Numbers

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The Natural Numbers

Overview

- Number Domains
  - Natural Numbers
  - Integer Numbers
  - Rational Numbers
  - Real Numbers
  - Complex Numbers
- Related Notions
  - Minimum and Maximum
  - Sum and Product
  - Binomials
  - Matrix Operations
  - Polynomial Operations

Natural Numbers

- The numbers of counting distinct objects.

  no object, one object, two objects, ...

- Axiomatic characterization.
  1. Describe properties of natural numbers.
  2. Peano axioms.
- Set-theoretic construction.
  1. Numbers are defined as sets.
  2. Definition satisfies Peano laws.

Two ways to introduce the natural numbers.

Peano Arithmetic

- Theory of natural numbers.
  - Object constant 0 (zero).
  - Unary function constant ' (successor).
- Axioms
  1. 0 is not the successor of any natural number:
  $$\forall x : x' \neq 0.$$
  2. Different natural numbers have different successors:
  $$\forall x, y : x' = y' \Rightarrow x = y.$$
  3. F holds for every number, if F holds for 0 and with every number also for its successor:
  $$(F[x \leftarrow 0] \wedge (\forall x : F \Rightarrow F[x \leftarrow x+1])) \Rightarrow \forall x : F.$$

Illustration

The natural numbers are a single infinite chain

$$0 \rightarrow 0' \rightarrow 0'' \rightarrow 0''' \rightarrow \ldots$$

1. Chain starts with 0.
2. Every application of ' yields a new natural number.
3. Every natural number is captured by the chain.

Construction from Sets

$$\begin{aligned} &0 := \emptyset; \\ &x' := x \cup \{x\}. \end{aligned}$$

Proof of first Peano law:

We prove $\forall x : x' \neq 0$. Take arbitrary x. By definition of 0 and $'$, we have to prove

$$x \cup \{x\} \neq \emptyset$$

which is true because $x \in (x \cup \{x\})$ but $x \notin \emptyset$.

Proof of second Peano law: see lecture notes.

Set of Natural Numbers

Definition: $\mathbb{N}$, the set of natural numbers

(omitted)

Proposition: $\mathbb{N}$ is the smallest set that satisfies the properties:

$$\begin{aligned} &0 \in \mathbb{N}; \\ &\forall x \in \mathbb{N} : x' \in \mathbb{N}; \\ &\forall F : \\ &\quad (F(0) \wedge \forall x \in \mathbb{N} : F(x) \Rightarrow F(x+1)) \Rightarrow \forall x \in \mathbb{N} : F(x). \end{aligned}$$

Third Peano law is a consequence of this proposition.

Auxiliary Notions

All further definitions work for both constructions of the naturals.

- Subsets of the natural numbers:

$$\begin{aligned}\mathbb{N}_{>0} &:= \{x \in \mathbb{N} : x > 0\};\\ \mathbb{N}_n &:= \{x \in \mathbb{N} : x < n\}.\end{aligned}$$

  e.g. $\mathbb{N}_3 = \{0, 1, 2\}$.

- Predecessor function:

$$x^- := \textbf{such } y : x = y'.$$

  e.g. $3^- = 2;\ 0^- = ?$.

Natural Number Arithmetic

Constants

$$1 := 0',\ 2 := 1';$$

Addition

$$x + y := \textbf{if } y = 0 \textbf{ then } x \textbf{ else } (x + y^-)'$$

Multiplication

$$x * y := \textbf{if } y = 0 \textbf{ then } 0 \textbf{ else } x + (x * y^-)$$

Total Order

$$\begin{aligned}&x \leq y :\Leftrightarrow\\ &\quad \textbf{if } x = 0 \textbf{ then } \mathrm{T}\\ &\quad \textbf{else if } y = 0 \textbf{ then } \mathrm{F}\\ &\quad \textbf{else } x^- \leq y^-\end{aligned}$$

Termination function $r(x, y) := y$ for recursive definitions.

Example

$$3 := 2'; 4 := 3'; 5 := 4'$$

$$\begin{aligned}&3 < 5 :\Leftrightarrow\\ &\quad 3^- < 5^- \Leftrightarrow 2 < 4 \Leftrightarrow\\ &\quad 2^- < 4^- \Leftrightarrow 1 < 3 \Leftrightarrow\\ &\quad 1^- < 3^- \Leftrightarrow 0 < 2 \Leftrightarrow\\ &\quad \mathrm{T}\end{aligned}$$

Recursive unfolding of definitions.

Natural Number Laws

For all natural numbers x and y, we have:

Addition

$$\begin{aligned}x + 0 &= x,\\ x + y' &= (x + y)';\end{aligned}$$

Multiplication

$$\begin{aligned}x * 0 &= 0,\\ x * y' &= x + (x * y);\end{aligned}$$

Total Order

$$\begin{aligned}0 \leq x &\Leftrightarrow \mathrm{T},\\ x \leq 0 &\Leftrightarrow x = 0,\\ x' \leq y' &\Leftrightarrow x \leq y.\end{aligned}$$

Natural Number Laws

For all natural numbers x, y, z, we have:

$$
\begin{aligned}
x+0 &= x,\\
x*1 &= x,\\
x+y &= y+x,\\
x*y &= y*x,\\
x+(y+z) &= (x+y)+z,\\
x*(y*z) &= (x*y)*z,\\
x*(y+z) &= (x*y)+(x*z),\\
x &\leq x,\\
(x\leq y \land y \leq x) &\Rightarrow x=y,\\
(x\leq y \land y \leq z) &\Rightarrow x\leq z.
\end{aligned}
$$

Order Predicates

In every domain with a binary relation $\leq$:

$$
\begin{aligned}
x<y &:\Leftrightarrow x\leq y \land x\neq y;\\
x>y &:\Leftrightarrow x \not\leq y;\\
x\geq y &:\Leftrightarrow x \not< y.
\end{aligned}
$$

We often write $a\leq x<b$ to denote $x\leq a \land x<b$ and similar for all other combinations of the order predicates.

