to be prepared for 03.12.2019

Exercise 1. Consider the partial order $\leq_\pi$ on $\mathbb{N}^n$ defined as

$$(a_1,\ldots,a_n) \leq_\pi (b_1,\ldots,b_n) \iff a_i \leq b_i \ \forall i \in \{1,\ldots,n\}.$$

Prove that any set $X \subseteq \mathbb{N}^n$ contains a finite set $Y \subseteq X$ such that

$$\forall x \in X \ \exists y \in Y \text{ with } y \leq_\pi x.$$

Exercise 2. In Lemma 4.2.14 it is claimed, that

$$g_1 \longrightarrow_F g_2 \ \Rightarrow \ a \cdot s \cdot g_1 \longrightarrow_F a \cdot s \cdot g_2.$$

Verify this statement on the basis of the following example:
$R = K[x,y]$, $F = \{x^2y^2 + y - 1,\ x^2y + x\}$, $g_1 = x^5y^5$, $s = x^3y^3$.

Exercise 3. Consider the polynomial $f = 2xy^2 - xy + x^3$ in the ring $K[x,y]$. Find admissible orderings $<_1$ and $<_2$ so that $\mathrm{lpp}_{<_1}(f) \neq \mathrm{lpp}_{<_2}(f)$.

Exercise 4. Fix an admissible ordering and consider an ideal $I \subseteq K[x_1,\ldots,x_n]$. Suppose that $f \in K[x_1,\ldots,x_n]$.

1. Show that f can be written in the form $f = g + r$ with $g \in I$ and no term of r is divisible by any element of $\mathrm{lpp}(I)$.
2. Given two expressions $f = g + r = g' + r'$ as in part 1, prove that $g = g'$ and $r = r'$.

Exercise 5.

1. Consider the system of equations

$$\begin{aligned} 2x^4 - 3x^2y + y^4 - 2y^3 + y^2 &= 0 \\ 4x^3 - 3xy &= 0 \\ 4y^3 - 3x^2 - 6y^2 + 2 &= 0. \end{aligned}$$

   Compute all solutions.

2. The same for

$$\begin{aligned} 1 + 8xy + 2y^2 + 8xy^3 + y^4 - 16x^2 &= 0 \\ 8x + 4y + 24xy^2 + 4y^3 &= 0 \\ 8y + 8y^3 - 32x &= 0. \end{aligned}$$

This can be done by hand or using a computer algebra system.