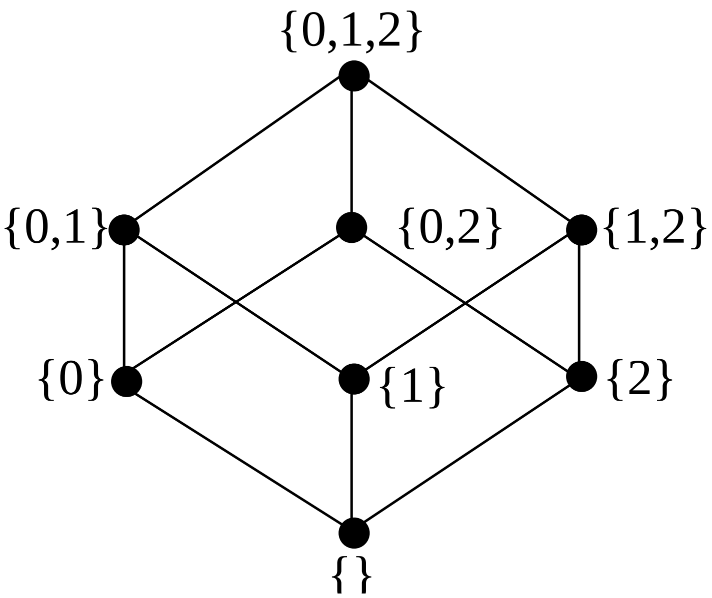
Example

This Hasse diagram illustrates the total order $\leq$ on $\mathbb{N}_5$.



Example

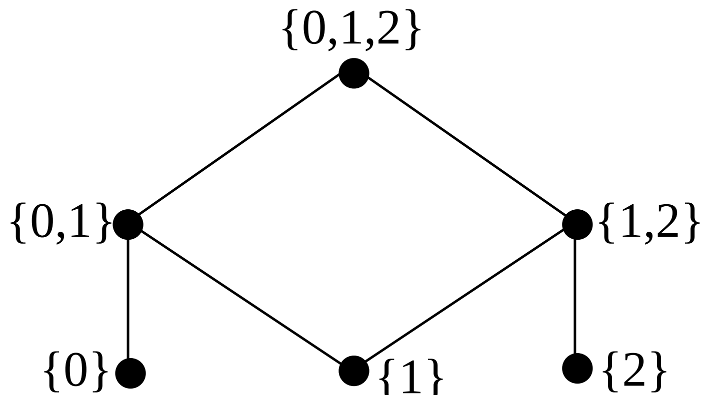
This Hasse diagram illustrates the partial order $\subseteq$ on $\mathbb{P}(\{1, 2, 3\})$:



Example

This Hasse diagram illustrates the partial order $\subseteq$ on

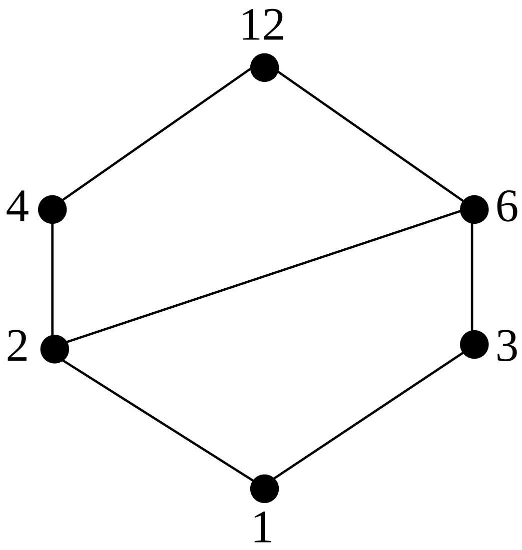
$$\{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 1, 2\}\}:$$



Example

Let $F_n := \{m \in \mathbb{N} : m|n\}$ (the factors of n).

