to be prepared for 01.12.2022

Exercise 33. Let $f, g \in \mathbb{F}[x, y]$, $\deg_x f = n$, $\deg_x g = m$ and $d \in \mathbb{N}$ so that $\deg_y f, \deg_y g \leq d$. Prove that

$$\deg_y \operatorname{res}_x(f, g) \leq (n+m)d.$$

Exercise 34. Let $f = \sum_{k=0}^{n} f_k x^k \in \mathbb{C}[x]$ with $\deg f = n \geq 0$. Considering f as an element of $\mathbb{C}^{n+1}$ we may assign to f the usual norms

$$||f||_p = \left(\sum_{k=0}^{n} |f_k|^p \right)^{1/p} \quad \text{for } p \geq 1 \text{ and } ||f||_\infty = \max_k |f_k|.$$

Check the following relations:

1. $||f||_\infty \leq ||f||_2 \leq \sqrt{n+1}||f||_\infty$;
2. $||f||_2 \leq ||f||_1 \leq (n+1)||f||_\infty$.

Exercise 35. Let $f \in \mathbb{C}[x]$ and $z \in \mathbb{C}$. Prove that

$$||(x-z)f||_2 = ||(\overline{z}x - 1)f||_2.$$

You may use $|w|^2 = w\overline{w}$ for $w \in \mathbb{C}$ and that conjugation $w \mapsto \overline{w}$ is an automorphism of the field $\mathbb{C}$.

Exercise 36. For a complex polynomial $f \in \mathbb{C}[x]$ written as product of linear factors $f = f_n(x - z_1) \cdots (x - z_n)$, where $z_k \in \mathbb{C}$, the real number $M(f)$ is

$$M(f) = |f_n| \cdot \prod_{k=1}^{n} \max(1, |z_k|).$$

Prove the following statements.

1. $f, g \in \mathbb{C}[x] \Rightarrow M(fg) = M(f) \cdot M(g)$ and $M(f) \geq |\operatorname{lc}(f)|$;
2. $f \in \mathbb{C}[x] \Rightarrow M(f) \leq ||f||_2$.

Hint to point 2:
Sort the zeros $z_1, \ldots, z_n$ of f so that $|z_j| \geq |z_{j+1}|$ $(j = 1, \ldots, n-1)$ and let $k = \max\{j : |z_j| > 1\}$. Then show that $M(f)$ equals the absolute value of the leading coefficient of the polynomial

$$g = \operatorname{lc}(f) \cdot \prod_{j=1}^{k} (\overline{z}x - 1) \cdot \prod_{j=k+1}^{n} (x - z_j)$$

so that $M(f)^2 \leq ||g||_2^2$. Then by repeatedly applying the identity in Ex. 35 prove that $||g||_2^2 = ||f||_2^2$.

Exercise 37. If $f, h \in \mathbb{C}[x]$ and $h|f$, prove that

$$||h||_1 \leq 2^{\deg h} M(h) \leq 2^{\deg h} \left| \frac{\operatorname{lc}(h)}{\operatorname{lc}(f)} \right| \cdot ||f||_2.$$

Hint: Write $h = \operatorname{lc}(h) \prod_j (x - z_j)$ as product of linear factors and note that each z_j is a zero of f. Derive from this that $M(h) \leq M(f)$. Then express the coefficients of h as (the elementary symmetric) functions in the z_j.

Exercise 38. Let now $f, g, h \in \mathbb{Z}[x]$ (integer coefficients) with degress
$n = \deg f \geq 1$, $m = \deg g$, $k = \deg h$, and assume that $gh|f$ in $\mathbb{Z}[x]$. Prove that

$$||g||_1 ||h||_1 \leq 2^{m+k} ||f||_2 \leq (n+1)^{1/2} \cdot 2^{m+k} ||f||_\infty.$$

Derive from this that

$$||h||_\infty \leq ||h||_2 \leq 2^k ||f||_2 \leq 2^k ||f||_1 \text{ and } ||h||_\infty \leq ||h||_2 \leq (n+1)^{1/2} \cdot 2^k ||f||_\infty.$$