Logic Evaluator

```
pred N(x) <=> Nat(x);

fun N0 = 0;
fun '(x: N) = +(x, 1);
fun ^-(x: N) = such(n in nat(0, x): =(x, '(n)), n);

fun N1 = '(N0);
fun N2 = '(N1);

fun +N(x: N, y: N) recursive y =
  if(=(y, N0), x, '(+N(x, ^-(y))));
fun *N(x: N, y: N) recursive y =
  if(=(y, N0), N0, +N(x, *N(x, ^-(y))));
pred <=N(x: N, y: N) recursive y <=>
  if(=(x, N0), true, if(=(y, N0), false, <=N(^-(x), ^-(y))));
```


Difference

Definition: z is a difference of x and y if $x=y+z$.

$$x-y := \textbf{such } z : x=z+y.$$

- Difference is not defined for every x and y:

  There is no z with $1=z+2$, thus $1-2$ is undefined.
- If a difference exists, it is unique:

  $$\forall x,y,z_0,z_1 : (x=z_0+y \land x=z_1+y) \Rightarrow z_0=z_1.$$
- If $x\geq y$, the difference of x and y is defined:

  $$\forall x,y : x\geq y \Rightarrow x=(x-y)+y.$$

Quotient and Remainder

Definition: quotient and remainder

$$\begin{aligned} x \operatorname{div} y &:= \textbf{such } q : \exists r : r < y \wedge x = (q * y) + r; \\ x \operatorname{mod} y &:= \textbf{such } r : \exists q : r < y \wedge x = (q * y) + r. \end{aligned}$$

Examples:

- $5 \operatorname{div} 3 = 1$, $5 \operatorname{mod} 3 = 2$.
- $15 \operatorname{div} 6 = 2$, $15 \operatorname{mod} 6 = 3$.
- $1 \operatorname{div} 3 = 0$, $1 \operatorname{mod} 3 = 1$.
- $0 \operatorname{div} 3 = 0$, $1 \operatorname{mod} 3 = 0$.

Properties of Quotient and Remainder

- Quotient and remainder are not defined for every x and y:

  $(x \operatorname{div} 0)$ and $(x \operatorname{mod} 0)$ are undefined for every x.
- If quotient respectively remainder exist, they are unique.

  $$\begin{aligned} &\forall x, y, q_0, q_1, r_0 < y, r_1 < y : \\ &\quad (x = (q_0 * y) + r_0 \wedge x = (q_1 * y) + r_1) \Rightarrow (q_0 = q_1 \wedge r_0 = r_1). \end{aligned}$$
- If the divisor is not null, quotient and remainder exist:

  $$\forall x, y \neq 0 : (\exists q, r : r < y \wedge x = (q * y) + r).$$
- We thus have the following relationship:

  $$\forall x, y \neq 0 : x = (x \operatorname{div} y) * y + (x \operatorname{mod} y).$$

Exponentiation

$$\begin{aligned} &\cdot^{\cdot} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \\ &x^n := \textbf{if } n = 0_{\mathbb{N}} \textbf{ then } 1 \textbf{ else } x * x^{n^-}. \end{aligned}$$

Termination function: $r(x, n) := n$

Example:

$\underline{5^3} = 5*(\underline{5^2}) = 5*(5*(\underline{5^1})) = 5*(5*(5*(\underline{5^0}))) = 5*(5*(5*(1))) = 5 * 5 * 5$.

More Notions

- x divides y if $x * z = y$ for some z:

  $$x|y :\Leftrightarrow \exists z : x * z = y.$$
- The greatest common divisor of x and y is the largest number that divides both x and y:

  $$\gcd(x, y) := \textbf{such } z : z|x \wedge z|y \wedge (\forall w : (w|x \wedge w|y) \Rightarrow w \leq z).$$
- The least common multiple of x and y is the smallest number that both x and y divide:

  $$\operatorname{lcm}(x, y) := \textbf{such } z : x|z \wedge y|z \wedge (\forall w : (x|w \wedge y|w) \Rightarrow z \leq w).$$

Examples

- $1|18,\ 2|18,\ 3|18,\ 6|18,\ 9|18,\ 18|18.$
- $1|24,\ 2|24,\ 3|24,\ 4|24,\ 6|24,\ 8|24,\ 12|24,\ 24|24.$
- $\gcd(18, 24) = 6.$
- $\gcd(16, 27) = 1.$
- $\mathrm{lcm}(4, 6) = 12.$
- $\mathrm{lcm}(8, 12) = 24.$

More Notions

- Two numbers are relatively prime if their gcd is 1:

$$x \text{ and } y \text{ are relatively prime} :\Leftrightarrow \gcd(x, y) = 1.$$

- A number greater than 1 is prime if its only divisors are 1 and itself:

$$x \text{ is prime } :\Leftrightarrow x > 1 \wedge (\forall y : y|x \Rightarrow (y = 1 \vee y = x)).$$

- 16 and 27 are relatively prime.
- (Only) the underlined numbers are prime:

$$0, 1, \underline{2}, \underline{3}, 4, \underline{5}, 6, \underline{7}, 8, 9, 10, \underline{11}, 12, \underline{13}, 14, 15, 16, \underline{17}, \ldots$$

Logic Evaluator

```
pred divides(x, y) <=> exists(z in nat(N0, y): =(*N(x, z), y));

fun gcd(x, y) =
  let(m = if(=(x, N0), y, x):
    such(z in nat(N0, m):
      and(divides(z, x), divides(z, y),
        forall(w in nat(+N(z, N1), m):
          or(not(divides(w, x)), not(divides(w, y))))),
      z));

pred isprime(x) <=>
  and(not(<=N(x, N1)),
    forall(y in nat(N0, x):
      implies(divides(y, x), or(=(y, N1), =(y, x)))));
```


The Integer Numbers

Motivation

- Not every pair of elements has a difference in $\mathbb{N}$:
  - $\neg\exists x : 0 = x + 1$.
  - $x - y :=$ **such** $z : x = z + y$.
  - $0 - 1$ is undefined.
- Introduce a set $\mathbb{Z}$ of integer numbers such that
  1. $\mathbb{N}$ can be "embedded" into $\mathbb{Z}$, and
  2. for all integers a and b there is an integer x with $a = x + b$ (and consequently $a - b$ is defined).

Set-theoretic construction on top of $\mathbb{N}$.

