WS 2023

Symbolic Summation and Integration (selected slides)

Carsten Schneider

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

JOHANNES KEPLER UNIVERSITÄT LINZ

Lecture 1: October 4, 2023

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\mathrm{den}(f')$ is minimal

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\mathrm{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\mathrm{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Example:

$$f = \frac{2x^5 + 6x^4 + 8x^3 + 5x^2 + 6x + 4}{x(x+2)\left(x^2+1\right)\left(x^2+2x+2\right)}$$

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\mathrm{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Example:

$$f = \frac{1}{x} + \frac{1}{x+2} - \frac{1}{x^2+1} + \frac{1}{(x+1)^2+1}$$

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\mathrm{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Example:

$$f = \frac{1}{x} + \frac{1}{x+2} - \frac{1}{x^2+1} + \frac{1}{(x+1)^2+1}$$

$$\frac{p(x+r)}{q(x+r)} = \frac{p(x)}{q(x)} + \gamma(x+1) - \gamma(x)$$

with

$$\gamma(x) = \sum_{i=0}^{r-1} \frac{p(x+i)}{q(x+i)}$$

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\operatorname{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Example:

$$f = \frac{1}{x} + \frac{1}{x} - \frac{1}{x^2+1} + \frac{1}{(x+1)^2+1} + g(x+1) - g(x)$$

with $g(x) = \frac{1}{x} + \frac{1}{x+1}$

$$\frac{p(x+r)}{q(x+r)} = \frac{p(x)}{q(x)} + \gamma(x+1) - \gamma(x)$$

with

$$\gamma(x) = \sum_{i=0}^{r-1} \frac{p(x+i)}{q(x+i)}$$

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\mathrm{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Example:

$$f = \frac{1}{x} + \frac{1}{x} - \frac{1}{x^2+1} + \frac{1}{(x+1)^2+1}$$
$$+ g(x+1) - g(x)$$

with $g(x) = \frac{1}{x} + \frac{1}{x+1}$

$$\frac{p(x+r)}{q(x+r)} = \frac{p(x)}{q(x)} + \gamma(x+1) - \gamma(x)$$

with

$$\gamma(x) = \sum_{i=0}^{r-1} \frac{p(x+i)}{q(x+i)}$$

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\mathrm{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Example:

$$f = \frac{1}{x} + \frac{1}{x} - \frac{1}{x^2+1} + \frac{1}{x^2+1} + g(x+1) - g(x)$$

with $g(x) = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x^2+1}$

$$\frac{p(x+r)}{q(x+r)} = \frac{p(x)}{q(x)} + \gamma(x+1) - \gamma(x)$$

with

$$\gamma(x) = \sum_{i=0}^{r-1} \frac{p(x+i)}{q(x+i)}$$

Refined telescoping (Abramov, 1975)

Given $f(x) \in \mathbb{K}(x)$;

find $g(x) \in \mathbb{K}(x)$ and $f'(x) \in \mathbb{K}(x)$ proper such that

$$g(x+1) - g(x) + f'(x) = f(x)$$

and such that the degree of $\operatorname{den}(f')$ is minimal

$$\downarrow$$

$$g(n+1) - g(1) + \sum_{k=1}^{n} f'(k) = \sum_{k=1}^{n} f(k)$$

Example:

$$f = \overbrace{\frac{2}{x}}^{=f'} + g(x+1) - g(x)$$

with $g(x) = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x^2+1}$

$$\rightarrow$$

$$\sum_{k=1}^{n} \frac{2k^5+6k^4+8k^3+5k^2+6k+4}{k(k+2)(k^2+1)(k^2+2k+2)}$$

$$||$$

$$-\frac{n(7+12n+8n^2+2n^3)}{(1+n)(2+n)(2+2n+n^2)} + 2\sum_{k=1}^{n} \frac{1}{k}$$

Lecture 10: January 10, 2024

Telescoping

GIVEN $f(k) = S_1(k)$.
FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Sigma computes

$$g(k) = (S_1(k) - 1)k.$$

Telescoping

GIVEN $f(k) = S_1(k)$.
FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$
$$= (S_1(n+1) - 1)(n+1).$$

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference ring for the summand

Consider a ring

$$\mathbb{A}$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference ring for the summand

Consider a ring

$$\mathbb{A} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference ring for the summand

Consider a ring

$$\mathbb{A} := \mathbb{Q}(x)$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(x) = x + 1, \qquad\qquad \mathcal{S}\, k = k + 1,$$

Telescoping in the given difference ring

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference ring for the summand

