Introduction to Logic Programming

Foundations, First-Order Language

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What is a Logic Program

Logic program is a set of certain formulas of a first-order language.

In this lecture: syntax and semantics of a first-order language.

Introductory Examples

Representing “John loves Mary”: $loves(John, Mary)$.

$loves$: a binary predicate (relation) symbol.

Intended meaning: The object in the first argument of $loves$ loves the object in its second argument.

$John$, $Mary$: constants.

Intended meaning: To denote persons John and Mary, respectively.

Introductory Examples

$father$: A unary function symbol.

Intended meaning: The father of the object in its argument.

John's father loves John: $loves(father(John), John)$.

First-Order Language

Syntax

Semantics

Syntax

Alphabet

Terms

Formulas

Alphabet

A first-order alphabet consists of the following disjoint sets of symbols:

- A countable set of variables $\mathcal{V}$.
- For each $n \geq 0$, a set of n-ary function symbols $\mathcal{F}^n$. Elements of $\mathcal{F}^0$ are called constants.
- For each $n \geq 0$, a set of n-ary predicate symbols $\mathcal{P}^n$.
- Logical connectives $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$.
- Quantifiers $\exists$, $\forall$.
- Parenthesis '(', ')', and comma ','.

Notation:

- x, y, z for variables.
- f, g for function symbols.
- a, b, c for constants.
- p, q for predicate symbols.

Terms

Definition

- A variable is a term.
- If $t_1, \ldots, t_n$ are terms and $f \in \mathcal{F}^n$, then $f(t_1, \ldots, t_n)$ is a term.
- Nothing else is a term.

Notation:

- s, t, r for terms.

Example

- $plus(plus(x, 1), x)$ is a term, where $plus$ is a binary function symbol, 1 is a constant, x is a variable.
- $father(father(John))$ is a term, where $father$ is a unary function symbol and $John$ is a constant.

Formulas

Definition

- If $t_1, \ldots, t_n$ are terms and $p \in \mathcal{P}^n$, then $p(t_1, \ldots, t_n)$ is a formula. It is called an atomic formula.
- If A is a formula, $(\neg A)$ is a formula.
- If A and B are formulas, then $(A \vee B)$, $(A \wedge B)$, $(A \Rightarrow B)$, and $(A \Leftrightarrow B)$ are formulas.
- If A is a formula, then $(\exists x.A)$ and $(\forall x.A)$ are formulas.
- Nothing else is a formula.

Notation:

- A, B for formulas.

Eliminating Parentheses

- Excessive use of parentheses often can be avoided by introducing binding order.
- $\neg, \forall, \exists$ bind stronger than $\vee$.
- $\vee$ binds stronger than $\wedge$.
- $\wedge$ binds stronger than $\Rightarrow$ and $\Leftrightarrow$.
- Furthermore, omit the outer parentheses and associate $\vee, \wedge, \Rightarrow, \Leftrightarrow$ to the right.

Eliminating Parentheses

Example

The formula

$$(\forall y.(\forall x.((p(x)) \wedge (\neg r(y))) \Rightarrow ((\neg q(x)) \vee (A \vee B)))))$$

due to binding order can be rewritten into

$$(\forall y.(\forall x.(p(x) \wedge \neg r(y) \Rightarrow \neg q(x) \vee (A \vee B))))$$

which thanks to the convention of the association to the right and omitting the outer parentheses further simplifies to

$$\forall y.\forall x.(p(x) \wedge \neg r(y) \Rightarrow \neg q(x) \vee A \vee B).$$

Example

Translating English sentences into first-order logic formulas:

1. Every rational number is a real number.

$$\forall x.(rational(x) \Rightarrow real(x))$$

2. There exists a number that is prime.

$$\exists x.\, prime(x)$$

3. For every number x, there exists a number y such that $x < y$.

$$\forall x.\, \exists y.\, x < y$$

Assume:

- $rational, real, prime$: unary predicate symbols.
- $<$: binary predicate symbol.

Example

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Example

Translating English sentences into first-order logic formulas:
For each natural number there exists exactly one immediate successor natural number.

$$\forall x.(\exists y.succ(x,y) \wedge \forall z.(succ(x,z) \Rightarrow y \doteq z)))$$

Assume:

- $succ$: binary predicate symbol for immediate successor.
- $\doteq$: binary predicate symbol for equality.

