to be prepared for 10.11.2009

Exercise 8. Let R be a commutative ring with 1. Demonstrate that the following statements are equivalent:

1. Every ideal in R is generated by a finite set.
2. There are no infinite strictly ascending chains of ideals in R.
3. Every nonempty set S of ideals contains a maximal element (i.e. an ideal $a \in S$ such that $\forall b \in S$, if $a \subseteq b$ then $a = b$.

Exercise 9. Let R be a ring of prime characteristic p and $a, b \in R$. Prove:

$$\begin{aligned}(a+b)^p &= a^p + b^p\\ (a+b)^{p^n} &= a^{p^n} + b^{p^n}, \text{ for } n \in \mathbb{N}.\end{aligned}$$

Exercise 10. For $m \in \mathbb{Z}$ let $\mathbb{Z}_m$ denote the residue class ring $\mathbb{Z}/m\mathbb{Z}$. Give a proof for the following statement: If m has a prime decomposition $m = p_1^{k_1} \cdots p_t^{k_t}$ ($k_i > 0,\ p_i$ distinct primes) then there is an isomorphism of rings

$$\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_t^{k_t}}.$$

Exercise 11. Let I be a unique factorization domain, and $f, g \in I[x]$. Write $f \sim g$ if there is a unit ε with $g = \varepsilon f$. Prove the following:

1. $\mathrm{cont}(fg) \sim \mathrm{cont} \cdot (f)\mathrm{cont}(g)$
2. $\mathrm{pp}(fg) \sim \mathrm{pp}(f) \cdot \mathrm{pp}(g)$

Exercise 12. Let R be a commutative ring with 1, and $S \subseteq R$ a multiplicative monoid (that means, $1 \in S$ and the product of elements coming from S lies in S). On the set $R \times S$ define the relation

$$(r_1, s_1) \sim (r_2, s_2) \iff \exists s_3 \in S \text{ such that } s_3 s_2 r_1 = s_3 s_1 r_2.$$

1. Verify that this is an equivalence relation.
2. Let $R[S^{-1}]$ denote the quotient $R \times S/\sim$, write $\frac{r}{s}$ for the equivalence class of the pair (r, s) and define addition and multiplpication on $R[S^{-1}]$ by

   $$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}, \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

   Verify that these are well defined operations turning $R[S^{-1}]$ into a commutative ring with 1.
3. Define the map

   $$\eta\colon R \longrightarrow R[S^{-1}],\ r \mapsto \frac{r}{1}.$$

   Make sure that this is actually a well defined homomorphism of rings.
4. Give a description of η's kernel. Formulate conditions on the monoid S that warrant R being embedded in $R[S^{-1}]$. Is it possible that $R \cong R[S^{-1}]$?

Exercise 13.

1. Explain why the Euclidean algorithm can be considered as a special case of Gröbner base algorithm.

2. Considering a system of linear equations in indeterminates $x_1, \ldots, x_n$, discuss the relation between Gauss elimination and Gröbner bases.

Exercise 14. The graduated reverse lexicographic ordering on power products of $x_1, \ldots, x_n <_{\text{grlex}}$ is defined by

$$s <_{\text{grlex}} t \quad \text{iff} \quad \begin{array}{lll} \deg(s) < \deg(t) & \text{or} & \\ \deg(s) = \deg(t) & \text{and} & t <_{\text{lex},\pi} s; \end{array}$$

where π is the permutation on n letters given by $\pi(j) = n - j + 1$ and $<_{\text{lex},\pi}$ is the lexicographic order wrto. π. Prove that $<_{\text{grlex}}$ is an admissible ordering.

Exercise 15. Consider the polynomials

$$\begin{array}{rcl} f_1(x,y) & = & x^2y + xy + 1, \\ f_2(x,y) & = & y^2 + x + y \end{array}$$

in $\mathbb{Z}_3[x,y]$. Compute a Gröbner basis for the ideal $\langle f_1, f_2 \rangle$ w.r.t. the graduated lexicographical ordering with $x < y$. Show intermediate results.

Exercise 16. Let $<_1$ be an admissible ordering on $X_1 = [x_1, \ldots, x_i]$ and $<_2$ an admissible ordering on $X_2 = [x_{i+1}, \ldots, x_n]$. Show that the product ordering $<_{prod,i,<_1,<_2}$ on $X = [x_1, \ldots, x_n]$ is an admissible ordering.

Exercise 17. $R[x_1, \ldots, x_n] = R[X]$ denote the polynomial ring in n indeterminates over a commutative ring with 1. Any admissible ordering $<$ on the monoid of power products $[X]$ induces a partial order $<<$ on $R[X]$ in the following way:

$$f << g \quad \text{iff} \quad \begin{array}{l} f = 0 \text{ and } g \neq 0 \text{ or} \\ f \neq 0,\ g \neq 0 \text{ and } \operatorname{lpp}(f) < \operatorname{lpp}(g) \text{ or} \\ f \neq 0,\ g \neq 0,\ \operatorname{lpp}(f) = \operatorname{lpp}(g) \text{ and } \operatorname{red}(f) << \operatorname{red}(g). \end{array}$$

Prove that $<<$ is a Noetherian partial order on $R[X]$. (Lemma 8.2.4.)

Exercise 18. Consider the partial order $\leq_\pi$ on $\mathbb{N}^n$ defined as

$$(a_1, \ldots, a_n) \leq_\pi (b_1, \ldots, b_n) \iff a_i \leq b_i \ \forall i \in \{1, \ldots, n\}.$$

Prove that any set $X \subseteq \mathbb{N}^n$ contains a finite set $Y \subseteq X$ such that

$$\forall x \in X \exists y \in Y \text{ with } y \leq_\pi x.$$

Exercise 19. Use Gröbner bases to find the implicit representation of the parametrized surface

$$\begin{array}{rcl} x & = & 1 + s + t + st \\ y & = & 2 + s + st + t^2 \\ z & = & s + t + s^2 \end{array}$$