Consider a ring

$$\mathbb{A} := \mathbb{Q}(x)[h]$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(x) = x + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{x+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = h.}$$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)x \in \mathbb{A}.$$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)x \in \mathbb{A}.$$

This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)x \in \mathbb{A}.$$

This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

FIND $g \in \mathbb{Q}(x)[h]$:

$$\sigma(g) - g = h.$$

FIND $g \in \mathbb{Q}(x)[h]$:

$$\sigma(g) - g = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$\forall g \in \mathbb{Q}(x)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(x)[h]$:

$$\sigma(g) - g = h.$$

Degree bound: COMPUTE $b \geq 0$:

$b = 2$

$$\forall g \in \mathbb{Q}(x)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(x)[h]$:

$$\sigma(g) - g = h.$$

Degree bound: COMPUTE $b \geq 0$:

$b = 2$

$$\forall g \in \mathbb{Q}(x)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

Polynomial Solution: FIND

$$g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h].$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\sigma(g) - g = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\big[\sigma(g_2\, h^2 + g_1 h + g_0)\big] \\ -\big[g_2\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\big[\sigma(g_2\, h^2) + \sigma(g_1 h + g_0)\big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\Big[\sigma(g_2)\,\sigma(h^2) + \sigma(g_1 h + g_0)\Big] \\ -\Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\big[\sigma(g_2)\,\sigma(h)^2 + \sigma(g_1 h + g_0)\big] \\ -\big[g_2\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{x+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(x)[h]$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{x+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\,h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 h + g_0)\Big] \\ -\big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\Big[\sigma(c)\big(h + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 h + g_0)\Big] \\ -\big[c\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{x+1}\big)^2+\sigma(g_1 h+g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\Big[c\big(h+\tfrac{1}{x+1}\big)^2+\sigma(g_1 h+g_0)\Big] - \big[c\, h^2 + g_1 h + g_0\big] = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\big[\tfrac{2h(x+1)+1}{(x+1)^2}\big]$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(x+1)+1}{(x+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{x+1}}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{x+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\, h^2 + g_1 h + g_0\Big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(x+1)+1}{(x+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{x+1}}$$

$c = 0$, $g_1 = x + d$, $d \in \mathbb{Q}$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{x+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\Big[\tfrac{2h(x+1)+1}{(x+1)^2}\Big]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{x+1}}$$

$c = 0$, $g_1 = x + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{x+1}}$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(x)[h]$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{x+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

g=hx-x

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\left[\tfrac{2h(x+1)+1}{(x+1)^2}\right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{x+1}$$

$c = 0,\quad g_1 = x + d,\quad d \in \mathbb{Q}$

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{x+1}$$

$\leftarrow$

$g_0 = -x$, $d = 0$

Telescoping in the given difference ring

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = h.}$$

We compute

$$g = (h-1)x \in \mathbb{A}.$$

This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

Lecture 13: January 31, 2024

Summation paradigm 1:

Recurrence solving

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$\Big[a_d(n)S^d + a_{d-1}(n)S^{d-1} + \cdots + a_0(n)I\Big]A(n)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$
$$||$$
$$\Big[a_d(n)S^d + a_{d-1}(n)S^{d-1} + \cdots + a_0(n)I\Big]A(n)$$

Hyper

$$\prod_{j=\lambda}^{n} b_1(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$\left[\Big(\tilde{a}_{d-1}(n)S^{d-1} + \tilde{a}_{d-2}(n)S^{d-2} + \cdots + \tilde{a}_0(n)I\Big)\Big(S - b_1(n)\Big)\right]A(n)$$

$$\prod_{j=\lambda}^{n} b_1(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$\Big[\Big(\tilde{a}_{d-1}(n)S^{d-1} + \tilde{a}_{d-2}(n)S^{d-2} + \cdots + \tilde{a}_0(n)I\Big)\Big(S - b_1(n)\Big)\Big]A(n)$$

Hyper

$$\prod_{j=\lambda}^{n} b_2(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$\Big[\Big(\tilde{a}_{d-2}(n)S^{d-2} + \tilde{a}_{d-3}(n)S^{d-3} + \cdots + \tilde{a}_0(n)I\Big)\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)\Big]A(n)$$

$$\prod_{j=\lambda}^{n} b_2(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$c(n)\Big(S - b_d(n)\Big)\ldots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$
$$||$$
$$c(n)\Big(S - b_d(n)\Big)\ldots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$c(n)\Big(S - b_d(n)\Big)\ldots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