Example

Translating English sentences into first-order logic formulas:

There is no natural number whose immediate successor is 0.

$$\neg\exists x.\, succ(x, zero)$$

Assume:

- $zero$: constant for 0.
- $succ$: binary predicate symbol for immediate successor.

Semantics

Meaning of a first-order language consists of an universe and an appropriate meaning of each symbol.

This pair is called structure.

Structure fixes interpretation of function and predicate symbols.

Meaning of variables is determined by a variable assignment.

Interpretation of terms and formulas.

Structure

Structure: a pair (D, I).

D is a nonempty universe, the domain of interpretation.

I is an interpretation function defined on D that fixes the meaning of each symbol associating

- to each $f \in \mathcal{F}^n$ an n-ary function $f_I : D^n \to D$,
  (in particular, $c_I \in D$ for each constant c)
- to each $p \in \mathcal{P}^n$ different from $\doteq$, an n-ary relation p_I on D.

Variable Assignment

A structure $\mathcal{S} = (D, I)$ is given.

Variable assignment $\sigma_{\mathcal{S}}$ maps each $x \in \mathcal{V}$ into an element of D: $\sigma_{\mathcal{S}}(x) \in D$.

Given a variable x, an assignment $\vartheta_{\mathcal{S}}$ is called an x-variant of $\sigma_{\mathcal{S}}$ iff $\vartheta_{\mathcal{S}}(y) = \sigma_{\mathcal{S}}(y)$ for all $y \neq x$.

Interpretation of Terms

A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.

Value of a term t under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$, $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t)$:

- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(x) = \sigma_{\mathcal{S}}(x)$.
- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(f(t_1, \ldots, t_n)) = f_I(Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t_1), \ldots, Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t_n))$.

Interpretation of Formulas

A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.

Value of an atomic formula under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ is one of $true, false$:

- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(s \doteq t) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(s) = Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t)$.
- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(p(t_1, \ldots, t_n)) = true$ iff
  $(Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t_1), \ldots, Val_{\mathcal{S},\sigma_{\mathcal{S}}}(t_n)) \in p_I$.

Interpretation of Formulas

A structure $\mathcal{S} = (D, I)$ and a variable assignment $\sigma_{\mathcal{S}}$ are given.

Values of compound formulas under $\mathcal{S}$ and $\sigma_{\mathcal{S}}$ are also either *true* or *false*:

- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\neg A) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$.
- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \vee B) = true$ iff
  $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \wedge B) = true$ iff
  $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = true$ and $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \Rightarrow B) = true$ iff
  $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = false$ or $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B) = true$.
- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A \Leftrightarrow B) = true$ iff $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(A) = Val_{\mathcal{S},\sigma_{\mathcal{S}}}(B)$.
- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\exists x.A) = true$ iff
  $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for some x-variant $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.
- $Val_{\mathcal{S},\sigma_{\mathcal{S}}}(\forall x.A) = true$ iff
  $Val_{\mathcal{S},\vartheta_{\mathcal{S}}}(A) = true$ for all x-variants $\vartheta_{\mathcal{S}}$ of $\sigma_{\mathcal{S}}$.

Interpretation of Formulas

A structure $\mathcal{S} = (D, I)$ is given.

The value of a formula A under $\mathcal{S}$ is either $true$ or $false$:

- $Val_{\mathcal{S}}(A) = true$ iff $Val_{\mathcal{S}}, \sigma_{\mathcal{S}}(A) = true$ for all $\sigma_{\mathcal{S}}$.

$\mathcal{S}$ is called a model of A iff $Val_{\mathcal{S}}(A) = true$.

Written $\models_{\mathcal{S}} A$.

Example

Formula: $\forall x.(p(x) \Rightarrow q(f(x), a))$.

Define $\mathcal{S} = (D, I)$ as

- $D = \{1, 2\}$,
- $a_I = 1$,
- $f_I(1) = 2, f_I(2) = 1$,
- $p_I = \{2\}$,
- $q_I = \{(1, 1), (1, 2), (2, 2)\}$.

If $\sigma_{\mathcal{S}}(x) = 1$, then $Val_{\mathcal{S},\sigma_S}(\forall x.(p(x) \Rightarrow q(f(x), a))) = true$.

If $\sigma_{\mathcal{S}}(x) = 2$, then $Val_{\mathcal{S},\sigma_S}(\forall x.(p(x) \Rightarrow q(f(x), a))) = true$.

Hence, $\models_{\mathcal{S}} A$.

