Name: ..... 26 Jan 2010

Studienkennzahl: .....

Matrikelnummer: .....

Final Exam in **Computer Algebra** (326.017)
(no books allowed)

(1) Consider the following polynomials in $\mathbb{Q}[x, y]$:

$$a(x, y) = xy^2 + 2y^2 - xy - 2, \qquad b(x, y) = xy + 2y + x - 4$$

(a) Compute $\mathrm{res}_y(a, b) \in \mathbb{Q}[x]$, the resultant of a and b w.r.t. y;
(b) Can the solution $x = -2$ of $\mathrm{res}_y(a, b)$ be extended to a solution of the system $a(x, y) = 0 = b(x, y)$? Explain this situation.
(c) Can the solution $x = 1$ of $\mathrm{res}_y(a, b)$ be extended to a solution of the system $a(x, y) = 0 = b(x, y)$? Explain this situation.

(2) Consider the polynomials $a(x, y)$ and $b(x, y)$ of Question (1). We want to determine a Gröbner basis G for the ideal generated by $\{a, b\}$ w.r.t. the lexicographic order with $x < y$:
$\mathrm{spol}(a, b)$ can be reduced to $c(x, y) = 4y + x - 5$, so we add c to the basis;
$\mathrm{spol}(a, c)$ can be reduced to $d(x, y) = x^2 - 7x + 6$, so we add d to the basis;
Complete the algorithm and determine G.

(3) Consider the following polynomial ideals I, J in $\mathbb{Z}_5[x, y]$:
$I = \langle G \rangle,\ G = \{xy + x + 1, y + 4x^2, x^3 + x + 1\}$; G is a Gröbner basis for I w.r.t. the lexicographic term ordering with $x < y$,
$J = \langle H \rangle,\ H = \{xy + x + 1, y^2 + x + y, x^2 + 4y\}$; H is a Gröbner basis for J w.r.t. the graduated lexicographic term ordering with $x > y$.
(a) Is $h = xy + y^2 + 2 \in I$? Is $h \in J$?
(b) Are I and J the same ideal ($I = J$)? Why, or why not?

(4) Let K be a field, X a finite set (of variables). Let $\longrightarrow$ be reduction w.r.t. an admissible ordering $>$ on $[X]$. Let $a \in K^*$ (i.e. a a non-zero constant), $s \in [X]$, $F \subseteq K[X]$, $g_1, g_2, h \in K[X]$. Prove:
(a) *If* $g_1 \longrightarrow_F g_2$ *then* $a \cdot s \cdot g_1 \longrightarrow_F a \cdot s \cdot g_2$.
(b) *If* $g_1 \longrightarrow_F g_2$ *then* $g_1 + h \downarrow_F^* g_2 + h$.

(5) Consider the following polynomials in $\mathbb{Q}[x]$:

$$f(x) = x^3 - 3x + 2, \quad g(x) = x^3 - 1\ .$$

For a prime number p let $f_{(p)}, g_{(p)}$ be the images of f, g modulo p, respectively. Check whether

$$\deg(\gcd(f, g)) \ = \ \deg(\gcd(f_{(p)}, g_{(p)})$$

for $p = 2$ and for $p = 3$. Show the intermediate results and explain the situation.

(6) **Optional:** Recall that a commutative ring R with 1 is **Noetherian** iff there is no infinite strictly increasing chain of ideals of the form $I_1 \subset I_2 \subset \cdots \subset R$.
Give an example of a non-Noetherian commutative ring with 1.