$$L_2(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i1+1} b_1(j-1)}$$

Recurrence solving

Special case: homogeneous recurrences with $a_i(n) \in \mathbb{K}[n]$

$$a_d(n)A(n+d) + a_{d-1}(n)A(n+d-1) + \cdots + a_0(n)A(n) = 0$$

$$||$$

$$c(n)\Big(S - b_d(n)\Big)\ldots\Big(S - b_2(n)\Big)\Big(S - b_1(n)\Big)A(n)$$

$$L_1(n) = \prod_{j=\lambda}^{n} b_1(j-1)$$

d linearly independent solutions

$$L_2(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i1+1} b_1(j-1)}$$

$$\vdots$$

$$L_d(n) = \prod_{j=\lambda}^{n} b_1(j-1) \sum_{i_1=\lambda}^{n-1} \frac{\prod_{j=\lambda}^{i_1} b_2(j-1)}{\prod_{j=\lambda}^{i_1+1} b_1(j-1)} \cdots \sum_{i_{d-1}=\lambda}^{i_{d-2}-1} \frac{\prod_{j=\lambda}^{i_{d-1}} b_d(j-1)}{\prod_{j=\lambda}^{i_{d-1}+1} b_{d-1}(j-1)}$$

Summation paradigm 2:

Telescoping

The naive Ansatz (M. Karr, 1981)

GIVEN a $\Pi\Sigma$-ring/field $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.
($\mathbb{A}$ an integral domain)

FIND, in case of existence, $g \in \mathbb{A}$:

$$\sigma(g) - g = f.$$

Still a naive Ansatz (in rings) for simplification

GIVEN an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

FIND, in case of existence, $g\in\mathbb{A}$:

$$\sigma(g)-g=f.$$

Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

FIND, in case of existence, $g \in \mathbb{A}$:

$$\sigma(g) - g = f.$$

Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^{n}\Big(\frac{-2+k}{10(1+k^2)}+\frac{(1-4k-2k^2)H_k}{10(1+k^2)(2+2k+k^2)}+\frac{(1-4k-2k^2)H_k^{(3)}}{5(1+k^2)(2+2k+k^2)}\Big)$$

$$=?$$

Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = degrees in denominators minimal

Example:

$$\sum_{k=1}^{n}\Big(\frac{-2+k}{10(1+k^2)}+\frac{(1-4k-2k^2)H_k}{10(1+k^2)(2+2k+k^2)}+\frac{(1-4k-2k^2)H_k^{(3)}}{5(1+k^2)(2+2k+k^2)}\Big)$$

$$=\frac{n^2+4n+5}{10(n^2+2n+2)}H_n-\frac{(n-1)(n+1)}{5(n^2+2n+2)}H_n^{(3)}-\frac{2}{5}\sum_{k=1}^{n}\frac{1}{k^2}$$

Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}=?$$

Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}=\frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^{3}+\frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right)+\frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

depth 3 depth 1

Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$ with $f \in \mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E} > \mathbb{A}$ with $g \in \mathbb{E}$:

$$\sigma(g) - g = f.$$

appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a} (-1)^k H_k^2 \binom{n}{k} = ?$$

Symbolic simplification of sums

1. FIND an appropriate $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$ with $f\in\mathbb{A}$.

2. FIND an appropriate extension $\mathbb{E}>\mathbb{A}$ with $g\in\mathbb{E}$:

$$\sigma(g)-g=f.$$

appropriate = sum representations with minimal number of objects

Example:

$$\sum_{k=0}^{a}(-1)^k H_k^2\binom{n}{k} = -\frac{1}{n}\sum_{i=1}^{a}\frac{(-1)^i}{i}\binom{n}{i} - (a-n)\big(n^2H_a^2+2nH_a+2\big)\frac{(-1)^a\binom{n}{a}}{n^3}-\frac{2}{n^2}$$

Summation paradigm 3:

Creative telescoping

Simplify

$$\sum_{k=1}^{n} \binom{n}{k} S_1(k)$$

where $S_1(k) = \sum_{i=1}^{k} \frac{1}{i}$

Simplify

$$A(n) := \sum_{k=1}^{n} \binom{n}{k} S_1(k).$$

A difference ring for the summand

Consider the ring

$$\mathbb{A} := \mathbb{Q}(n)(k)[b, b^{-1}][h]$$

with the automorphism $\sigma : \mathbb{A} \to \mathbb{A}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}(n),$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(b) = \frac{n-k}{k+1}\, b, \qquad \mathcal{S} \binom{n}{k} = \frac{n-k}{k+1} \binom{n}{k}$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Creative telescoping

REPRESENT $f(n,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k}$$

Creative telescoping

REPRESENT $f(n,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = f_0}$$

☹

Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\binom{n}{k}}{n+1-k}$$

FIND $g \in \mathbb{A}$:

$$\boxed{\sigma(g) - g = f_0}$$

Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \quad\longleftrightarrow\quad h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\,\binom{n}{k}}{n+1-k} \quad\longleftrightarrow\quad \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