Validity, Unsatisfiability

A formula A is valid, if $\models_{\mathcal{S}} A$ for all $\mathcal{S}$.

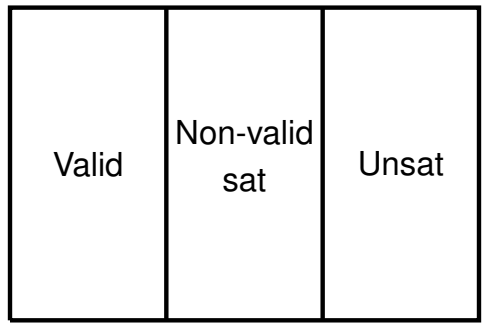
Written $\models A$.

A formula A is unsatisfiable, if $\models_{\mathcal{S}} A$ for no $\mathcal{S}$.

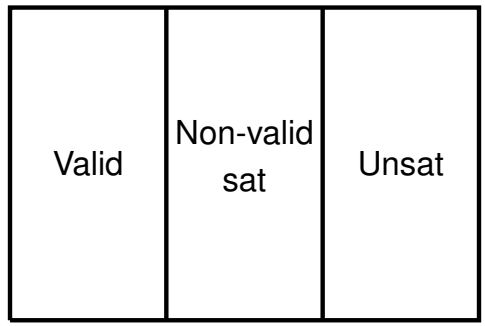
If A is valid, then $\neg A$ is unsatisfiable and vice versa.

The notions extend to (multi)sets of formulas.

For $\{A_1, \ldots, A_n\}$, just formulate them for $A_1 \wedge \cdots \wedge A_n$.



Validity, Unsatisfiability



- $\forall x.p(x) \Rightarrow \exists y.p(y)$ is valid.
- $p(a) \Rightarrow \neg\exists x.p(x)$ is satisfiable non-valid.
- $\forall x.p(x) \wedge \exists y.\neg p(y)$ is unsatisfiable.

Logical Consequence

Definition

A formula A is a logical consequence of the formulas $B_1, \ldots, B_n$, if every model of $B_1 \wedge \cdots \wedge B_n$ is a model of A.

Example

- $mortal(socrates)$ is a logical consequence of $\forall x.(person(x) \Rightarrow mortal(x))$ and $person(socrates)$.
- $cooked(apple)$ is a logical consequence of $\forall x.(\neg cooked(x) \Rightarrow tasty(x))$ and $\neg tasty(apple)$.
- $genius(einstein)$ is not a logical consequence of $\exists x.person(x) \wedge genius(x)$ and $person(einstein)$.

Logic Programs

Logic programs: finite non-empty sets of formulas of a special form, called program clauses.

Program clause: $\forall x_1 \ldots \forall x_k . B_1 \wedge \cdots \wedge B_n \Rightarrow A$, where

- $k, n \geq 0$,
- A and the B's are atomic formulas,
- $x_1, \ldots, x_k$ are all the variables which occur in $A, B_1, \ldots, B_n$.

Usually written in the inverse implication form without quantifiers and conjunctions:

$$A \Leftarrow B_1, \ldots, B_n$$

Goal

Goals or queries of logic programs: formulas of the form

$$\exists x_1 \ldots \exists x_k . B_1 \wedge \cdots \wedge B_n,$$

where

- $k, n \geq 0$,
- the B's are atomic formulas,
- $x_1, \ldots, x_k$ are all the variables which occur in $B_1, \ldots, B_n$.

Usually written without quantifiers and conjunction:

$$B_1, \ldots, B_n$$

The problem is to find out whether a goal is a logical consequence of the given logic program or not.

The Problem and the Idea

Let P be a program and G be a goal.

Problem: Is G a logical consequence of P?

Idea: Try to show that the set of formulas $P \cup \{\neg G\}$ is inconsistent.

How? This we will learn in this course.

Example

Let P consist of the two clauses:

- $\forall x.mortal(x) \Leftarrow person(x)$.
- $person(socrates)$.

Goal: $G = \exists x.mortal(x)$.

$\neg G$ is equivalent to $\forall x.\neg mortal(x)$.

The set

$$\{\forall x.mortal(x) \Leftarrow person(x),\ person(socrates),\ \forall x.\neg mortal(x)\}$$

is inconsistent.

Hence, G is a logical consequence of P.

We can even compute the witness term for the goal:
$x = socrates$.

How? This we will learn in this lecture.