Exercises discussed on November 29, 2011

31. Determine all sequences that are at the same time C-finite and hypergeometric.

32. Determine the asymptotics of

$$\frac{3^n}{4n+1}\binom{3n}{n+1}^2\binom{6n}{2n}^{-1}.$$

33. Determine the hypergeometric function representation of

    (a) $\frac{1}{x}\log(1+x) = \sum_{n\geq 0}\frac{(-1)^n}{n+1}x^n$

    (b) $\cos(x) = \sum_{n\geq 0}\frac{(-1)^n}{(2n)!}x^{2n}$

34. Show that in $\mathbb{Q}[\![x]\!]$ the hypergeometric function $y(x) = {}_2F_1\left(\begin{matrix} a & b \\ c & \end{matrix};x\right)$ satisfies the differential equation:

$$x(1-x)y''(x) + (c-(a+b+1)x)y'(x) - aby(x) = 0.$$

35. Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ have the hypergeometric series representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix} -n & n+\alpha+\beta+1 \\ \alpha+1 & \end{matrix};\frac{1-x}{2}\right).$$

Show that the derivative of Jacobi polynomials is again a Jacobi polynomial with shifted parameters, i.e., show that

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

Chebyshev polynomials of the first kind $T_n(x)$ are special instances of Jacobi polynomials. Which parameters α,β do they correspond to?