to be prepared for 19.11.2019

Exercise 1. Let R be a commutative ring with 1. Demonstrate that the following statements are equivalent:

1. Every ideal in R is generated by a finite set.
2. There are no infinite strictly ascending chains of ideals in R.
3. Every nonempty set S of ideals contains a maximal element (i.e. an ideal $a \in S$ such that $\forall b \in S$, if $a \subseteq b$ then $a = b$.

Exercise 2. Use resultants to find the implicit representation, i.e. a polynomial equation just in x, y, and z of the parametrized surface

$$\begin{aligned} x &= 1 + s + t + st \\ y &= 2 + s + st + t^2 \\ z &= s + t + s^2 \end{aligned}$$

Exercise 3.

1. Find a rational parametrization of the circle $x^2 + y^2 = 1$.
2. Compute a rational parametrization for the sphere $x^2 + y^2 + z^2 = 1$.

Exercise 4. Let $<$ be an admissible ordering on $[X]$. Prove the following statements.

1. If $s, t \in [X]$ and s divides t then $s \leq t$.
2. $<$ (or actually $>$) is Noetherian, i.e. there are no infnite chains of the form $t_0 > t_1 > t_2 > \cdots$, and consequently every subset of $[X]$ has a smallest element.

Exercise 5. The graduated reverse lexicographic ordering on power products of $x_1, \ldots, x_n$ $<_{\text{grlex}}$ is defined by

$$s <_{\text{grlex}} t \quad \text{iff} \quad \deg(s) < \deg(t) \quad \text{or} \quad \deg(s) = \deg(t) \quad \text{and} \quad t <_{\text{lex},\pi} s;$$

where π is the permutation on n letters given by $\pi(j) = n - j + 1$ and $<_{\text{lex},\pi}$ is the lexicographic order wrto. π. Prove that $<_{\text{grlex}}$ is an admissible ordering.

Exercise 6. Let $<_1$ be an admissible ordering on $X_1 = [x_1, \ldots, x_i]$ and $<_2$ an admissible ordering on $X_2 = [x_{i+1}, \ldots, x_n]$. Show that the product ordering $<_{prod,i,<_1,<_2}$ on $X = [x_1, \ldots, x_n]$ is an admissible ordering.