to be prepared for 05.11.2024

Exercise 18. Let R be a Euclidean domain. Prove the following:

1. If $m_1, \ldots, m_n \in R \setminus 0$ are pairwise coprime and $M = \prod_{i=1}^{n-1} m_i$. Then m_n and M are relatively prime.

2. Assume that $r, r' \in R$, and $m_1, m_2 \in R \setminus 0$ are coprime. Then $r \equiv r' \mod m_1$ and $r \equiv r' \mod m_2$ if and only if $r \equiv r' \mod m_1 m_2$.

Exercise 19. Use the facts formulated in the previous exercise for developing a recursive algorithm that computes a solution of a Chinese remainder problem in a Euclidean domain.

Exercise 20. Solve the Chinese remainder problem

$$\begin{array}{rcl} r \equiv & 62 & \mod 79 \\ r \equiv & 66 & \mod 83 \\ r \equiv & 72 & \mod 89 \end{array}$$

over the integers.

Exercise 21. Let I be a unique factorization domain. Every $f \in I[x]$ can be decomposed into "content $\times$ primitive part"

$$f = \operatorname{cont}(f)\operatorname{pp}(f)$$

where $\operatorname{cont}(f) \in I$ and $\operatorname{pp}(f) \in I[x]$ is primitive, i.e., the GCD of all coefficients is 1. This decomposition is unique up to multiplication by units.

Given $f, g \in I[x]$, we write $f \sim g$ if there is a unit ε with $g = \varepsilon f$. Then prove the following:

1. $\operatorname{cont}(fg) \sim \operatorname{cont}(f) \cdot \operatorname{cont}(g)$

2. $\operatorname{pp}(fg) \sim \operatorname{pp}(f) \cdot \operatorname{pp}(g)$.

Exercise 22. As a consequence of the previous exercise, demonstrate that, over a unique factorization domain, the product of primitive polynomials is primitive.