Introduction to Sequent Calculus

Tudor Jebelean
RISC–Linz
Johannes Kepler University

October 25, 2019

Abstract

This is a tutorial introduction to sequent calculus, starting from a certain natural style of proving, and based on an example. We present this as a an exercise in constructing the abstract model "sequent calculus" of the of the real world method of proving logical statements in "natural style".

1 Introduction

The purpose of mathematical logic is to study the real–world[1] activity of proving mathematical statements and to construct and analyse abstract models of it.

In this presentation we analyse a concrete example of proof developed in the "natural style" which is used by mathematicians, we identify the essential aspects of the proof method, and we introduce sequent calculus as the mathematical model which can be constructed as an abstraction of it.

2 Proving in Natural Style

2.1 Proof Example

Let us consider the propositional formula:
(G0) $((A \Rightarrow C) \vee (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C)$.

The proof this formula, by the intuitive method which is "built in" our brain, as the result of our mathematical training, may look as follows:

[1] For proving (G0), we assume:

(A1) $(A \Rightarrow C) \vee (B \Rightarrow C)$

and we prove:

(G1) $(A \wedge B) \Rightarrow C$.

[1] Mathematics operates on abstract models, however the activity of mathematicians takes place in the real world, which therefore also includes the proofs they develop.

[2] For proving (G1), we assume:

(A2) $A \wedge B$

and we prove:

(G2) C.

[3] From (A2) we obtain:

(A2.1) A

and

(A2.2) B.

[4] We prove (G2) by case distinction, using the disjunction (A1).

Case 1:

(A3) $A \Rightarrow C$.

[5] From (A2.1), by (A3), using modus ponens we obtain the goal (G2).

Case 2:

(A4) $B \Rightarrow C$.

[6] From (A2.2), by (A4), using modus ponens we obtain the goal (G2).

This style of proof is usually called *natural style*, because it is typically used by humans.

This example and the subsequent proof may look unusual and somewhat artificial, because in mathematical texts we do not really encounter theorems which look like (G0), neither do we find proofs which have all the steps presented in detail like in the proof above. However, the example captures the essence of the method, and it is simple enough to be presented and analysed in detail in a relatively small space. In mathematical texts the steps of the proofs are not different from the ones above, only the *presentation* of the proof is different, usually abbreviated by omitting the steps which are considered obvious for the intented audience. However for the purpose of constructing a mathematical model of this process, we need to exhibit every single detail.

Analysing of this proof we can identify the essential elements of the proof method: *proof situation, proof step, proof tree*, and *inference rule*.

2.2 Proof Situations

First, we notice that at each moment in the proof there is a certain "goal" which has to be proven, and a certain set of "assumptions" which can be used in the proof (forming a kind of "knowledge base" of propositions assumed to be true at that moment). For instance, at the very beginning of the proof, the goal is (G0), and there are no assumptions. After the first proof step [1], the goal is (G1) and there is only one assumption (A1). After the second proof step [2], the goal is (G2), and the assumptions are (A1) and (A2). Thus, at each moment in the course of the proof, one has a *proof situation*, composed from a set of assumptions and a goal, which may be represented as:

$$\begin{array}{|c|}\hline \text{goal} \\ \hline \text{assumption 1} \\ \text{assumption 2} \\ \ldots \\ \hline \end{array}$$

For instance, the initial proof situation in the proof example is represented by:

$$\text{(a)} \begin{array}{|c|}\hline ((A \Rightarrow C) \vee (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C) \\ \hline \\ \hline \end{array}$$

The proof situation in after step [1] is:

$$\text{(b)} \begin{array}{|c|}\hline (A \wedge B) \Rightarrow C \\ \hline (A \Rightarrow C) \vee (B \Rightarrow C) \\ \hline \end{array}$$

After step [2]:

$$\text{(c)} \begin{array}{|c|}\hline C \\ \hline A \wedge B \\ (A \Rightarrow C) \vee (B \Rightarrow C) \\ \hline \end{array}$$

After step [3]:

$$\text{(d)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ (A \Rightarrow C) \vee (B \Rightarrow C) \\ \hline \end{array}$$

After step [4] we have two proof situations:

$$\text{(e)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ A \Rightarrow C \\ \hline \end{array} \qquad \text{(f)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ B \Rightarrow C \\ \hline \end{array}$$

These two are modified by steps [5] and [6], respectively, into:

$$\text{(g)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ C \\ \hline \end{array} \qquad \text{(h)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ C \\ \hline \end{array}$$

The proof situations above have the property that the goal is among the assumptions, thus the proof stops successfully. We can call these proof situations *final*. There may be other kind of proof situations which we consider successfull, in particular the one where there are two contradictory assumptions. As we know from our experience, it may also happen that the proof stops because there are no possible ways to continue the proof, and in this case the proof does not succeed[2]. These are also final proof situations.