Definition

Idea:

- Representation: let $\langle a, b\rangle$ denote the difference between a and b.
- Normalize: $\langle a, 0_\mathbb{N}\rangle$ for non-negative ints, $\langle 0_\mathbb{N}, a\rangle$ for negative ones.

$$
\begin{aligned}
\mathbb{Z} &:= \mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{<0};\\
\mathbb{Z}_{\geq 0} &:= \{\langle x, 0_\mathbb{N}\rangle : x \in \mathbb{N}\};\\
\mathbb{Z}_{<0} &:= \{\langle 0_\mathbb{N}, x\rangle : x \in \mathbb{N}\backslash\{0_\mathbb{N}\}\};
\end{aligned}
$$

Constructor function:

$$
\begin{aligned}
&\mathrm{I} : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}\\
&\mathrm{I}(x, y) := \textbf{if } x \geq_\mathbb{N} y \textbf{ then } \langle x -_\mathbb{N} y, 0_\mathbb{N}\rangle \textbf{ else } \langle 0_\mathbb{N}, y -_\mathbb{N} x\rangle;
\end{aligned}
$$

Example

- The difference of 5 and 3 is denoted by
$$\mathrm{I}(5_\mathbb{N}, 3_\mathbb{N}) = \langle 2_\mathbb{N}, 0_\mathbb{N}\rangle = 2_\mathbb{Z}.$$
- The difference of 3 and 5 is denoted by
$$\mathrm{I}(3_\mathbb{N}, 5_\mathbb{N}) = \langle 0_\mathbb{N}, 2_\mathbb{N}\rangle = -2_\mathbb{Z}.$$

Now it remains to define the arithmetic operations.

Integer Arithmetic

Constants

$$0 := \mathrm{I}(0_\mathbb{N}, 0_\mathbb{N});\ \ 1 := \mathrm{I}(1_\mathbb{N}, 0_\mathbb{N});\ \ 2 := \mathrm{I}(2_\mathbb{N}, 0_\mathbb{N}).$$

Basic Arithmetic

$$
\begin{aligned}
x + y &:= \mathrm{I}(x_0 +_\mathbb{N} y_0, x_1 +_\mathbb{N} y_1);\\
x * y &:= \mathrm{I}((x_0 *_\mathbb{N} y_0) +_\mathbb{N} (x_1 *_\mathbb{N} y_1), (x_0 *_\mathbb{N} y_1) +_\mathbb{N} (x_1 *_\mathbb{N} y_0))\\
-x &:= \langle x_1, x_0\rangle;\\
x - y &:= x + (-y).
\end{aligned}
$$

Total Order

$$x \leq y :\Leftrightarrow (x_0 + y_1 <_\mathbb{N} y_0 + x_1).$$

Examples

- $-2 = -\langle 2_\mathbb{N}, 0_\mathbb{N}\rangle = \langle 0_\mathbb{N}, 2_\mathbb{N}\rangle$.
- $(-2) + 1 = (-\langle 2_\mathbb{N}, 0_\mathbb{N}\rangle) + \langle 1_\mathbb{N}, 0_\mathbb{N}\rangle = \langle 0_\mathbb{N}, 2_\mathbb{N}\rangle + \langle 1_\mathbb{N}, 0_\mathbb{N}\rangle = \mathrm{I}(0_\mathbb{N} + 1_\mathbb{N}, 2_\mathbb{N} + 0_\mathbb{N}) = \mathrm{I}(1_\mathbb{N}, 2_\mathbb{N}) = \langle 0_\mathbb{N}, 1_\mathbb{N}\rangle = -\langle 1_\mathbb{N}, 0_\mathbb{N}\rangle = -1$.
- $2 * 3 = \langle 2_\mathbb{N}, 0_\mathbb{N}\rangle * \langle 3_\mathbb{N}, 0_\mathbb{N}\rangle = \mathrm{I}((2_\mathbb{N} *_\mathbb{N} 3_\mathbb{N}) + (0_\mathbb{N} *_\mathbb{N} 0_\mathbb{N}), (2_\mathbb{N} *_\mathbb{N} 0_\mathbb{N}) + (3_\mathbb{N} *_\mathbb{N} 0_\mathbb{N})) = \mathrm{I}(6_\mathbb{N}, 0_\mathbb{N}) = \langle 6_\mathbb{N}, 0_\mathbb{N}\rangle = 6$.
- $(-2) * 3 = (-\langle 2_\mathbb{N}, 0_\mathbb{N}\rangle) * \langle 3_\mathbb{N}, 0_\mathbb{N}\rangle = \langle 0_\mathbb{N}, 2_\mathbb{N}\rangle * \langle 3_\mathbb{N}, 0_\mathbb{N}\rangle = \mathrm{I}((0_\mathbb{N} *_\mathbb{N} 3_\mathbb{N}) + (2_\mathbb{N} *_\mathbb{N} 0_\mathbb{N}), (0_\mathbb{N} *_\mathbb{N} 0_\mathbb{N}) + (2_\mathbb{N} *_\mathbb{N} 3_\mathbb{N})) = \mathrm{I}(0_\mathbb{N}, 6_\mathbb{N}) = \langle 0_\mathbb{N}, 6_\mathbb{N}\rangle = -\langle 6_\mathbb{N}, 0_\mathbb{N}\rangle = -6$.

Substituting and evaluating the definitions.

More Arithmetic

$$|x| := \textbf{if } 0 \le x \textbf{ then } x \textbf{ else } -x;$$

$$\mathrm{sign}(x) := \textbf{if } x = 0 \textbf{ then } 0 \textbf{ else if } 0 \le x \textbf{ then } 1 \textbf{ else } -1;$$

$$\begin{aligned} &x \operatorname{div} y := \textbf{ such } q : \exists r : \\ &\quad |r| < |y| \land x = q * y + r \land (\mathrm{sign}(r) = 0 \lor \mathrm{sign}(r) = \mathrm{sign}(y)); \end{aligned}$$

$$\begin{aligned} &x \bmod y := \textbf{ such } r : \exists q : \\ &\quad |r| < |y| \land x = q * y + r \land (\mathrm{sign}(r) = 0 \lor \mathrm{sign}(r) = \mathrm{sign}(y)). \end{aligned}$$

Integer Laws

Same laws that also hold in $\mathbb{N}$.

