1 Theorem and proof

Satz (LRP-Zerlegung)**.** *For every regular matrix A there exist (square) matrices L, R, P with $L_{ii} = 1$ for all $i = 1, \dots, \dim(L)$ and*

$$P \cdot A = L \cdot R, \tag{1}$$

where L is a lower and R is an upper triangular matrix and P is a permutation matrix.

Proof. We prove (1) by induction on the dimension of A.

Let first $\dim(A) = 1$. Then A is an upper triangular matrix and (1) holds with $L = P = \mathsf{E}$ and $R = A$.

Let now $\dim(A) > 1$. Then

$$I = \left\{ i \mid A_{i1} \neq 0 \right\} \neq \emptyset, \tag{2}$$

because otherwise the first column of A would only consist of zeros in contradiction to the regularity of A. We choose $i \in I$ und swap the first and the i-th row of A, i.e.

$$\bar{A} := \bar{V} \cdot A \quad \text{with } \bar{V} = \mathsf{V}^{(1,i,\dim(A))},$$

and get $\bar{A}_{11} \neq 0$. Now we can multiply with a Frobenius-Matrix F, such that the first column of the result will contain only zeros under the diagonal, i.e. $F \cdot \bar{A}$ has the form

$$F \cdot \bar{A} = \left(\begin{array}{c|ccc} \bar{A}_{11} & \bar{A}_{12} & \dots & \bar{A}_{1n} \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array} \right). \tag{3}$$

Due to the regularity of F, also $F \cdot \bar{A}$ and thus A' are regular. Since $\dim(A') = \dim(A) - 1$ there exist, by induction hypothesis, L', R' and P' with $P' \cdot A' = L' \cdot R'$ and with $\bar{P} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P' \end{array} \right)$ we get from (3)

$$\bar{P} \cdot F \cdot \bar{A} = \bar{P} \cdot \left(\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{1,2:n} \\ \hline 0 & A' \end{array} \right) = \left(\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{1,2:n} \\ \hline 0 & L' \cdot R' \end{array} \right) = \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & L' \end{array} \right)}_{=:\bar{L}} \cdot \underbrace{\left(\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{1,2:n} \\ \hline 0 & R' \end{array} \right)}_{=:R}.$$

Now we know (additional knowledge!) that there is a Frobenius-matrix $\bar{F}$ with $\bar{F} \cdot \bar{P} = \bar{P} \cdot F$, such that

$$\bar{F} \cdot \bar{P} \cdot \bar{V} \cdot A = \bar{P} \cdot F \cdot \bar{A} = \bar{L} \cdot R.$$

Summarizing all this, we have shown (1) with

$$L := \bar{F}^{-1} \cdot \bar{L} \quad \text{and} \quad P := \bar{P} \cdot \bar{V}. \tag{4}$$

L is a lower triangular matrix with $L_{ii} = 1$ for all $i = 1, \dots, \dim(A)$, because L is a product of lower triangular matrices with 1's along the diagonal. P is a product of two permutation matrices and hence a permutation matrix. □

2 Definitions and additional knowledge

Definition (Permutationmatrix and elementary permutationmatrix)**.** A matrix $P \in \mathbb{R}^{m\times m}$ is called a *permutationsmatrix* iff it results from exchanging columns in E. Swapping just the k-th and l-th column gives the *elementare permutationmatrix*

$$\mathsf{V}^{(k,l,m)} = \begin{matrix} \\ \\ k \rightarrow \\ \\ l \rightarrow \\ \\ \end{matrix} \begin{pmatrix} & \overset{k}{\downarrow} & & \overset{l}{\downarrow} & \\ \ddots & & & & \\ & 0 & & 1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & \ddots \end{pmatrix}.$$

Definition (Frobenius-matrix)**.** A matrix $F \in \mathbb{R}^{m\times m}$ *is a Frobenius-Matrix* iff it differs from E in at most one column under the main diagonal.

