EXERCISES-09

(1) Show that S_n is a group if the group operation is the function composition.

(2) Let S_4 acts on $\binom{[4]}{2}$ by $\phi\left(f,\{x,y\}\right) := \{f(x), f(y))\}$. Show that this indeed a group action. In particular, check that for $f \in S_4$ and $\{x,y\} \in \binom{[4]}{2}$, we have $\{f(x), f(y))\} \in \binom{[4]}{2}$.

(3) Let $G = S_4$ and $X = \{0,1\}^{\binom{[4]}{2}} = \left\{ f \mid f : \binom{[4]}{2} \to \{0,1\} \right\}$. Define $\phi : G \times X \to X$ by $\phi(\sigma, f) := g$ with $g\left(\sigma\{x,y\}\right) := f\left(\{x,y\}\right)$, or equivalently, $g\left(\{x,y\}\right) := f\left(\{\sigma^{-1}(x), \sigma^{-1}(y)\}\right)$. Here $\sigma\{x,y\} = \{\sigma(x), \sigma(y)\}$. Show that ϕ is a group action.

(4) Let X be defined as above and f_1 be an element of X given by

$$f_1 := \begin{pmatrix} \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Compute the orbit of f_1; i.e., compute

$$S_4 f_1 = \left\{\phi(\sigma, f_1) \mid \sigma \in S_4\right\}.$$