TSW: Exam Proof Training with Solutions: Sample 1

Subject 1. Prove: $((A \wedge B) \Rightarrow C) \;\Leftrightarrow\; ((A \Rightarrow C) \vee (B \Rightarrow C))$

Solution: We prove the above in both directions.

- Direction $\Rightarrow$: We assume: $(A \wedge B) \Rightarrow C$ (1)
  and we prove: $(A \Rightarrow C) \vee (B \Rightarrow C)$.
  For proving the above we assume: $\neg(B \Rightarrow C)$ (2)
  and we prove: $A \Rightarrow C$.
  For proving the above we assume: A
  and we prove: C.
  From (2) we obtain: B and $\neg C$.
  From A and B by (1) we obtain the goal.

- Direction $\Leftarrow$: We assume: $(A \Rightarrow C) \vee (B \Rightarrow C)$ (1)
  and we prove: $(A \wedge B) \Rightarrow C$.
  For proving the above we assume: A and B
  and we prove C.
  We prove the above by case distinction using (1):

  - Case: $A \Rightarrow C$:
    From A by the above we obtain the goal.
  - Case: $B \Rightarrow C$:
    From B by the above we obtain the goal.

Subject 2. Prove: $(\underset{x}{\forall}(P[x] \Rightarrow Q)) \;\Rightarrow\; ((\underset{x}{\exists} P[x]) \Rightarrow Q)$

Solution: For proving the above we assume: $\underset{x}{\forall}(P[x] \Rightarrow Q)$ (1)
and we prove: $(\underset{x}{\exists} P[x]) \Rightarrow Q$.
For proving the above we assume: $\underset{x}{\exists} P[x]$ (2)
and we prove: Q.
By (2) we can take an x_0 such that: $P[x_0]$.
We instantiate (1) to: $P[x] \Rightarrow Q$. From $P[x_0]$ by the above we obtain the goal.

Subject 3. For each symbol occuring in the following formula, specify whether it is a: logical quantifier, logical connective, predicate symbol, function symbol, variable, or constant. (Note that functions and predicates can also be constant or variable.)

$$\underset{f}{\forall} C[f] \Leftrightarrow \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y}{\forall}\ (|x-y|<\delta \ \Rightarrow\ |f[x]-f[y]|<\epsilon)$$

Solution: We consider each symbol from left to right (without the punctuation symbols: comma, parantheses, and brackets)
$\forall$: logical quantifier
f : quantified variable function symbol
C : constant predicate symbol
f : variable function symbol
$\Leftrightarrow$: logical connective
$\forall$: universal quantifier
ϵ : quantified variable, also variable (as term)
$>$: constant predicate symbol
0 : constant
$\exists$: logical quantifier
δ : quantified variable, also variable (as term)
$>$: constant predicate symbol
0 : constant
$\forall$: logical quantifier
x : quantified variable
y : quantified variable
$|$: part of a constant function symbol together with the next $|$
x : variable
$-$: constant function symbol
y : variable
$|$: part of a constant function symbol together with the previous $|$
$<$: constant predicate symbol
δ : variable
$\Rightarrow$: logical connective
$|$: part of constant function symbol
f : variable function symbol
x : variable
$-$: constant function symbol
f : variable function symbol
y : variable
$|$: part of constant function symbol
$<$: constant predicate symbol
ϵ : variable.

Subject 4. In the previous definition, C denotes uniform continuity, f denotes a real function of real argument, and x, y, δ and ϵ denote real numbers. Formulate and prove the statement that the sum of uniformly continuos functions is uniformly continuos. Emphasize the properties from the theory of real numbers which are necessary for this proof (definition of $f_1 + f_2$, properties of minimum and of absolute value, etc.).

Solution: See a similar proof in: `Buchberger-Predicate-Logic-2006.pdf` under `http://www.risc.jku.at/education/courses/ws2010/tsw/`

Subject 5. Based on the previous definition, show that the product of uniformly continuous functions is **not** uniformly continuous, by using a counterexample (e. g. both functions are the identity).

Solution: We need to proof that:

$$\underset{f,g}{\forall} (C[f] \wedge C[g]) \Rightarrow C[f * g]$$

is false, that means we need to prove the negation of it:

$$\underset{f,g}{\exists} C[f] \wedge C[g] \wedge \neg C[f * g].$$

For this we must find witnesses for f and g which satisfy the above conjunction.
We take f and g as being the identity function.
For the proof of $C[f]$ by definition see the similar but more complicated proof in the material referenced above.
We prove now $\neg C[h]$, where $h[x] = x^2$.
We take the definition of $C[h]$:

$$\underset{\epsilon>0}{\forall} \underset{\delta>0}{\exists} \underset{x,y}{\forall} (|x - y| < \delta \;\Rightarrow\; |x^2 - y^2| < \epsilon)$$

and we negate it:

$$\underset{\epsilon>0}{\exists} \underset{\delta>0}{\forall} \underset{x,y}{\exists} (|x - y| < \delta \;\wedge\; |x^2 - y^2| \geq \epsilon)$$