Proposition:

$$\forall x \in \mathbb{Z}, y \in \mathbb{Z} : x + y = y + x.$$

Proof: Take arbitrary $x \in \mathbb{Z}, y \in \mathbb{Z}$. We have

$$\begin{aligned} x + y &= \text{(definition of } +) \\ \mathrm{I}(x_0 +_\mathbb{N} y_0, x_1 +_\mathbb{N} y_1) &= \text{(commutativity of } +_\mathbb{N}) \\ \mathrm{I}(y_0 +_\mathbb{N} x_0, y_1 +_\mathbb{N} x_1) &= \text{(definition of } +) \\ y + x. & \end{aligned}$$

Difference

Proposition: For every integer x and y the difference is defined:

$$\forall x \in \mathbb{Z}, y \in \mathbb{Z} : x = (x - y) + y.$$

Proof: Take arbitrary $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. We have

$$\begin{aligned} (x - y) + y &= \text{(definition of } -) \\ (x + (-y)) + y &= \text{(associativity of } +) \\ x + ((-y + y)) &= (*) \\ x + 0 &= \text{(definition of } + \text{ and } 0) \\ x. & \end{aligned}$$

Proof (Continued)

(*) We show $-y + y = 0$:

$$\begin{aligned}
-y + y &= \text{(definition of } -\text{)} \\
\langle y_1, y_0 \rangle + y &= \text{(definition of } +\text{)} \\
\mathrm{I}(y_1 +_{\mathbb{N}} y_0, y_0 +_{\mathbb{N}} y_1) &= \text{(definition of } \mathrm{I}, \text{computation in } \mathbb{N}\text{)} \\
\langle 0_{\mathbb{N}}, 0_{\mathbb{N}} \rangle &= \text{(definition of } 0\text{)} \\
&\ 0.
\end{aligned}$$

Integer Numbers

- Difference of two numbers is always is well-defined.
- Definition on top of $\mathbb{N}$.
  - $\mathbb{Z} \subseteq \mathbb{N} \times \mathbb{N}$.
  - $\mathbb{N} \not\subseteq \mathbb{Z}$!
- We will see later how to "embed" $\mathbb{N}$ into $\mathbb{Z}$.
- We will see later a "better" construction of $\mathbb{Z}$.

The Rational Numbers

Motivation

- Not every pair of elements has a quotient in $\mathbb{Z}$:
  - $\neg\exists x : 2 = x * 3$.
  - $x/y :=$ **such** $z : x = z * y$.
  - $2/3$ is undefined.
- Introduce a set $\mathbb{Q}$ of rational numbers such that
  1. $\mathbb{Z}$ can be "embedded" into $\mathbb{Q}$, and
  2. for all rationals a and b there is an rational x with $a = x * b$ (and consequently a/b is defined).

Set-theoretic construction on top of $\mathbb{Z}$.

Definition

Idea:

- Representation: let $\langle a, b\rangle$ denote the quotient between a and b.
- Normalization: a and b are relatively prime and b is positive.

Conversion functions:

$$\begin{aligned}&\mathrm{Z} : \mathbb{N} \to \mathbb{Z}_{\geq 0},\ \mathrm{Z}(x) := \langle x, 0_{\mathbb{N}}\rangle;\\ &\mathrm{N} : \mathbb{Z} \to \mathbb{N},\ \mathrm{N}(x) := |x|_0;\end{aligned}$$

Set definition:

$$\begin{aligned}\mathbb{Q} := \{&\langle x, y\rangle : x \in \mathbb{Z} \wedge y \in \mathbb{Z}_{>0} \wedge\\ &\mathrm{N}(x) \text{ and } \mathrm{N}(y) \text{ are relatively prime}\}.\end{aligned}$$

Constructor function

$$\begin{aligned}&\tfrac{*}{*} : \mathbb{Z} \times \mathbb{Z}_{>0} \to \mathbb{Q}\\ &\frac{x}{y} := \langle \mathrm{sign}(x *_{\mathbb{Z}} y) *_{\mathbb{Z}} (|x| \ \mathrm{div}_{\mathbb{Z}}\ g), |y| \ \mathrm{div}_{\mathbb{Z}}\ g\rangle\\ &\quad \textbf{where } g = \mathrm{Z}(\gcd(\mathrm{N}(x), \mathrm{N}(y))).\end{aligned}$$

Example:

- $\frac{10_{\mathbb{Z}}}{6_{\mathbb{Z}}} = \langle 1_{\mathbb{Z}} *_{\mathbb{Z}} (10_{\mathbb{Z}} \ \mathrm{div}_{\mathbb{Z}}\ 2_{\mathbb{Z}}), 6_{\mathbb{Z}} \ \mathrm{div}_{\mathbb{Z}}\ 2\rangle = \langle 5_{\mathbb{Z}}, 3_{\mathbb{Z}}\rangle.$
- $\frac{-_{\mathbb{Z}}10_{\mathbb{Z}}}{6_{\mathbb{Z}}} = \langle -_{\mathbb{Z}}1_{\mathbb{Z}} *_{\mathbb{Z}} (10_{\mathbb{Z}} \ \mathrm{div}_{\mathbb{Z}}\ 2_{\mathbb{Z}}), 6_{\mathbb{Z}} \ \mathrm{div}_{\mathbb{Z}}\ 2\rangle \ = \langle -_{\mathbb{Z}}5_{\mathbb{Z}}, 3_{\mathbb{Z}}\rangle.$
- $\frac{-_{\mathbb{Z}}10_{\mathbb{Z}}}{-_{\mathbb{Z}}6_{\mathbb{Z}}} = \langle 1_{\mathbb{Z}} *_{\mathbb{Z}} (10_{\mathbb{Z}} \ \mathrm{div}_{\mathbb{Z}}\ 2_{\mathbb{Z}}), 6_{\mathbb{Z}} \ \mathrm{div}_{\mathbb{Z}}\ 2\rangle \ = \langle 5_{\mathbb{Z}}, 3_{\mathbb{Z}}\rangle.$

Numerator and Denominator

Let $r \in \mathbb{Q}$ and take x and y such that $r = \langle x, y\rangle$. We call x the numerator of r and y its denominator:

$$\begin{aligned}\mathrm{numerator}(r) &:= \textbf{such } x : \exists y : r \in \mathbb{Q} \wedge r = \langle x, y\rangle;\\ \mathrm{denominator}(r) &:= \textbf{such } y : \exists x : r \in \mathbb{Q} \wedge r = \langle x, y\rangle.\end{aligned}$$

Numerator and denominator are uniquely defined.

Rational Arithmetic

Definition:

See Lecture Notes!

Satisfies same laws as integer arithmetic.

Quotient

Proposition: For all rationals x and $y \neq 0$ the quotient is defined:

$$\forall x \in \mathbb{Q}, y \in \mathbb{Q}\backslash\{0\} : x = (x/y) * y.$$

Proof: see lecture notes.

Proposition: Between any two rational numbers, there is another rational number:

$$\forall x \in \mathbb{Q}, y \in \mathbb{Q} : x < y \Rightarrow \exists z \in \mathbb{Q} : x < z < y.$$

Proof: Take $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$ with $x < y$. Then $x < (x+y)/2 < y$.