We can see that a proof may have several branches, and the whole proof succeeds if all the branches finish with successful final proof situations.

What we can also notice in the schematic representation of the proof situation is the removal of some assumptions during the proof, at least in the sense that they are replaced by other assumptions. From the methodological point of view[3], it is good to notice that by trying to represent in a more exact way the process of proving, we identify certain operations which are not clearly described in the text. This exhibits the power of the systematic scientific analysis, and the necessity of it. Indeed, in human produced proofs one never mentions explicitely the operations of removal or replacement of some assumptions. It simply happens that some assumptions are never used again in the proof, like in our proof the assumption $(A \Rightarrow C) \vee (B \Rightarrow C)$ is not used after step [4] and the assumptions $A \Rightarrow C$ and $B \Rightarrow C$ are not used after steps [5] and [6]. In our schematic representation these assumptions are removed after using them, and in this particular proof they are not necessary anymore, but is this a general rule which can allways be applied? Does it happen in every proof in similar situations that these kind of assumptions are not necessary anymore? The answer is "yes", however this is not obvious in this moment, and here we already have a glimpse of the usefulness of constructing the abstract model of the process and studying it mathematically: by this investigation we will be able, among other, to provide an answer to these questions[4].

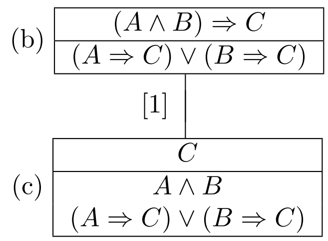
2.3 Proof Steps and Proof Tree

The proof proceeds in individual *proof steps*, each step consisting in creating one or more proof situations from the current proof situation. The new proof situation[s] may differ with respect to the goal, the set of assumptions, or both. For instance, step [2] creates one proof situation and modifies both the goal and the set of assumptions:

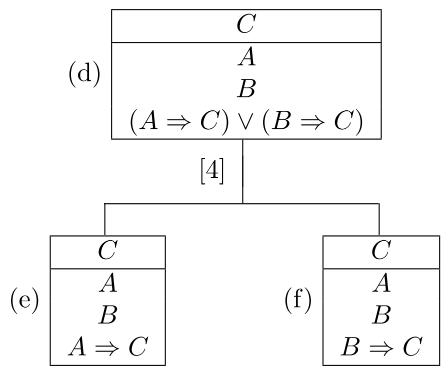
[2]We do not address here the situations when *we do not know* how to continue the proof.

[3]With respect to the methodology of analysing the proofs performed by humans in order to create an abstract model of the process.

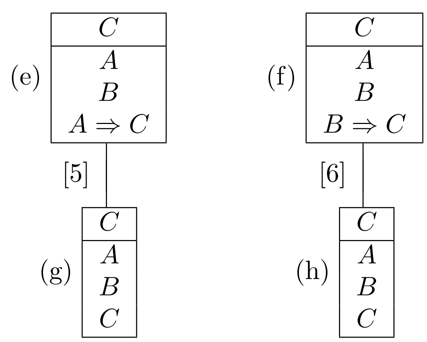
[4]The question about the removal of assumptions during proofs may appear insignificant for the proving practician, why should we care about them since we just do not use them anymore? Even from the theoretical point of view of classical logic the problem is not important, because classical logic is concerned with finding methods which make the proof *possible*, and not with the efficiency of these methods. However, when having in mind *automated reasoning*, where the efficiency of the methods is essential, it is quite important to remove unnecessary assumptions, because the mechanical prover has to take into account all the assumptions at every step for making a decision about the next operation to perform, and in fact the resource consumption depends exponentially on the number of assumptions.