The following Hasse diagram illustrates the partial order $|$ on F_{12}:



Least and Greatest Element

Definition: If x is an element of S such that x is less than or equal to any element of S, then x is called the least element of S:

$$x \text{ is least element of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\ x \in S \wedge \forall y \in S : x \preceq y.$$

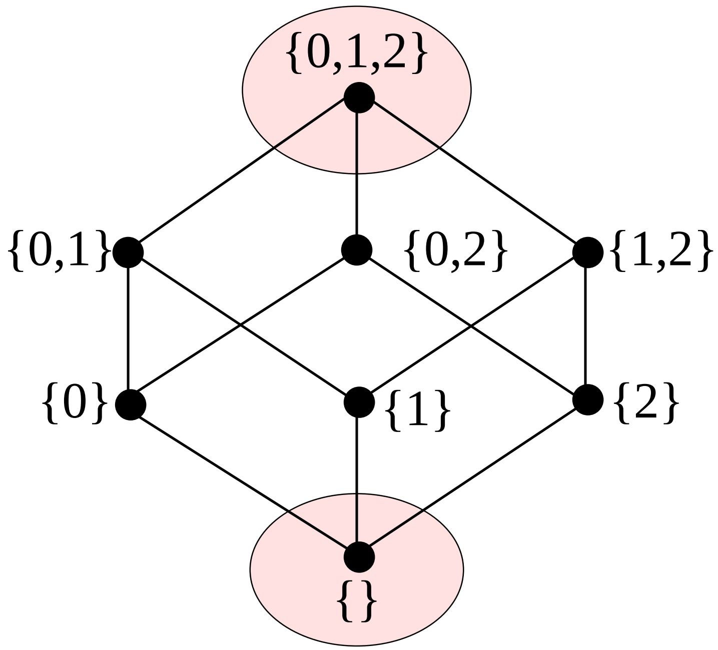
If x is an element of S such that x is greater than or equal to any element of S, then x is called the greatest element of S:

$$x \text{ is greatest element of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\ x \in S \wedge \forall y \in S : y \preceq x.$$

Least respectively greatest element is unique (if it exists).

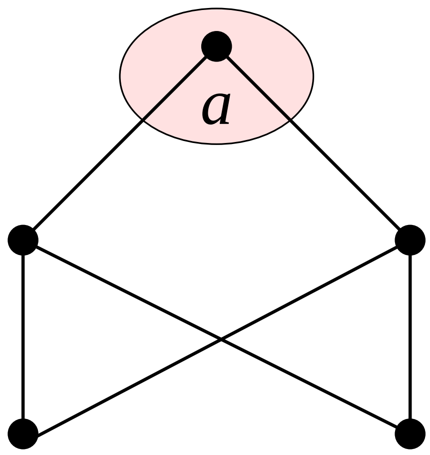
Example

$\emptyset$ is the least element of $\mathbb{P}(\{1,2,3\})$ with respect to $\subseteq$, $\{1,2,3\}$ is the greatest element:



Example

The following Hasse diagram denotes a partial order with greatest element a but without a least element:



Minimal and Maximal Element

Definition: If x is an element of S such that no element of S is smaller than x, then x is called a minimal element of S.

$$x \text{ is minimal element of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow$$
$$x \in S \wedge \forall y \in S : y \preceq x \Rightarrow y = x.$$

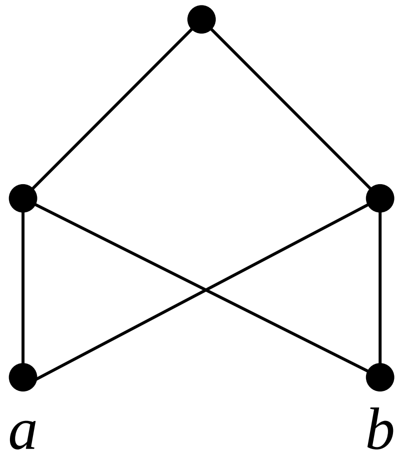
If x is an element of S such that no element of S is greater than x, then x is called a maximal element of S.

$$x \text{ is maximal element of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow$$
$$x \in S \wedge \forall y \in S : x \preceq y \Rightarrow x = y.$$

Minimal and maximal elements are not necessarily unique.