Rational Numbers

- Quotient of two numbers is always is well-defined.
- Between any two rationals, there is another rational.
- Definition on top of $\mathbb{Z}$.
  - $\mathbb{Q} \subseteq \mathbb{Z} \times \mathbb{Z}$.
  - $\mathbb{Z} \not\subseteq \mathbb{Q}$!
- We will see later how to "embed" $\mathbb{Z}$ into $\mathbb{Q}$.
- We will see later a "better" construction of $\mathbb{Q}$.

The Real Numbers

Motivation

- Not every element has a square root in $\mathbb{Q}$:
  - $\neg\exists x : x * x = 2$.
  - $\sqrt{x} :=$ **such** $z : x = z * z$.
  - $\sqrt{2}$ is undefined.
- Proof: see lecture notes.
- Introduce a set $\mathbb{R}$ of real numbers such that
  1. $\mathbb{Q}$ can be "embedded" into $\mathbb{R}$, and
  2. for every non-negative real number a there is a real number x with $a = x*x$ (and consequently $\sqrt{a}$ is defined).

Axiomatic characterization.

Theory of Reals

- Object constants 0 and 1.
- Unary function constant $^{-1}$.
- Binary function constants $+, -, *$.
- Binary predicate constant $\leq$.
- Axioms: see lecture notes.

Existence of Real Roots

Proposition: In $\mathbb{R}$ every non-negative number has an n-th root:

$$\forall a \in \mathbb{R}_{\geq 0}, n \in \mathbb{N}_{>0} : \exists x \in \mathbb{R} : x^n = a.$$

Definition:

$$\sqrt[n]{x} := \textbf{such } y : x^n = y$$
$$\sqrt{x} := \sqrt[2]{x}.$$

Consequence:

$$\forall a \in \mathbb{R}_{\geq 0}, n \in \mathbb{N}_{>0} : (\sqrt[n]{a})^n = a.$$

All roots of non-negative reals are well-defined.

Real Numbers

- Roots of non-negative reals are well defined.
- Axiomatic characterization.
- There are also “constructive” definitions of $\mathbb{R}$.
- We will see later how to “embed” $\mathbb{Q}$ into $\mathbb{R}$.

The Complex Numbers

Motivation

- Not every element has a square root in $\mathbb{R}$:
  - $\neg\exists x : x * x = -1$.
  - $\sqrt{x} :=$ **such** $z : x = z * z$.
  - $\sqrt{-1}$ is undefined.
- Proof: We prove $\forall x \in \mathbb{R} : x * x \neq -1$. Take arbitrary $x \in \mathbb{R}$. If $x \geq 0$, then $x * x \geq 0$. If $x < 0$, then also $x * x \geq 0$.
- Introduce a set $\mathbb{C}$ of complex numbers such that
  1. $\mathbb{R}$ can be "embedded" into $\mathbb{C}$, and
  2. for every complex number a there is a complex number x with $a = x * x$ (and consequently $\sqrt{a}$ is defined).

Set-theoretic definition on top of $\mathbb{R}$.

Definition

$$\mathbb{C} := \mathbb{R} \times \mathbb{R}.$$

Constructor Function:

$$\begin{aligned} &\_ + \_\mathrm{i} : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \\ &x + y\mathrm{i} := \langle x, y \rangle. \end{aligned}$$

Let $c \in \mathbb{C}$ and take x and y such that $c = \langle x, y \rangle$. We call x the real part of c and y its imaginary part:

$$\begin{aligned} \mathrm{real}(c) &:= \textbf{such } x : \exists y : c \in \mathbb{C} \wedge c = \langle x, y \rangle; \\ \mathrm{imaginary}(c) &:= \textbf{such } y : \exists x : c \in \mathbb{C} \wedge c = \langle x, y \rangle. \end{aligned}$$

Complex Number Operations

Constants

$$0 := 0_{\mathbb{R}} + 0_{\mathbb{R}}\mathrm{i};\ 1 := 1_{\mathbb{R}} + 0_{\mathbb{R}}\mathrm{i};\ 2 := 2_{\mathbb{R}} + 0_{\mathbb{R}}\mathrm{i}; i := 0_{\mathbb{R}} + 1_{\mathbb{R}}\mathrm{i}.$$

Arithmetic

$$\begin{aligned} x + y &:= (x_0 +_{\mathbb{R}} y_0) + (x_1 +_{\mathbb{R}} y_1)\mathrm{i}; \\ x - y &:= (x_0 -_{\mathbb{R}} y_0) + (x_1 -_{\mathbb{R}} y_1)\mathrm{i}; \\ x * y &:= ((x_0 *_{\mathbb{R}} y_0) -_{\mathbb{R}} (x_1 *_{\mathbb{R}} y_1)) + ((x_0 *_{\mathbb{R}} y_1) +_{\mathbb{R}} (x_1 *_{\mathbb{R}} y_0))\mathrm{i} \\ x / y &:= (((x_0 *_{\mathbb{R}} y_0) +_{\mathbb{R}} (x_1 *_{\mathbb{R}} y_1)) /_{\mathbb{R}}\ d) + \\ &\quad\ (((x_1 *_{\mathbb{R}} y_0) -_{\mathbb{R}} (x_0 *_{\mathbb{R}} y_1)) /_{\mathbb{R}}\ d)\mathrm{i} \\ &\qquad \textbf{where } d = (y_0 *_{\mathbb{R}} y_0) +_{\mathbb{R}} (y_1 *_{\mathbb{R}} y_1). \end{aligned}$$

Example

- $i * i = (0_{\mathbb{R}} + 1_{\mathbb{R}}\mathrm{i}) * (0_{\mathbb{R}} + 1_{\mathbb{R}}\mathrm{i}) = (0_{\mathbb{R}} -_{\mathbb{R}} 1_{\mathbb{R}}) + (0_{\mathbb{R}} +_{\mathbb{R}} 0_{\mathbb{R}})\mathrm{i} = (-_{\mathbb{R}} 1_{\mathbb{R}}) + 0_{\mathbb{R}}\mathrm{i} = -(1_{\mathbb{R}} + 0_{\mathbb{R}}\mathrm{i}) = -1$.
- $2 * 3 = (2_{\mathbb{R}} + 0_{\mathbb{R}}\mathrm{i}) * (3_{\mathbb{R}} + 0_{\mathbb{R}}\mathrm{i}) = (6_{\mathbb{R}} -_{\mathbb{R}} 0_{\mathbb{R}}) + (0_{\mathbb{R}} +_{\mathbb{R}} 0_{\mathbb{R}})\mathrm{i} = 6_{\mathbb{R}} + 0_{\mathbb{R}}\mathrm{i} = 6$.
- $(1_{\mathbb{R}} + 3_{\mathbb{R}}\mathrm{i}) * (2_{\mathbb{R}} + 4_{\mathbb{R}}\mathrm{i}) = (2_{\mathbb{R}} -_{\mathbb{R}} 12_{\mathbb{R}}) + (4_{\mathbb{R}} +_{\mathbb{R}} 6_{\mathbb{R}})\mathrm{i} = (-_{\mathbb{R}} 10_{\mathbb{R}}) + 10_{\mathbb{R}}\mathrm{i}$.

