Due date: 12.11.2019

Remark
The last three exercises are more challenging. You should, however, at the very least solve Exercise 1 and Exercise 2.

Exercise 1
Consider the polynomials $f(x), g(x) \in \mathbb{Q}[x]$, where

$$f(x) = x^4 - 2\,x^2 - 3\,x - 2 \quad \text{and} \quad g(x) = x^4 - x^3 - 3\,x^2 + 4\,x - 4.$$

(a) Plot the corresponding polynomial functions in a suitable range, i.e. choose an interval $[a, b] \subseteq \mathbb{R}$ and draw the graph of the functions

$$f : [a, b] \to \mathbb{R},\ v \mapsto f(v) \quad \text{and} \quad g : [a, b] \to \mathbb{R},\ v \mapsto g(v),$$

where $f(v), g(v)$ denote the evaluation of the polynomials $f(x), g(x)$ at the number v.

(b) Based on your observations of the previous item, do $f(x)$ and $g(x)$ have common roots? What is the expected value of the resultant in this case?

(c) Construct the Sylvester matrix $\mathrm{Syl}_x(f(x), g(x))$ and use a CAS to compute the resultant $\mathrm{res}_x(f(x), g(x))$. What is the resultant of $f(x)/(x-2)$ and $g(x)$?

(d) Notice that the antecedent resultant computations yield integers in both cases. This is not a coincidence: Given non-constant polynomials $p(x), q(x) \in \mathbb{Z}[x]$, explain why $\mathrm{res}_x(p(x), q(x)) \in \mathbb{Z}$.

Exercise 2
In this exercise you should verify two straightforward identities of resultants. Let $f(x), g(x) \in I[x]$ be non-constant polynomials over an integral domain I such that $\deg(f(x)) = m$ and $\deg(g(x)) = n$.

(a) Prove that

$$\mathrm{res}_x(f(x), g(x)) \ = \ (-1)^{n\,m}\, \mathrm{res}_x(g(x), f(x)).$$

Hint: How does the determinant of a matrix change when two rows are interchanged?

(b) Let $\lambda, \mu \in I \setminus \{0\}$. Show that

$$\mathrm{res}_x(\lambda \cdot f(x), \mu \cdot g(x)) \ = \ \lambda^n\, \mu^m\, \mathrm{res}_x(f(x), g(x)).$$

Exercise 3
Consider non-constant polynomials $f(x), g(x) \in F[x]$ over a field F. Let I denote the ideal $\langle f(x) \rangle \subseteq F[x]$ and define the multiplication map

$$\mu : F[x]\,/\,I \to F[x]\,/\,I,\ h(x) + I \mapsto g(x)\,h(x) + I.$$

Now show that

$$\mathrm{res}_x(f(x), g(x)) \ = \ \mathrm{lc}(f(x))^{\deg(g(x))}\, \det(\mu),$$

where $\mathrm{lc}(f(x))$ denotes the leading coefficient of $f(x)$.

Exercise 4

Let $f(x), g(x) \in F[x]$ be non-constant polynomials over a field F. Perform division with remainder of $f(x)$ by $g(x)$, i.e. $f(x) = q(x)\,g(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$. We omit the dependency on the variable x in the following for better readability. Show that

$$\operatorname{res}_x(f, g) \;=\; (-1)^{\deg(f)\deg(g)}\,\operatorname{lc}(g)^{\deg(f)-\deg(r)}\,\operatorname{res}_x(g, r)$$

whenever $\operatorname{res}_x(g, r)$ is defined.

Exercise 5

Consider non-constant polynomials $f(x), g(x) \in F[x]$. Denote by $\zeta_1, \dots, \zeta_m$ and $\eta_1, \dots, \eta_n$ the roots of $f(x)$ and $g(x)$ in their common splitting field, respectively. Prove that

$$\operatorname{res}_x(f, g) \;=\; \operatorname{lc}(f)^{\deg(g)}\,\operatorname{lc}(g)^{\deg(f)} \prod_{i=1}^{m} \prod_{j=1}^{n} (\zeta_i - \eta_j).$$