to be prepared for 20.10.2009

Exercise 1. An integral domain is a commutative ring $D \neq \{0\}$ without zero divisors, that means, $rs = 0 \Rightarrow r = 0 \vee s = 0$ $(\forall r, s \in D)$. Give a proof for the following statement.

1. If D is an integral domain, then also the polynomial ring $D[x]$.
2. Derive from this that - for arbitrary fields k - the ring $k[x_1, \ldots, x_n]$ is an integral domain.
3. Give similar arguments for the ring $D[[x]]$ of formal power series.

Exercise 2. Explain why the following is or is not a Euclidean domain.

1. $\mathbb{Z}$ with degree function $\delta(n) = |n|$;
2. $\mathbb{Q}$ with $\delta(r) = |r|$;
3. $\mathbb{Z}[i]$ with $\delta(z) = |z|^2$.

Exercise 3. Consider the polynomials

$$\begin{aligned} f &= x^6 - 2x^5 - x^4 - 4x^3 - 5x^2 - 2x - 3 \\ g &= 3x^6 - x^5 + 4x^4 - 2x^3 + 2x^2 - x + 1. \end{aligned}$$

f and g are elements of the ring $\mathbb{Z}[x]$. Apply the extended Euclidean algorithm in $\mathbb{Q}[x]$ to compute $\gcd(f, g)$ as a linear combination of f and g. Use a computeralgebra system of your choice.

Exercise 4. Prove that every Euclidean domain is a principal ideal domain, and that every principal ideal domain is a unique factorization domain.

Exercise 5. Let R be commutative ring, $R[x]$ the corresponding polynomial ring. Proof that $R[x]$ is a Euclidean domain if and only if R is a field.

Exercise 6. Consider the ring $\mathbb{Z}$ of integers and an ideal I generated by finitely many numbers $a_1, \ldots, a_n$. As $\mathbb{Z}$ is a principal ideal domain there must be a single generator b for I.

1. Describe a procedure for finding b when arbitrary generators $a_1, \ldots, a_n$ of I are given as an input.
2. Compute a single generator for the ideal $I = \langle 33600, 784080, 214500 \rangle$.

Exercise 7. Prove the following theorem: *If I is a unique factorization domain, then so is $I[x]$.*

Exercise 8. Let R be a commutative ring with 1. Demonstrate that the following statements are equivalent:

1. Every ideal in R is generated by a finite set.
2. There are no infinite strictly ascending chains of ideals in R.
3. Every nonempty set S of ideals contains a maximal element (i.e. an ideal $a \in S$ such that $\forall b \in S$, if $a \subseteq b$ then $a = b$.

Exercise 9. Let R be a ring of prime characteristic p and $a, b \in R$. Prove:

$$\begin{aligned} (a+b)^p &= a^p + b^p \\ (a+b)^{p^n} &= a^{p^n} + b^{p^n}, \text{ for } n \in \mathbb{N}. \end{aligned}$$