Projects for Computer Algebra
326.017, WS2009-10

October 7, 2009

1 Solving Polynomial Equations

1. Write down criteria for determining whether a system of polynomial equations in several indeterminants with coefficients in $\mathbb{C}$ has only finitely many solutions. Present the theory behind those criteria.

2. Implement an algorithm within a computer algebra system which decides whether a polynomial system has finitely many solutions and - in the affirmative case - determines all solutions to some specified precision.

Input:	$f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$, $d \in \mathbb{N}$
Output:	The list of all finitely many solutions of $f_1 = \cdots = f_r = 0$ in $\mathbb{C}^n$ in n-tuples of complex numbers with precision d
OR	**INFINITE**.

References: [15] [19] [14] [18]

2 Buchberger's Algorithm

1. Work out the general theory of dividing a polynomial $f \in k[x_1, \ldots, x_n]$ by a finite set of polynomials $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$.

2. Implement an algorithm which computes the reduced Gröbner basis of a finite set of polynomials with respect to a term order together with the additional information of representation.

Input:	$F = [f_1, \ldots, f_r] \in \mathbb{C}[x_1, \ldots, x_n]^r$, $<$ a term order
Output:	G reduced Gröbner basis of $\langle F \rangle$ wrto $<$ A matrix of polynomials such that $G = AF$.

References: [19] [14] [3]

3 Intersection of Surfaces

1. Work out the theory of surface intersections e.g. along the lines of [17] pages 230 cc.

2. Implement an algorithm which determines the plain projection of the intersection curve of two surfaces.
   Input: $F = [f_1, f_2] \in \mathbb{R}[x_1, x_2, x_3]^2$
   Output: The projection of $\mathbf{V}(f_1) \cap \mathbf{V}(f_2)$ onto $\mathbb{R}^2$.

References: [17] available online at

http : //www.cs.purdue.edu/homes/cmh/distribution/books/geo.html

4 Gröbner Basis Conversion

1. Give the theory of converting a Gröbner basis of a zero-dimensional ideal with respect to a given term order into a lexicographic Gröbner basis.

2. Implement an algorithm which does this.
   Input: $<$ term order of $\{x_1, \ldots, x_n\}$
   $i\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ permutation
   $F = \{f_1, \ldots, f_r\} \subset k[x_1, \ldots, x_n]$ Gröbner basis of the zero-dimensional ideal $\langle F \rangle$ wrto $<$
   Output: The reduced Gröbner basis of $\langle F \rangle$ wrto lex-order $x_{i_1} > \cdots > x_{i_n}$.

References: [16], [17] available online at

http : //www.cs.purdue.edu/homes/cmh/distribution/books/geo.html

5 Implicitization by Interpolation

Finding the implicit representation of a parametrized variety means finding the coefficients of a finite set of polynomials. So, if we know bounds for the degrees of the desired polynomials, we may evaluate the given parametrizing functions in some finite set of interpolation nodes, thereby obtaining a linear system L. A nontrivial solution of L yields an answer to the implicitization problem.

1. Work out the theory behind these remarks.

2. Implement an algorithm that does the job.
   Input: $(r_1(t), r_2(t))$ proper parametrization of an irreducible plane curve $\mathcal{C}$
   Output: The implicit representation of $\mathcal{C}$.

3. Run the algorithm on some examples.

References: [12], [9]

6 Implicitization by Gröbner Bases

1. Work out the theory of implicitizing rationally parametrized algebraic varieties (affine case, projective case), elimination ideals and the use of Groebner bases for implicitization.

2. Demonstrate your results by computing several examples.

References: [15], [5], [12]

7 Robotics and Motion Planning

1. Work out the theory of planar robots (joint space, configuration space, forward/inverse kinematic problem).

2. Demonstrate the theory by means of a planar robot with a fixed segment 1 and with n revolute joints linking segments of length $l_2, \ldots, l_n$. The 'hand' is segment $n+1$, attached to segment n by joint n. Determine the position of the hand as a function of joint settings.