In order to prove the above we take as witness for ϵ the value 1 and we prove:

$$\underset{\delta>0}{\forall} \underset{x,y}{\exists} (|x - y| < \delta \;\wedge\; |x^2 - y^2| \geq 1).$$

In order to prove the above we take an arbitrary but fixed $\delta > 0$ and we prove:

$$\underset{x,y}{\exists} (|x - y| < \delta \;\wedge\; |x^2 - y^2| \geq 1)$$

We need to find appropriate witnesses for x, y.
For this we will determine some convenient values for them using the inequations above. We take $x = y + \delta/2$ in order to fulfill the first inequality, and then we need an y such that:
$(x + y) * (x - y) \geq 1$, that is: $(2y + \delta/2) * \delta/2 \geq 1$, from which we can find y in function of δ using $\geq$ as equality.

Subject 6. In the following formulae, t stands for a tuple (i. e. list of elements). Examples of tuples are: $\langle\rangle$ (the empty list), $\langle a, b\rangle$ (a list with two elements).
The binary infix function $\smile$ concatenates two tuples. Examples:

$$\langle\rangle\smile\langle a, b, c\rangle = \langle a, b, c\rangle.$$

$$\langle a, b\rangle\smile\langle b, c\rangle = \langle a, b, b, c\rangle.$$

Consider the following definitions:

$$\begin{array}{ll}
(1) & F[\langle\rangle] = \langle\rangle \\
(2) & \underset{a\ t}{\forall\forall}\ F[\langle a\rangle\smile t] = F[t]\smile\langle a\rangle \\
(3) & \underset{s}{\forall}\ G[\ \langle\rangle, s] = s \\
(4) & \underset{a\ t,s}{\forall\forall}\ G[\langle a\rangle\smile t,\ s] = G[t,\ \langle a\rangle\smile s,]
\end{array}$$

Use these equalities as rewrite rules in order to compute the expressions: $F[\langle a, b, c\rangle]$ and $G[\langle a, b, c\rangle,\ \langle\rangle]$.

Solution:

$$F[\langle a, b, c\rangle] \underset{(2)}{=} F[\langle b, c\rangle]\smile\langle a\rangle \underset{(2)}{=} F[\langle c\rangle]\smile\langle b\rangle\smile\langle a\rangle \underset{(2)}{=}$$

$$F[\langle\rangle]\smile\langle c\rangle\smile\langle b\rangle\smile\langle b\rangle \underset{(1)}{=} \langle\rangle\smile\langle c\rangle\smile\langle b\rangle\smile\langle a\rangle = \langle c, b, a\rangle.$$

$$G[\langle a, b, c\rangle,\ \langle\rangle] \underset{(4)}{=} G[\langle b, c\rangle,\ \langle a\rangle] \underset{(4)}{=} G[\langle c\rangle,\ \langle b, a\rangle] \underset{(4)}{=} G[\langle\rangle,\ \langle c, b, a\rangle] \underset{(3)}{=} \langle c, b, a\rangle.$$

Subject 7. Using the formulae above, prove:

$$\underset{t}{\forall}\ F[t] = G[t,\ \langle\rangle].$$

Hint: prove first $\underset{t}{\forall}\ \underset{s}{\forall}\ F[t]\smile s = G[t,\ s]$. For proving the latter, consider the predicate $P[t]$ defined as $\underset{s}{\forall}\ F[t]\smile s = G[t,\ s]$ and use the induction principle for tuples in order to prove $\underset{t}{\forall}\ P[t]$. (One must prove $P[\langle\rangle]$ and $\underset{a\ t}{\forall\forall}\ (P[t] \Rightarrow P[\langle a\rangle\smile\ t])$.) Note that for proving equalities it is enough to transform both sides by using known equalities as rewrite rules – and, of course, if necessary, the appropriate properties of tuples).

Solution:
We prove $\underset{t}{\forall}\ \underset{s}{\forall}\ F[t]\smile s = G[t,\ s]$ by tuple induction on t :

- We prove $\underset{s}{\forall}\ F[\langle\rangle]\smile s = G[\langle\rangle,\ s]$.
  For this we take arbitrary but fixed s and using (1) and (3) we rewrite the goal into: $\langle\rangle\smile s = s$, which is true by the concatenation property of the empty tuple.

- For arbitrary but fixed a and t, we assume: $\underset{s}{\forall}\ F[t]\smile s = G[t,\ s]$ (5)
  and we prove: $\underset{s}{\forall}\ F[\langle a\rangle\smile t]\smile s = G[\langle a\rangle\smile t,\ s]$.
  For this we take arbitrary but fixed s_0 and using (2) and (4) we rewrite the goal into: $F[t]\smile\langle a\rangle\smile s_0 = G[t,\ \langle a\rangle\smile s_0]$, which is (5) instantiated with $\langle a\rangle\smile s_0$ for s.