Fundamental Theorem of Algebra

For every $a_0 \in \mathbb{C}, \ldots, a_{n-1} \in \mathbb{C}$, there exists an $x \in \mathbb{C}$ such that

$$a_0 * x^0 + \ldots + a_{n-1} * x^{n-1} = 0.$$

- $\mathbb{C}$ is complete with respect to $+$ and $*$.
  - Every equation with $+$ and $*$ has a solution in $\mathbb{C}$.

No further extension required.

Complex Square Root

$$\begin{array}{l}\sqrt{x} := \\ \quad \textbf{if } x_1 \geq_{\mathbb{R}} 0_{\mathbb{R}} \textbf{ then } u + v\mathrm{i} \textbf{ else } u + (-_{\mathbb{R}} v)\mathrm{i} \\ \quad \textbf{where} \\ \qquad u = \sqrt{\left(x_0 +_{\mathbb{R}} \sqrt{{x_0}^2 +_{\mathbb{R}} {x_1}^2}\right) /_{\mathbb{R}} 2_{\mathbb{R}}} \\ \qquad v = \sqrt{\left(-_{\mathbb{R}} x_0 +_{\mathbb{R}} \sqrt{{x_0}^2 +_{\mathbb{R}} {x_1}^2}\right) /_{\mathbb{R}} 2_{\mathbb{R}}}.\end{array}$$

Proposition: the (positive or negative) root of x squared equals x.

$$\begin{array}{l}\forall x \in \mathbb{C}: \\ \quad \textbf{let } r = \sqrt{x}: \\ \qquad x = r * r \land x = (-r) * (-r).\end{array}$$

Complex Conjugate

Definition: the complex conjugate.

$$\overline{x} := x_0 + (-_{\mathbb{R}} x_1)\mathrm{i}.$$

Example: $\overline{3_{\mathbb{R}} + 5_{\mathbb{R}}\mathrm{i}} = 3_{\mathbb{R}} + (-_{\mathbb{R}} 5_{\mathbb{R}})\mathrm{i}$.

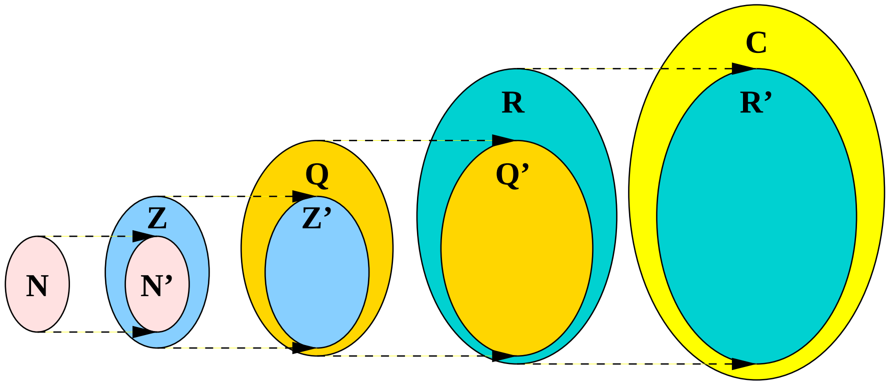
Proposition: For every $x \in \mathbb{C}, y \in \mathbb{C}, z \in \mathbb{C}$, the following holds:

$$\begin{aligned} \overline{\overline{x}} &= x, \\ \overline{x + y} &= \overline{x} + \overline{y}, \\ \overline{x * y} &= \overline{x} * \overline{y}, \\ y \neq 0 \Rightarrow \overline{x / y} &= \overline{x} / \overline{y}, \end{aligned}$$

Complex Numbers

- $\mathbb{C}$ is complete with respect to $+$ and $*$.
  - All equations have solutions in $\mathbb{C}$.
- Definition on top of $\mathbb{R}$.
  - $\mathbb{C} = \mathbb{R} \times \mathbb{R}$.
  - $\mathbb{R} \not\subseteq \mathbb{C}$!
  - Cartesian coordinates.
- We will see later how to "embed" $\mathbb{R}$ into $\mathbb{C}$.
- We will see later another definition of $\mathbb{C}$ (polar coordinates).

Relationship between Number Domains



Each domain has an "identical twin" in subsequent domain.

Related Notions

Minimum and Maximum Quantifier

Definition: If x is a variable and F is a formula, then the following are terms with bound variable x:

$$\min_x F$$
$$\max_x F$$

The value of the first term is the smallest value of x such that F holds; the value of the second term is the largest such value:

$$\min_x F := \textbf{such } x : F \wedge (\forall y : F[x \leftarrow y] \Rightarrow x \leq y);$$
$$\max_x F := \textbf{such } x : F \wedge (\forall y : F[x \leftarrow y] \Rightarrow x \geq y).$$

Quantifiers for every domain with a binary predicate $\leq$.

Minimum and Maximum Function

$$\min(S) := \min_x x \in S;$$
$$\max(S) := \max_x x \in S;$$

Examples:

- We have
$$\max_x(\mathrm{isprime}(x) \wedge x|100) = 5.$$
- The value of
$$\min(\{1/x : x \in \mathbb{N}_{>0}\})$$
is undefined, because for every x in $\{1/1, 1/2, 1/3, 1/4, \ldots\}$ there is always an y in this set with $y < x$, namely $1/(x+1)$.

