Due date: 17.12.2019

Exercise 1

Let F be a field and consider the linear polynomials

$$f_i = a_{i1}\,x_1 + \cdots + a_{in}\,x_n \in F[x_1, \dots, x_n],$$

where $1 \leq i \leq m$. Denote by $A = \left(a_{ij}\right)$ the $m \times n$-matrix whose rows are formed from coefficients of the f_i. Let $B = \left(b_{ij}\right)$ be the reduced row echelon matrix obtained from A. The goal of this exercise is to prove that the non-zero polynomials obtained from the rows of B constitute the normed, reduced Gröbner basis of the ideal $I = \langle f_1, \dots, f_m\rangle \subseteq F[x_1, \dots, x_n]$ with respect to the lexicographic ordering with $x_1 > \cdots > x_n$.

(a) Let l be the number of non-zero rows in B and consider the polynomials $g_1, \dots, g_l$, where

$$g_k = b_{k1}\,x_1 + \cdots + b_{kn}\,x_n$$

and $1 \leq k \leq l$. Show that $\langle g_1, \dots, g_l\rangle = I$.

(b) Show that $G = \{g_1, \dots, g_l\}$ is a Gröbner basis for the ideal I with respect to the aforementioned admissible ordering.

(c) Explain why the Gröbner basis G is normed and reduced.

Exercise 2

Consider the polynomials $f = y - x^2$ and $g = z - x^3$ in $\mathbb{Q}[x, y, z]$. Prove or disprove: $\{f, g\}$ is a Gröbner basis of the ideal $\langle f, g\rangle \subseteq F[x, y, z]$ with respect to the lexicographic ordering with $x > y > z$. What would the result be if we choose a permutation of the variables where x is not greater than y and z?

Exercise 3

Determine whether the polynomial $x\,y^3 + y^5 - z^3 - z^2$ is in the ideal $\left\langle y - x^3, x^2\,y - z\right\rangle \subseteq \mathbb{Q}[x, y, z]$.

Exercise 4

Given the polynomials $f, g, h \in \mathbb{Q}[x, y, z]$ such that

$$\begin{aligned} f &= x^4 - 2\,x + z + 1, \\ g &= x^2 + y^2 - 2, \\ h &= x^5 - 6\,x^3 + x^2 - 1. \end{aligned}$$

(a) Is the set $\{f, g, h\}$ a Gröbner basis of the ideal $I = \langle f, g, h\rangle$ with respect to the lexicographic ordering with $z > y > x$?

(b) How many (complex) solutions does the system $f = g = h = 0$ have?

(c) Give a basis of the $\mathbb{Q}$-vector space $\mathbb{Q}[x, y, z]/I$. How does the dimension of this vector space relate to the number of solutions of such a system?

Exercise 5

Let $G \subseteq F[x_1, \dots, x_n]$ and denote by I the ideal generated by G. Show that G is a Gröbner basis of I (with respect to some admissible ordering $>$) if and only if $\langle \text{initial}_>(G)\rangle = \langle \text{initial}_>(I)\rangle$, cf. Theorem 4.2.22 (f).