to be prepared for 19.01.2023

Exercise 49. Consider the partial order $\leq_\pi$ on $\mathbb{N}^n$ defined as

$$(a_1, \dots, a_n) \leq_\pi (b_1, \dots, b_n) \iff a_i \leq b_i \ \forall i \in \{1, \dots, n\}.$$

Prove that any set $A \subseteq \mathbb{N}^n$ contains a finite set $B \subseteq A$ such that

$$\forall_{a \in A} \exists_{b \in B} \text{ with } b \leq_\pi a.$$

Hint: You may proceed by applying the classical Hilbert Basis Theorem or by pure combinatorial observations.

Exercise 50. Let $<$ be a monomial order on $\mathbb{N}^n$, $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$ an ideal and $G \subseteq I$. Show that

$$\langle \mathrm{LT}(G) \rangle = \langle \mathrm{LT}(I) \rangle \iff \forall_{p \in I} \exists_{g \in G} \operatorname{lt}(g) | \operatorname{lt}(p).$$

Exercise 51. Given a monomial order $<$ on $\mathbb{N}^n$. A **Gröbner basis** for an ideal $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$ is a finite subset $G \subseteq I$ with the property $\langle \mathrm{LT}(G) \rangle = \langle \mathrm{LT}(I) \rangle$.

Let G be a Gröbner basis for $I \trianglelefteq \mathbb{F}[x_1, \dots, x_n]$ and $f \in \mathbb{F}[x_1, \dots, x_n]$. Prove that there exists a unique $r \in \mathbb{F}[x_1, \dots, x_n]$ such that

1. $r \equiv f \mod I$;
2. no term of r is divisible by any monomial in $\mathrm{LT}(G)$.

Exercise 52. Show that the result of applying the Euclidean Algorithm in $\mathbb{F}[x]$ to any pair of polynomials f, g is a Gröbner basis for $\langle f, g \rangle$.

Exercise 53. Consider linear polynomials in $\mathbb{F}[x_1, \dots, x_n]$

$$f_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad 1 \leq i \leq m$$

and let $A = (a_{ij})$ be the $m \times n$ matrix of their coefficients. Let B be the reduced row echelon matrix determined by A and let $g_1, \dots, g_r$ be the linear polynomials coming from the nonzero rows of B. Use lex order with $x_1 > \cdots > x_n$ and show that $\{g_1, \dots, g_r\}$ is a Groebner basis of $\langle f_1, \dots, f_m \rangle$.