to be prepared for 1.12.2009

Exercise 20. Prove the following theorem.
Let $F \subseteq k[x_1, \ldots, x_n]$. The ideal congruence modulo $\langle F \rangle$ equals the reflexive-transitive-symmetric closure of the reduction relation $\longrightarrow_F$, i.e., $\equiv_{\langle F \rangle} = \longleftrightarrow^\star_F$.

Exercise 21. Let I be an ideal in $K[x_1, \ldots, x_n]$, $F \subset K[x_1, \ldots, x_n]$ with $\langle F \rangle = I$. Prove the equivalence of the following statements.

1. F is a Gröbner basis for I.
2. $\forall f \in I$ we have that $f \longrightarrow^\star_F 0$.
3. $f \longrightarrow_F$ for every $f \in I \setminus 0$.
4. $\forall g \in I\ \forall h \in K[x_1, \ldots, x_n]$: if $g \longrightarrow^\star_F \underline{h}$ then $h = 0$.
5. $\forall g, h_1, h_2 \in K[x_1, \ldots, x_n]$: if $g \longrightarrow^\star_F \underline{h_1}$ and $g \longrightarrow^\star_F \underline{h_2}$ then $h_1 = h_2$.
6. $\langle \mathrm{in}(F) \rangle = \langle \mathrm{in}(I) \rangle$.

Exercise 22. Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal and G a Gröbner basis for I. Let $g, h \in G$ with $g \neq h$. Prove the following statements.

1. If $\mathrm{lpp}(g)|\mathrm{lpp}(h)$ then $G \setminus \{h\}$ is a Gröbner basis for I.
2. If $h \longrightarrow_g h'$ then $(G \setminus \{h\}) \cup \{h'\}$ is a Gröbner basis for I.

Exercise 23. Let K be an algebraically closed field, n a positive integer, H a hypersurface in $\mathcal{A}^n(K)$, the affine space of dimension n over K. Let $f \in K[x_1, \ldots, x_n] \setminus K$ be a defining polynomial of H, i.e.,

$$H = \{(a_1, \ldots, a_n) \mid f(a_1, \ldots, a_n) = 0\}.$$

Let $f = f_1^{m_1} \cdots f_r^{m_r}$ be the factorization of f into irreducible factors. Let I be the ideal of polynomials in $K[x_1, \ldots, x_n]$ vanishing on H. Show that $I = \langle f_1 \cdots f_r \rangle$.

Exercise 24. Let K be a field, $f, g \in K[x, y]$ relatively prime. Show that there are only finitely many points $(a_1, a_2) \in K^2$ with $f(a_1, a_2) = g(a_1, a_2) = 0$.

Exercise 25. Use Gröbner bases for solving over $\mathbb{C}$:

$$\begin{aligned}
f_1(x, y, z) &= xz - xy^2 - 4x^2 - \frac{1}{4} = 0, \\
f_2(x, y, z) &= y^2 z + 2x + \frac{1}{2} = 0, \\
f_3(x, y, z) &= x^2 z + y^2 + \frac{1}{2}x = 0.
\end{aligned}$$

Exercise 26. Determine the singularities of the curve

$$x^6 + 3x^4y^2 - 4x^2y^2 + 3x^2y^4 + y^6 = 0$$

in the projective plane $\mathbb{P}^2(\mathbb{C})$.

Exercise 27. Give an answer to the following problem:
Given two ideals $I, J \subseteq k[x_1, \dots, x_n]$ in terms of generators $I = \langle f_1, \dots, f_r \rangle$, $J = \langle g_1, \dots, g_s \rangle$, find generators of the intersection $I \cap J$.

Exercise 28. Let I denote the ideal in $\mathbb{Q}[x, y, z]$ generated by the polynomials

$$\begin{aligned} f_1 &= xz - 3x^2 + x + 6x^3 + 1 \\ f_2 &= x^2 + y^2 - 2 \\ f_3 &= x^5 - 6x^3 + x^2 - 1. \end{aligned}$$

Compute the dimension of $\mathbb{Q}[x, y, z]/I$.

Exercise 29.

1. Find a rational parametrization of the circle $x^2 + y^2 = 1$.
2. Compute a rational parametrization for the sphere $x^2 + y^2 + z^2 = 1$.