Step [4] creates 2 proof situations and modifies only the set of assumptions:



Steps [5] and [6] are similar, each uses two assumptions and simplifies an assumption:



If we combine pictorially all proof steps, then we obtain the *proof tree* presented in Fig. 1.

Formally, a proof tree is a graph having as nodes the proof situations and as arcs the proof steps. The root is the initial proof situation and the leaves are the final proof situations.

2.4 Inference Rules

Each step of the proof follows a certain rule. For instance, steps [1] and [2] follow the rule:

(R1) In order to prove that a statement γ_1 implies a statement γ_2

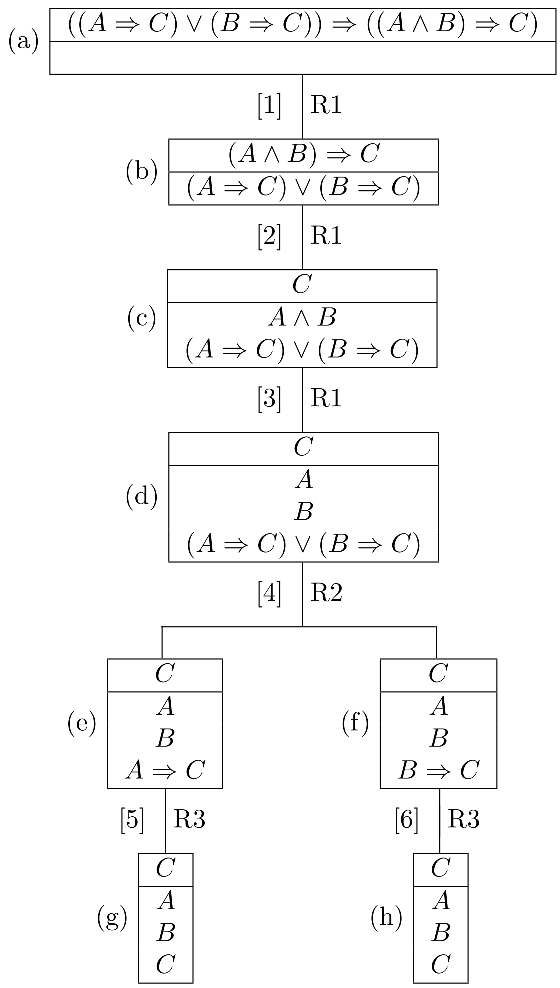


Figure 1: Proof tree.

assume γ_1 and prove γ_2.

In order to create a formal model of the inference rules, we need to represent proof situations in an abstract way, by emphasizing the elements which are important for the application of the inference rules.

For instance, a proof situation which is appropriate for the application of rule (R1) can be represented as:

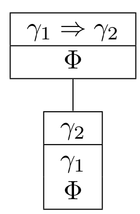
$$\begin{array}{|c|}\hline \gamma_1 \Rightarrow \gamma_2 \\ \hline \Phi \\ \hline \end{array}$$

where γ_1, γ_2 are individual formulae composing the goal and Φ is a (possibly empty) set of formulae — the assumptions. Likewise, the proof situation created by rule (R1) from this schematic proof situation may be represented as:

$$\begin{array}{|c|}\hline \gamma_2 \\ \hline \gamma_1 \\ \Phi \\ \hline \end{array}$$

which suggests that the goal now is the right hand side of the implication which was the goal of the previous proof situation, while the new set of assumptions consists from the old set of assumptions to which we add the left hand side of the previous goal.

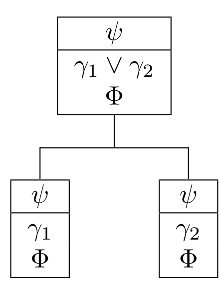
Thus inference rule (R1) may be represented as:



Step [4] follows the rule:

> (R2) If an assumption is a disjunction, create a proof situation for every disjunct, adding it to the current assumptions, and keep the same goal.

By the same principle, this inference rule may be represented as:



Steps [5] and [6] follow the rule:

(R3) If an assumption is an implication and the LHS[5] of it is also an assumption, then replace the implication by the RHS[6] of it.

