# to be prepared for 13.10.2022

**Exercise 1.** Consider the integers $a = 215712,\ b = 739914$. Determine the gcd of $a$ and $b$ and integers $s, t$ such that $\gcd(a, b) = sa + tb$.

**Exercise 2.** Compute the GCD of

$$\begin{aligned} f(x) &= 6x^5 + 2x^4 - 19x^3 - 6x^2 + 15x + 9 \\ g(x) &= 5x^4 - 4x^3 + 2x^2 - 2x - 2\ . \end{aligned}$$

**Exercise 3.** Consider the polynomials $f_1, \ldots, f_m$ in the ring $K[x_1, \ldots, x_n]$. Prove that the set

$$\Big\{ \sum_{j=1}^{m} h_j f_j \mid \forall_{1 \le j \le m}\, h_j \in K[x_1, \ldots, x_n] \Big\}$$

is the smallest ideal in $K[x_1, \ldots, x_n]$ that contains the set $\{f_1, \ldots, f_m\}$.

**Exercise 4.**

1. Consider the system of equations

$$\begin{aligned} 2x^4 - 3x^2y + y^4 - 2y^3 + y^2 &= 0 \\ 4x^3 - 3xy &= 0 \\ 4y^3 - 3x^2 - 6y^2 + 2 &= 0. \end{aligned}$$

   Compute all solutions.

2. The same for

$$\begin{aligned} 1 + 8xy + 2y^2 + 8xy^3 + y^4 - 16x^2 &= 0 \\ 8x + 4y + 24xy^2 + 4y^3 &= 0 \\ 8y + 8y^3 - 32x &= 0. \end{aligned}$$

This can be done by hand or using a computer algebra system.

**Exercise 5.** Let $R$ be a commutative ring with 1, and $S \subseteq R$ a multiplicative monoid. Consider the equivalence relation on $R \times S$ given by

$$(r_1, s_1) \sim (r_2, s_2) \iff \exists s_3 \in S \text{ such that } s_3 s_2 r_1 = s_3 s_1 r_2.$$

1. Let $R[S^{-1}]$ denote the quotient $R \times S / \sim$, write $\frac{r}{s}$ for the equivalence class of the pair $(r, s)$ and define addition and multiplication on $R[S^{-1}]$ by

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}, \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

   Verify that these are well defined operations turning $R[S^{-1}]$ into a commutative ring with 1.

2. Define the map

$$\eta \colon R \longrightarrow R[S^{-1}],\ r \mapsto \frac{r}{1}.$$

   Check that this is a well defined homomorphism of rings.

3. Give a description of $\eta$'s kernel. Formulate conditions on the monoid $S$ that make $R$ embedded in $R[S^{-1}]$.

Note that, for an integral domain $R$ and monoid $S = R \setminus 0$, this construction provides the usual quotient field $Q(R)$.

**Exercise 6.** We can check that the polynomial ring $K[x]$ over a field $K$ is an integral domain by considering the degree function.

1. Prove, by a similar argument, that also the ring $K[[x]]$ of formal power series is an integral domain. Moreover show that K[[x]] is an Euclidean domain by introducing an appropriate function and verifying the required properties.

2. Show that $K[[x]]$ is a local ring.