Sum Quantifier

Definition: If x is a variable, F is a formula and T is a term, then the following is a term with bound variable x:

$$\sum_{x,F} T.$$

The value of this term is 0, if F does not hold for any x; otherwise it is, for every x that satisfies F, the sum of the value of T and of the value of the term for all other x:

$$(\forall x : \neg F) \Rightarrow \textstyle\sum_{x,F} T = 0;$$
$$(\forall y : F[x \leftarrow y] \Rightarrow \textstyle\sum_{x,F} T = T[x \leftarrow y] + \sum_{x,F \wedge x \neq y} T).$$

Examples

$$\sum_{1 \leq i \leq n} i^2 = \sum_{i,(i \in \mathbb{N} \wedge 1 \leq i \wedge i \leq n)} i^2$$

$$\sum_{1 \leq i \leq 0} i^2 = 0$$

$$\sum_{1 \leq i \leq 5} i^2 = 1^2 + \sum_{2 \leq i \leq 5} i^2$$

$$\sum_{1 \leq i \leq 5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2;$$

Examples

$$\sum_{1 \leq i \leq 9} (x-i)^2 = (x-1)^2 + \sum_{2 \leq i \leq 9} (x-i)^2$$

$$\sum_{1 \leq i \leq n} x^i = \sum_{1 \leq i \leq n \wedge \mathrm{iseven}(i)} x^i + \sum_{1 \leq i \leq n \wedge \mathrm{isodd}(i)} x^i$$

Identities which are true for every x.

Example: Decimal Number Representation

Let $a := [3, 1, 2, 9, 0, 7]$. We have

$$\sum_{0 \leq i \leq 5} a_i * 10^i = 709213.$$

In general, for any finite sequence d of "decimal digits" the term

$$\sum_{0 \leq i < \mathrm{length}(d)} d_i * 10^i$$

denotes the value of this sequence in the decimal number system.

Example: Binary Number Representation

Likewise, for any finite sequence b of binary digits 0 and 1, the value

$$\sum_{0 \leq i < \mathrm{length}(b)} b_i * 2^i$$

denotes the value of this sequence in the binary number system, e.g., the value of $[0,1,1,0,1]$ is

$$1 * 2^4 + 0 * 2^3 + 1 * 2^2 + 1 * 2^1 + 0 * 2^0 = 22.$$

Generalization to any number base.

Multiple Variable Bindings

$$\sum_{1 \leq i \leq 5, 1 \leq j \leq 3} i * j = 1*1 + 1*2 + 1*3 + \sum_{2 \leq i \leq 5, 1 \leq j \leq 3} i * j$$

$$\sum_{1 \leq i \leq 3, 1 \leq j \leq i} i * j = 1*1 + 2*1 + 2*2 + 3*1 + 3*2 + 3*3.$$

Bound variables have to be deduced from context.

Sum Identities

For all vars i and j and formulas F (in which j does not occur freely), G (in which i does not occur freely), and H and terms T and U:

$$\sum_{i,F} T * \sum_{j,G} U = \sum_{i,F} \sum_{j,G} T * U.$$

$$\sum_{i,F} \sum_{j,G} T = \sum_{j,G} \sum_{i,F} T = \sum_{i,j,F \wedge G} T.$$

$$\sum_{i,F} T + \sum_{i,H} T = \sum_{i,F \vee H} T + \sum_{i,F \wedge H} T.$$

Sum Identities

Furthermore, if term C is a term in which i does not occur freely:

$$\sum_{i,F} C * T = C * \sum_{i,F} T.$$

$$\sum_{i,F} C = n * C$$

(where n is the number of i for which F holds).

Examples

$$\sum_{1\le i\le n} x^i * \sum_{1\le j\le m} x^j = \sum_{1\le i\le n}\sum_{1\le j\le m} x^{i+j} = \sum_{1\le i\le n\wedge 1\le j\le m} x^{i+j}.$$

$$\sum_{1\le i\le n}\sum_{1\le j\le m} i * x^j = \sum_{1\le i\le n} (i * \sum_{1\le j\le m} x^j).$$

Many more identities can be deduced from basic definition.

Product Quantifier

If x is a variable, F is a formula and T is a term, then the following is a term with bound variable x:

$$\prod_{x,F} T.$$

The value of this term is 1, if F does not hold for any x; otherwise it is, for every x that satisfies F, the product of the value of T and of the value of the term for all other x:

$$\begin{aligned}(\forall x : \neg F) &\Rightarrow \textstyle\prod_{x,F} T = 1;\\ (\forall y : F[x \leftarrow y] &\Rightarrow \textstyle\prod_{x,F} T = T[x \leftarrow y] * \prod_{x,F\wedge x\neq y} T).\end{aligned}$$

Example

$$\prod_{1\le i\le n} i^2 = \prod_{i,(i\in\mathbb{N}\wedge 1\le i\wedge i\le n)} i^2$$

$$\prod_{1\le i\le 0} i^2 = 1$$

$$\prod_{1\le i\le 5} i^2 = 1^2 * \prod_{2\le i\le 5} i^2$$

$$\prod_{1\le i\le 5} i^2 = 1^2 * 2^2 * 3^2 * 4^2 * 5^2;$$

Product Identities

See lecture notes.

Factorial

Definition: The factorial of a natural number n is the product of all non-zero numbers less than or equal n:

$$n! := \prod_{1 \leq i \leq n} i.$$

Handy notation for a particular product.

Binomial

Definition: The binomial coefficient (Binomialkoeffizient) "n choose k" of two natural numbers n and k:

$$\binom{n}{k} := \textbf{if } 0 \leq k \leq n \textbf{ then } \frac{n!}{k! * (n-k)!} \textbf{ else } 0.$$

Proposition: We have for every n and k with $0 \leq k \leq n$

$$\binom{n}{k} = \frac{\prod_{n-k+1 \leq i \leq n} i}{\prod_{1 \leq i \leq k} i}.$$

Important notion in combinatorics (the math of "counting things").

Motivation

$\binom{n}{k}$ is the number of ways

- to choose a k element set
- from an n-element set.

Example:

The set $\{0, 1, 2, 3\}$ has $6 = \binom{4}{2}$ subsets with 2 elements:

$$\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}.$$

Binomial Identities

For every $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $0 \leq k \leq n$:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1},$$

$$\binom{n}{k} = \binom{n}{n-k},$$

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Pascal's Triangle

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \\
\multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
\multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}
$$

Construction

This triangle is bounded by sides of 1 and where every interior element is the sum of both parents:

$$
\begin{array}{ccc}
 & \binom{n}{k} \quad \binom{n}{k+1} & \\
\cdots\cdots & \binom{n+1}{k+1} & \cdots\cdots\cdot
\end{array}
$$

Quick construction of binomial values.

