

to be prepared for 10.11.2022

Exercise 21. Consider the polynomials $f, g \in \mathbb{Q}[x]$

$$\begin{aligned} h &:= x^8 + x^6 - 3x^4 - 3x^3 + 8x^2 + 2x - 5 \\ f &:= 3x^6 + 5x^4 - 4x^2 - 9x + 2. \end{aligned}$$

Compute polynomials $r, t \in \mathbb{Q}[x]$ such that

$$r \equiv tf \pmod{h} \text{ with } \deg r < 4 \text{ and } \deg t \leq 4.$$

Hint: Write down the extended remainder sequence produced when executed the extended Euclidean algorithm on the input (h, f) . Decide which of the triples (r_i, s_i, t_i) determine rational function approximations, i.e., a pair (r, t) of polynomials in $\mathbb{Q}[x]$ such that $\deg r < n$, $\deg t \leq n - \deg r$ and so that

$$\text{GCD}(t, h) = 1 \text{ and } rt^{-1} \equiv f \pmod{h}.$$

Exercise 22. Compute a $(3, 4)$ -Pade approximant for the sine function, that is, compute polynomials $r, t \in \mathbb{Q}[x]$ with $\deg r < 3$, $\deg t \leq 4$ such that t has a nonzero constant coefficient and $\frac{r}{t} \equiv T_7 \pmod{x^{3+4}}$ where T_7 is the degree 7 Taylor polynomial of $\sin x$ with center 0.

Exercise 23. Let F be a field, $(u_0, v_0), \dots, (u_{n-1}, v_{n-1})$ n points in the plane $F \times F$, where the u_i are pairwise distinct, and $f \in F[x]$ an approximating polynomial, i.e., $\forall_i f(u_i) = v_i$ and $\deg f < n$. Apply the extended Euclidean algorithm to the polynomials $h := (x - u_0) \cdots (x - u_{n-1})$ and f , thereby producing the remainder sequence (r_j, s_j, t_j) . In this situation prove the following:

If $k \in \{0, \dots, n\}$ and, in the course of the algorithm, j is the first index with $\deg r_j < k$, then

$$\deg t_j \leq n - k \text{ and } \forall_{0 \leq i < n} r_j(u_i) = t_j(u_i)v_i.$$

If in addition r_j and t_j are coprime, then $t_j(u_i) \neq 0$ and $\frac{r_j(u_i)}{t_j(u_i)} = v_i$ for all $0 \leq i < n$.