We may represent this inference rule as:

$$\begin{array}{|c|}\hline \psi \\ \hline \gamma_1 \Rightarrow \gamma_2 \\ \gamma_1 \\ \Phi \\ \hline \end{array}$$

$$\Big|$$

$$\begin{array}{|c|}\hline \psi \\ \hline \gamma_2 \\ \gamma_1 \\ \Phi \\ \hline \end{array}$$

By inspecting the schematic proof tree of the proof example, we can see that the proof steps are *instances* of the inference rules.

Note also that the successfull proof situations where the goal is obtained as one of the assumptions can be represented as:

$$\begin{array}{|c|}\hline \gamma \\ \hline \gamma \\ \Phi \\ \hline \end{array}$$

and indeed the final nodes (the leaves) of the proof tree are instances of this schematic proof situation.

In some proofs we may have a successfull proof situation by “contradictory assumptions” (both a formula and its negation are in the set of assumptions), this can be represented as:

$$\begin{array}{|c|}\hline \psi \\ \hline \gamma \\ \neg\gamma \\ \Phi \\ \hline \end{array}$$

3 Sequent Calculus

3.1 Basic Elements

Now we construct the formal model for the elements identified above.

The idea of *proof situation* is modeled by the notion of *sequent.* A sequent is a pair $\langle \Phi, \Psi \rangle$, where Φ and Ψ are sets of formulae. The traditional notation used in mathematical logic for concrete sequents is:

$$\varphi_1, \ldots, \varphi_n \ \vdash \ \psi_1, \ldots, \psi_m,$$

[5] Left hand side.
[6] Right hand side.

where $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m$ are individual formulae. (When one of the sets is empty, then nothing is written on the corresponding side of $\vdash$.) We call the formulae on the left hand side *assumptions* and the ones on the right hand side *goals.* (Other names used in presentations are *premises/conclusions* and *antecedents/postcedents.*)

In this presentation[7], we define the semantics of sequents by considering that a sequent as above is the abbreviation of the formula

$$\varphi_1 \wedge \ldots \wedge \varphi_n \ \vdash \ \psi_1 \wedge \ldots \wedge \psi_m.$$

More abstractly, $\Phi \vdash \Psi$ is the abbreviation of $(\wedge\Phi) \Rightarrow (\vee\Psi)$, where $\wedge$ ($\vee$) is the conjunction (disjunction) of all the formulae in the respective sets. This also expresses the situations when one of the sets is empty: the empty set of assumptions (goals) represents the constant $\mathbf{T}(\mathbf{F})$[8].

Note that the abstract model is a little more general than the proof situation, because we may have *several goals*, while in the natural style proving we only have one goal at a time. Also, the model allows to have no goal at all (when the set of goals is empty). By the semantics defined above, this correspons to the proof situations when we are looking for a contradiction (one may also say that the goal is the truth constant $\mathbf{F}$, for instance when one applies the inference rule "proof by contradiction").

In order to represent schematically the inference rules, we represent abstract sequents as

$$\varphi_1, \varphi_2, \ldots, \Phi_1, \Phi_2, \ldots \vdash \psi_1, \psi_2, \ldots, \Psi_1, \Psi_2, \ldots,$$

where $\varphi_1, \varphi_2, \ldots, \psi_1, \psi_2, \ldots$ are either variables (at meta level) representing individual formulae, either logical constructions over such variables and constants, representing formulae which have a certain structure (like e. g. $\gamma_1 \Rightarrow \gamma_2$), while $\Phi_1, \Phi_2, \ldots, \Psi_1, \Psi_2, \ldots$ are sets of formulae. Each list of symbols is an abreviation for the respective set of formulae, thus the sequent represented is:

$$\{\varphi_1, \varphi_2, \ldots\} \cup \Phi_1 \cup \Phi_2 \cup \ldots \ \vdash \ \{\psi_1, \psi_2, \ldots\} \cup \Psi_1 \cup \Psi_2 \cup \ldots$$

In the formal model, the inference rules are represented by *sequent rules.* A sequent rule is a pair $\langle\{S_1, S_2, \ldots\}, S_0\rangle$ between a set of sequents and a sequent and is traditionally represented as:

$$\frac{S_1, S_2, \ldots}{S_0}$$

For the semantics of a sequent rule we adopt $(\wedge\{S_1, S_2, \ldots\}) \Rightarrow S_0$, and note that when the set is empty (in the traditional notation there will be nothing written above the line), then semantic of the sequent rule is $\mathbf{T} \Rightarrow S_0$, thus the validity of the sequent reduces to the validity of S_0 : such sequents are also called *axioms* and they model the successful terminal proof situations.