3. Consider a concrete planar robot with 3 revolute joints linking 4 segments of length 1, followed by one prismatic joint taking length values from the interval $[0, 1]$, linking the 4th segment to the hand. Solve the inverse kinematic problem for this robot. Describe possible kinematic singularities.

References: [15]

8 Geometric Theorem Proving

1. Develope the theory of proving theorems of plane geometry by using Gröbner bases: Translation of geometric statements into polynomial equations, definition of strict/generic consequence from a set of hypotheses, sufficient conditions for being a strict/generic consequence.

2. Prove Pappus's Hexagon Theorem along these lines.

3. Write an algorithm that detects whether a given geometric statement follows (strictly or generically) from a given system of geometric statements.

References: [15]

References

[1] T. Becker and V. Weisspfennig. *Gröbner Bases. A Computational Approach to Commutative Algebra.* Springer 1993.

[2] B. Buchberger. Gröbner basis: an algorithmic method in polynomial ideal theory. *Multidimensional Systems Theory* (1985), N. K. Bose ed., 184–232.

[3] B. Buchberger and F. Winkler. Gröbner Bases and Applications. *London Math. Soc. Lecture Notes 251* (1998).

[4] D. Cox, J. Little and D. O'Shea. *Ideals, Varieties, and Algorithms.* Springer Verlag, New York (1996).

[5] D. Cox, J. Little and D. O'Shea. Using Algebraic Geometry. Springer Verlag, New York (1998).

[6] G. Fix, Chih-Ping Hsu and Tie Luo. Implicitization of Rational Parametric Surfaces. *Journal of Symbolic Computation 21* (1996), 329–336.

[7] Xiao-Shan Gao and Shang-Ching Chou. Implicitization of Rational Parametric Equations. *Journal of Symbolic Computation 14* (1992), 459–470.

[8] D. Manocha and J.F. Canny. Algorithms for Implicitizing Rational Parametric Surfaces. *Computer Aided Geometric Design 9* (1992), 25–50.

[9] A. Marco and J. J. Martinez. Using polynomial interpolation for implicitizing algebraic curves. Computer Aided Geometric Design 18(4) (2001), 309–319.

[10] J. Schicho. Rational Parametrization of Real Algebraic Surfaces. *RISC-report, RISC-Linz 98-01* (1998).

[11] T.W. Sederberg. Improperly parametrized rational curves. *Computer Aided Geometric Design 3* (1986), 67–75.

[12] J.R. Sendra and F. Winkler. A symbolic algorithm for the parametrization of algebraic plane curves. *Tech. Rep. 89-41.1, RISC-Linz* (1998).

[13] J.R. Sendra and F. Winkler. Parametrization of Algebraic Curves over Optimal Field Extensions. *Journal of Symbolic Computation 23* (1997), 191–207.

[14] T. Becker and V. Weispfenning. *Gröbner bases - a computational approach to commutative algebra*. Graduate Texts in Mathematics. Springer Verlag, 1993.

[15] D. Cox, J. Little, and D. ÓShea. *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics. Springer Verlag, 1992.

[16] J.C. Faugère, P. Gianni, D. Lazard, and T. Mora. Efficient computation of zero-dimensional Groebner bases by change of ordering. *Journal of Symbolic Computation*, 16:329–344, 1993.

[17] C. M. Hoffmann. *Geometric and Solid Modelling – an Introduction*. Morgan Kauffmann Publisher, San Mateo, California, 1989.

[18] Q.-N. Tran and F. Winkler, editors. *Applications of the Gröbner Bases Method*, volume 30 of *special issue of the J. of Symbolic Computation*, Oct. 2000.

[19] F. Winkler. Polynomial algebrithms in computer algebra. *Springer-Verlag Wien New York*, 1996.