FIND $g \in \mathbb{A}$ and $c_0, c_1 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g) - g = c_0 f_0 + c_1 f_1}$$

☹

Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\,\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\,\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

FIND $g \in \mathbb{A}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g) - g = c_0 f_0 + c_1 f_1 + c_2 f_2}$$

☺

Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

FIND $g \in \mathbb{A}$ and $c_0, c_1, c_2 \in \mathbb{Q}(n)$:

$$\boxed{\sigma(g) - g = c_0 f_0 + c_1 f_1 + c_2 f_2}$$

We compute

$$c_0 := 4\,(1+n), \quad c_1 := -2\,(3+2\,n), \quad c_2 := 2+n,$$

$$g := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)h)\,b)}{(1-k+n)\,(2-k+n)}.$$

Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

This gives

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k)}$$

with

$$c_0(n) := 4\,(1+n), \quad c_1(n) := -2\,(3+2\,n), \quad c_2(n) := 2+n,$$

$$g(n,k) := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)S_1(k))\binom{n}{k})}{(1-k+n)\,(2-k+n)}.$$

Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\,\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\,\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

This gives

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)f(n+1,k)+c_2(n)f(n+2,k)}$$

Summing over k from 0 to n gives

$$\boxed{g(n,n+1)-g(n,0)} = \boxed{\begin{array}{l} c_0(n)\,\mathsf{A}(n)+ \\ c_1(n)\,\big[\mathsf{A}(n+1)-f(n+1,n+1)\big] \\ c_2(n)\,\big[\mathsf{A}(n+2)-f(n+2,n+1)-f(n+2,n+2)\big]. \end{array}}$$

for $A(n)=\sum_{k=0}^{n}\binom{n}{k}S_1(k)$

Creative telescoping

REPRESENT $f(n+i,k)$ in $\mathbb{A}$:

$$f(n,k) = S_1(k)\binom{n}{k} \longleftrightarrow h\,b =: f_0$$

$$f(n+1,k) = \frac{(n+1)\,S_1(k)\,\binom{n}{k}}{n+1-k} \longleftrightarrow \frac{(n+1)\,h\,b}{n+1-k} =: f_1$$

$$f(n+2,k) = \frac{(n+1)\,(n+2)\,S_1(k)\,\binom{n}{k}}{(n+1-k)\,(n+2-k)} \longleftrightarrow \frac{(n+1)\,(n+2)\,h\,b}{(n+1-k)\,(n+2-k)} =: f_2.$$

This gives

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k)}$$

Summing over k from 0 to n gives

$$\boxed{1 = 4(1+n)A(n) - 2(3+2n)A(n+1) + (2+n)A(n+2)}$$

for $A(n) = \sum_{k=0}^{n}\binom{n}{k}S_1(k)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N)=\sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N),\dots,a_d(N),h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N)+\dots+a_d(N)F(N+d)=h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \cdots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$S_{2,1}(n) = \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}$$ (harmonic sums)

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \cdots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{k=1}^{n} \frac{2^k}{k} \sum_{i=1}^{k} \frac{2^{-i}}{i} \sum_{j=1}^{i} \frac{S_1(j)}{j}$$

(generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \cdots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{k=1}^{n} \frac{1}{(1+2k)^2} \sum_{j=1}^{k} \frac{1}{j^2} \sum_{i=1}^{j} \frac{1}{1+2i}$$

(cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N)=\sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N),\dots,a_d(N),h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N)+\dots+a_d(N)F(N+d)=h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{j=1}^{n}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}$$

(binomial sums)

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \cdots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{h=1}^{n} 2^{-2h}(1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2 \binom{2k}{k}}$$

(generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N) = \sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N), \dots, a_d(N), h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N) + \dots + a_d(N)F(N+d) = h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

A more general example:

$$\sum_{k=1}^{n} \left(\prod_{i=1}^{k} \frac{1+i+i^2}{i+1} \right) \sum_{j=1}^{k} \frac{1}{j\binom{4j}{3j}^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(N)=\sum_{k=0}^{N} f(N,k);$$

$f(N,k)$: indefinite nested product-sum in k;
N: extra parameter

FIND a recurrence for $F(N)$

2. Recurrence solving

GIVEN a recurrence

$a_0(N),\dots,a_d(N),h(N)$:
indefinite nested product-sum expressions.

$$a_0(N)F(N)+\dots+a_d(N)F(N+d)=h(N);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a "closed form"

F(N)=combined solutions in terms of indefinite nested sums.