[7]There are different definitions of the semantics of sequents, we adopt here this particular one because it simplifies reasoning about sequents.

[8] *True* (*False*).

A *calculus* (in the sense of sequent calculus) is a collection of sequent rules, and constitutes a proof method for the validity of sequents.

With this notation, the inference rule (R1) is modeled as the sequent rule:

$$\dfrac{\Phi, \gamma_1 \ \vdash\ \gamma_2, \Psi}{\Phi \ \vdash\ \gamma_1 \Rightarrow \gamma_2, \Psi} \vdash\Rightarrow$$

and the inference rule (R2) is modeled as the sequent rule:

$$\dfrac{\Phi, \gamma_1 \ \vdash\ \Psi \qquad \Phi, \gamma_2 \ \vdash\ \Psi}{\Phi, \gamma_1 \vee \gamma_2 \ \vdash\ \Psi} \vee\vdash$$

In contrast to the natural–style inference rules, note the addtitional Ψ in the goals, these are the rest of the goal formulae, which are not changed by the application of the rule.

The successful situation (goal is assumed) in the example proof is modeled by the sequent rule:

$$\dfrac{}{\Phi, \gamma \ \vdash\ \gamma, \Psi} \text{a}$$

As for inference rules in natural–style proving, the sequent rules can be instantiated in order to realize inferences on sequents. Some inferences corresponding to the proof steps [2], [4], and after step [5] in the example proof, which are instances of the sequent rules "$\vdash\Rightarrow$", "$\vee\vdash$", and "a", respectively, are:

$$\dfrac{(A \Rightarrow C) \vee (B \Rightarrow C),\ A \wedge B \ \vdash\ C}{(A \Rightarrow C) \vee (B \Rightarrow C) \ \vdash\ (A \wedge B) \Rightarrow C} \vdash\Rightarrow$$

$$\dfrac{A \Rightarrow C,\ A \wedge B \ \vdash\ C \qquad B \Rightarrow C,\ A \wedge B \ \vdash\ C}{(A \Rightarrow C) \vee (B \Rightarrow C),\ A \wedge B \ \vdash\ C} \vee\vdash$$

$$\dfrac{}{A,\ B,\ C \ \vdash\ C} \text{a}$$

The natural–style proof trees are represented in the model by sequent proof trees, which are graphs of sequents, and whose arcs correspond to instances of sequent rules, as for example in:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{S_7} \quad \dfrac{}{S_9}}{S_5}}{S_4} \qquad \dfrac{\dfrac{\dfrac{}{S_8}}{S_6} \quad \dfrac{}{S_2}}{S_3}}{S_1}}{S_0}$$

In this example, the sequents S_2, S_7, S_8, and S_9 are axioms, and the tree constitutes a sequent calculus proof of the validity of the root node S_0.

A sequent proof for our example is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A,B,C \;\vdash\; C}\,\mathrm{a}}{A,B,A\Rightarrow C \;\vdash\; C}\,\mathrm{MP} \qquad \dfrac{\dfrac{}{A,B,C \;\vdash\; C}\,\mathrm{a}}{A,B,B\Rightarrow C \;\vdash\; C}\,\mathrm{MP}}{A,B,(A\Rightarrow C)\vee(B\Rightarrow C) \;\vdash\; C}\,\vee\vdash}{A\wedge B,(A\Rightarrow C)\vee(B\Rightarrow C) \;\vdash\; C}\,\wedge\vdash}{(A\Rightarrow C)\vee(B\Rightarrow C) \;\vdash\; (A\wedge B)\Rightarrow C}\,\vdash\Rightarrow}{\vdash\; (A\Rightarrow C)\vee(B\Rightarrow C)\Rightarrow(A\wedge B)\Rightarrow C}\,\vdash\Rightarrow$$

Suggested exercise: Repeat the same development for the formulae:
$((A \wedge B) \Rightarrow C) \Rightarrow ((A \Rightarrow C) \vee (B \Rightarrow C))$ and
$((A \Rightarrow C) \wedge (B \Rightarrow C)) \Leftrightarrow ((A \vee B) \Rightarrow C)$.