Example

The following Hasse diagram denotes a partial order with two minimal elements a and b:



Lower and Upper Bound

Definition: A value x is a lower bound of S, if it is less than or equal to any element of S:

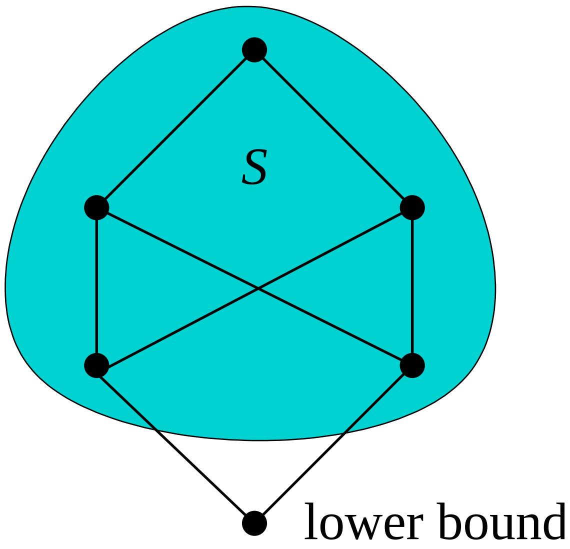
$$\begin{aligned}&x \text{ is lower bound of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\&\quad \forall y \in S : x \preceq y.\end{aligned}$$

A value x is an upper bound of S, if it is greater than or equal to any element of S:

$$\begin{aligned}&x \text{ is upper bound of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\&\quad \forall y \in S : y \preceq x.\end{aligned}$$

A lower (upper) bound of a set S need not be element of S.

Example



Infimum and Supremum

A value x is a infimum of S, if it is the greatest lower bound of S:

$$\begin{aligned}&x \text{ is infimum of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\&\quad x \text{ is lower bound of } S \text{ w.r.t. } \preceq \;\wedge\\&\quad \forall y : y \text{ is lower bound of } S \text{ w.r.t. } \preceq \Rightarrow y \preceq x.\end{aligned}$$

A value x is a supremum of S, if it is the least upper bound of S:

$$\begin{aligned}&x \text{ is supremum of } S \text{ w.r.t. } \preceq \;:\Leftrightarrow\\&\quad x \text{ is upper bound of } S \text{ w.r.t. } \preceq \;\wedge\\&\quad \forall y : y \text{ is upper bound of } S \text{ w.r.t. } \preceq \Rightarrow x \preceq y.\end{aligned}$$

Infimum respectively supremum of a set S need not be element of S.

Order Laws

Proposition: Let $\preceq$ be a partial order on S. Then for every x, the following holds:

$$\begin{array}{l}
x \text{ is least (greatest) element of } S \text{ w.r.t. } \preceq \;\Rightarrow \\
\quad x \text{ is minimal (maximal) element of } S \text{ w.r.t. } \preceq, \\
x \text{ is least (greatest) element of } S \text{ w.r.t. } \preceq \;\Rightarrow \\
\quad x \text{ is infimum (supremum) of } S \text{ w.r.t. } \preceq, \\
x \text{ is lower (upper) bound of } S \text{ w.r.t. } \preceq \wedge\, x \in S \;\;\Rightarrow \\
\quad x \text{ is least (greatest) element of } S \text{ w.r.t. } \preceq \;.
\end{array}$$

Summary

- Partial Order

  reflexive, antisymmetric, transitive.

- Quasi Order

  irreflexive, antisymmetric, transitive.

- Total Order

  Partial order that compares all elements.

- Hasse Diagram

  Compact visualization of partial order.

- Related Notions