to be prepared for 26.01.2023

Exercise 51. Consider the partial order $\leq_\pi$ on $\mathbb{N}^n$ defined as

$$(a_1, \dots, a_n) \leq_\pi (b_1, \dots, b_n) \iff a_i \leq b_i \ \forall i \in \{1, \dots, n\}.$$

Prove that any set $A \subseteq \mathbb{N}^n$ contains a finite set $B \subseteq A$ such that

$$\forall_{a \in A} \exists_{b \in B} \text{ with } b \leq_\pi a.$$

Hint: You may proceed by applying the classical Hilbert Basis Theorem or by pure combinatorial observations.

Exercise 52. Given a monomial order $<$ on $\mathbb{N}^n$. A **Gröbner basis** for an ideal $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$ is a finite subset $G \subseteq I$ with the property $\langle \mathrm{LT}(G) \rangle = \langle \mathrm{LT}(I) \rangle$.

Let G be a Gröbner basis for $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$ and $f \in \mathbb{F}[x_1, \dots, x_n]$. Prove that there exists a unique $r \in \mathbb{F}[x_1, \dots, x_n]$ such that

1. $r \equiv f \mod I$;
2. no term of r is divisible by any monomial in $\mathrm{LT}(G)$.

Exercise 53. Consider linear polynomials in $\mathbb{F}[x_1, \dots, x_n]$

$$f_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad 1 \leq i \leq m$$

and let $A = (a_{ij})$ be the $m \times n$ matrix of their coefficients. Let B be the reduced row echelon matrix determined by A and let $g_1, \dots, g_r$ be the linear polynomials coming from the nonzero rows of B. Use lex order with $x_1 > \cdots > x_n$ and show that $\{g_1, \dots, g_r\}$ is a Gröbner basis of $\langle f_1, \dots, f_m \rangle$.

> **Notation:** We write $\mathrm{M}(f)$ for the set of all monomials appearing with a nonzero coefficient in a polynomial f. Given a monomial order, $\mathrm{lm}(f)$ is the leading monomial of f, i.e., $\mathrm{lm}(f) = \max \mathrm{M}(f)$. For a set $G \subseteq \mathbb{F}[x_1, \dots, x_n]$, $\mathrm{LM}(G) = \{\mathrm{lm}(g) \mid g \in G\}$. As usual, the leading term of f is $\mathrm{lt}(f) = \mathrm{lc}(f)\mathrm{lm}(f)$.

Exercise 54. A set $G \subseteq \mathbb{F}[x_1, \dots, x_n] \setminus 0$ is called a **reduced Gröbner basis** (w.r.t. some monomial order) provided that

1. G is a Gröbner basis for $\mathbb{F}[x_1, \dots, x_n]\, G$;
2. $\forall_{g \in G}\, \mathrm{lc}(g) = 1$;
3. $\forall_{g \in G}\, \mathrm{M}(g) \cap \langle \mathrm{LM}(G \setminus \{g\}) \rangle = \emptyset$.

Let G be a Gröbner basis for the ideal $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$. Describe an algorithm which, starting from G, produces a reduced Gröbner basis for I.

Exercise 55. Let W be a set, ordered linearly by some relation $<$ and let $P_{\text{fin}}(W)$ denote the set of finite subsets of W. For $A, B \in P_{\text{fin}}(W)$ define

$$A < B \iff \max(A \Delta B) \in B \tag{1}$$

where $A \Delta B = A \setminus B \cup B \setminus A$ is the symmetric difference.

Show that:

1. (1) is a linear order on $P_{\text{fin}}(W)$ that extends both, the (partial) order of containment ($A \subset B$) and, via embedding $w \mapsto \{w\}$, the (linear) order $<$.

2. If $<$ is a well-order on W then (1) is a well-order on $P_{\text{fin}}(W)$.