Matrix Operations

See lecture notes.

Polynomials

A polynomial over the reals is an infinite sequence of real numbers, the coefficients, of which only finitely many are different from 0:

$$p \text{ is polynomial } :\Leftrightarrow p : \mathbb{N} \to \mathbb{R} \land (\exists k \in \mathbb{N} : \forall i \geq k : p_i = 0).$$

The degree of a polynomial is zero, if all coefficients are zero; otherwise, it is the index of the largest non-zero coeffient:

$$
\begin{array}{l}
\deg(p) := \\
\quad \textbf{if } \forall i \in \mathbb{N} : p_i = 0 \\
\quad\quad \textbf{then } 0 \\
\quad\quad \textbf{else } (\textbf{such } k \in \mathbb{N} : p_k \neq 0 \land (\forall i > k : p_i = 0)).
\end{array}
$$

Special Polynomials

$$\text{Poly} := \{p \in \mathbb{N} \to \mathbb{R} : p \text{ is polynomial}\}.$$

Definition: constant and variable polynomial

$$\begin{aligned}
&\cdot_{\text{Poly}} : \mathbb{R} \to \text{Poly}\\
&c_{\text{Poly}} := \textbf{such } p \in \text{Poly} : p_0 = c \wedge (\forall i > 0 : p_i = 0)
\end{aligned}$$

$$\mathsf{x} := \textbf{such } p \in \text{Poly} : p_0 = 0 \wedge p_1 = 1 \wedge (\forall i > 1 : p_i = 0)$$

$$\begin{aligned}
3_{\text{Poly}} &= [3, 0, 0, 0, 0, \ldots]\\
\mathsf{x} &= [0, 1, 0, 0, 0, \ldots]
\end{aligned}$$

Polynomial Operations

$$\begin{aligned}
&+ : \text{Poly} \times \text{Poly} \to \text{Poly}\\
&(p + q)_i := p_i + q_i
\end{aligned}$$

$$\begin{aligned}
&- : \text{Poly} \times \text{Poly} \to \text{Poly}\\
&(p - q)_i := p_i - q_i
\end{aligned}$$

$$\begin{aligned}
&- : \text{Poly} \to \text{Poly}\\
&(-p)_i := -(p_i)
\end{aligned}$$

$$\begin{aligned}
&* : \text{Poly} \times \text{Poly} \to \text{Poly}\\
&(p * q)_i := \textstyle\sum_{0 \leq j \leq i} p_j * q_{i-j}
\end{aligned}$$

Examples

$$\begin{aligned}
3_{\text{Poly}} &= [3, 0, 0, 0, 0, \ldots]\\
\mathsf{x} &= [0, 1, 0, 0, 0, \ldots]\\
\mathsf{x} + 3 &= [3, 1, 0, 0, 0, \ldots]\\
\mathsf{x} * \mathsf{x} &= [0, 0, 1, 0, 0, \ldots]\\
\mathsf{x} * \mathsf{x} + 2 * \mathsf{x} + 1 &= [1, 2, 1, 0, 0, \ldots]\\
(\mathsf{x} + 1) * (\mathsf{x} + 2) &= [2, 3, 1, 0, 0, \ldots]
\end{aligned}$$

Terms are just convenient notations to describe polynomials and compute with them.

Relationship to Reals

For all real numbers a and b we have

$$\begin{aligned}
a_{\text{Poly}} + b_{\text{Poly}} &= (a +_{\mathbb{R}} b)_{\text{Poly}},\\
a_{\text{Poly}} - b_{\text{Poly}} &= (a -_{\mathbb{R}} b)_{\text{Poly}},\\
-a_{\text{Poly}} &= (-_{\mathbb{R}} a)_{\text{Poly}},\\
a_{\text{Poly}} * b_{\text{Poly}} &= (a *_{\mathbb{R}} b)_{\text{Poly}}.
\end{aligned}$$

A property like $1 + 1 = 2$ also holds for polynomials 1_{Poly} and 2_{Poly} and $+$ interpreted as the polynomial addition.

Polynomial Evaluation

Definition: polynomial evaluation

$$
\begin{aligned}
&[\,] : \text{Poly} \times \mathbb{R} \ \to \mathbb{R} \\
&p[a] := \textstyle\sum_{0 \leq i \leq \deg(p)} p_i *_{\mathbb{R}} a^i.
\end{aligned}
$$

Example: $p := 2 + 3 * x + 4 * x * x$:

$$
\begin{aligned}
&p = [2, 3, 4, 0, 0, 0, 0, \ldots] \\
&p[5] = 2 * 5^0 + 3 * 5^1 + 4 * 5^2 = 117.
\end{aligned}
$$

Polynomial Evaluation

Proposition: For all polynomials p and q and all reals c and a:

$$
\begin{aligned}
c_{\text{Poly}}[a] &= c, \\
\mathsf{x}[a] &= a, \\
(p + q)[a] &= p[a] +_{\mathbb{R}} q[a], \\
(p * q)[a] &= p[a] *_{\mathbb{R}} q[a].
\end{aligned}
$$

When evaluating a polynomial p on a real a, we substitute a_{Poly} for every occurrence of x in p and then use arithmetic on reals.

$$
\begin{aligned}
(\mathsf{x} + 1)[2] &= 2 + 1 \ = 3 \\
(\mathsf{x} * \mathsf{x} + 2 * \mathsf{x} + 1)[3] &= 3 * 3 + 2 * 3 + 1 \ = 16
\end{aligned}
$$

Typical Notation

- No fixed "polynomial variable" x.
- Polynomial domains $\mathbb{R}[x]$, $\mathbb{Q}[y]$, $\mathbb{C}[z]$.
  - Coefficient domain $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$.
  - Polynomial variable x, y, z.
  - $\mathbb{Q}[y]$: $y = [0, 1, 0, 0, 0, \ldots]$.
- Multivariate polynomials $\mathbb{R}[x, y]$
  - $\mathbb{R}[x, y] = (\mathbb{R}[x])[y]$.
  - Coefficients are themselves polynomials.

Generalization to arbitrary number of variables.

Summary

- Number domains.
  - Construction.
  - Basic operations.
  - Basic laws.
  - Relationship.
- Minimum and Maximum.
- Sum and Product.
- Binomials.
- Matrix operations.
- Polynomials.