For the special Frobenius-matrix

$$F := \begin{pmatrix} 1 & & & \\ -\frac{A_{21}}{A_{11}} & 1 & & \\ \vdots & & \ddots & \\ -\frac{A_{m1}}{A_{11}} & & & 1 \end{pmatrix} \tag{5}$$

we have

$$F \cdot A = F \cdot \begin{pmatrix} A_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{m\bullet} \end{pmatrix} = \begin{pmatrix} A_{1\bullet} \\ A_{2\bullet} - \frac{A_{21}}{A_{11}} A_{1\bullet} \\ \vdots \\ A_{m\bullet} - \frac{A_{m1}}{A_{11}} A_{1\bullet} \end{pmatrix}, \tag{6}$$

and thus $(F \cdot A)_{\bullet 1} = A_{11}\mathsf{e}_1$, in other words the first column of $F \cdot A$ has only zeros under the diagonal.

Satz. *Let* $P = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P' \end{array}\right) \in \mathbb{R}^{n\times n}$ *with* $P' \in \mathbb{R}^{n-1\times n-1}$ *and* $F \in \mathbb{R}^{n\times n}$ *a Frobenius-matrix. Then*

$$\bar{F} \cdot P = P \cdot F \quad \textit{with } \bar{F} = \left(P \cdot F_{\bullet 1} \;\middle|\; \begin{array}{c} 0 \\ \hline \mathsf{E}^{(n-1)} \end{array} \right).$$

3 Induction over the dimension

Why and how does "induction over the dimension of a matrix" work? The statement we want to prove has the form

$$\forall_A \text{ is-square-matrix}(A) \implies B$$

with an arbitrary formula B. The definition of the "is-square-matrix"-property is

$$\begin{aligned} \text{is-square-matrix}(A) :&\iff \exists_{n\in\mathbb{N}}\ \text{is-square-matrix-of-dim}(A,n) \\ \text{is-square-matrix-of-dim}(A,n) :&\iff A\colon \mathbb{N}_n \to \mathbb{N}_n \end{aligned}$$

Now,

$$\begin{aligned} &\forall_A\ \text{is-square-matrix}(A) \implies B \iff \\ &\qquad \iff \forall_A \exists_{n\in\mathbb{N}}\ \text{is-square-matrix-of-dim}(A,n) \implies B \\ &\qquad \iff \forall_A (\neg\exists_{n\in\mathbb{N}}\ \text{is-square-matrix-of-dim}(A,n)) \vee B \\ &\qquad \iff \forall_A (\forall_{n\in\mathbb{N}} \neg\ \text{is-square-matrix-of-dim}(A,n)) \vee B \\ &\qquad \iff \forall_A \forall_{n\in\mathbb{N}} (\neg\ \text{is-square-matrix-of-dim}(A,n) \vee B) \\ &\qquad \iff \forall_A \forall_{n\in\mathbb{N}}\ \text{is-square-matrix-of-dim}(A,n) \implies B \\ &\qquad \iff \forall_{n\in\mathbb{N}} \forall_A\ \text{is-square-matrix-of-dim}(A,n) \implies B. \qquad (7) \end{aligned}$$

In (7), we can now use normal *induction over n in* $\mathbb{N}$, i.e.

Induction Base: for $n = 1$ prove

$$\forall_A\ \text{is-square-matrix-of-dim}(A,1) \implies B$$

Induction Step: for $n > 1$ a.b.f. prove

$$\forall_A\ \text{is-square-matrix-of-dim}(A,n) \implies B$$

under the

Induction hypothesis:

$$\forall_A\ \text{is-square-matrix-of-dim}(A,n-1) \implies B$$

Therefore, of course, all variants of induction known for $\mathbb{N}$ apply also to induction over the dimension of a matrix. In a similar fashion, "induction over the degree of a polynomial" or "induction over the cardinality of a finite set" can be derived from induction in $\mathbb{N}$, because both properties "is-polynomial" and "is-finite-set" are defined based on $\exists_{n\in\mathbb{N}} \ldots$.