Free resolution of an ideal/module

$R = K[X]$, M an R-module (e.g. I ideal)

we construct a finite free resolution of M:

R^{s_i} are free modules

$$0 \longrightarrow R^{s_i} \xrightarrow{\varphi_i} R^{s_{i-1}} \xrightarrow{\varphi_{i-1}} \dots \xrightarrow{\varphi_2} R^{s_1} \xrightarrow{\varphi_1} R^{s_0} \xrightarrow{\varphi_0} M \longrightarrow 0$$

s.t $\operatorname{im}(\varphi_{i-1}) = \ker(\varphi_i)$

So this is an exact sequence

we start with basis $(f_1, \dots, f_{s_0})$ of M

and basis $(g_1, \dots, g_{s_1})$ of $\operatorname{Syz}(f_1, \dots, f_{s_0})$

$$\text{let } \varphi_0 : R^{s_0} \longrightarrow M$$
$$(r_1, \dots, r_{s_0}) \longmapsto \sum r_i f_i \in M$$

$$\text{let } \varphi_1 : R^{s_1} \longrightarrow R^{s_0}$$
$$(r_1, \dots, r_{s_1}) \longmapsto \sum r_i \cdot g_i \in \operatorname{Syz}(f) \subseteq R^{s_0}$$

we have $\operatorname{im}(\varphi_1) = \operatorname{Syz}(f) = \ker(\varphi_0)$

now we continue to construct a basis for $\operatorname{Syz}(g)$ (2nd syz. module)

etc. D. Hilbert has proved in "Über die Theorie der algebraischen Formen", Math. Ann. 36, 473 – 534 (1890)
that the sequence of syzygy modules for an ideal I is finite.

CLO 2 Chap. 5 & 6

(1.1) Definition. A *module over a ring R* (or R-module) is a set M together with a binary operation, usually written as addition, and an operation of R on M, called (scalar) multiplication, satisfying the following properties.

a. M is an abelian group under addition. That is, addition in M is associative and commutative, there is an additive identity element $0 \in M$, and each element $f \in M$ has an additive inverse $-f$ satisfying $f + (-f) = 0$.
b. For all $a \in R$ and all $f, g \in M$, $a(f+g) = af + ag$.
c. For all $a, b \in R$ and all $f \in M$, $(a+b)f = af + bf$.
d. For all $a, b \in R$ and all $f \in M$, $(ab)f = a(bf)$.
e. If 1 is the multiplicative identity in R, $1f = f$ for all $f \in M$.

(1.1) Definition. Consider a sequence of R-modules and homomorphisms

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\varphi_{i+1}} M_i \xrightarrow{\varphi_i} M_{i-1} \longrightarrow \cdots$$

a. We say the sequence is *exact at* M_i if $\operatorname{im}(\varphi_{i+1}) = \ker(\varphi_i)$.
b. The entire sequence is said to be *exact* if it is exact at each M_i which is not at the beginning or the end of the sequence.

(1.9) Definition. Let M be an R-module. A *free resolution* of M is an exact sequence of the form

$$\cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0,$$

where for all i, $F_i \cong R^{r_i}$ is a free R-module. If there is a ℓ such that $F_{\ell+1} = F_{\ell+2} = \cdots = 0$, but $F_\ell \neq 0$, then we say the resolution is *finite*, of *length* ℓ. In a finite resolution of length ℓ, we will usually write the resolution as

$$0 \to F_\ell \to F_{\ell-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

Example (CLO p. 237 ff)

$F = (f_1, f_2, f_3)$

Here is a simple example. Let $I = \langle x^2 - x, xy, y^2 - y\rangle$ in $R = k[x, y]$. In geometric terms, I is the ideal of the variety $V = \{(0,0), (1,0), (0,1)\}$ in k^2. We claim that I has a presentation given by the following exact sequence:

(1.7) $$R^2 \xrightarrow{\psi} R^3 \xrightarrow{\varphi} I \to 0,$$

where φ is the homomorphism defined by the 1×3 matrix

$$A = (\, x^2 - x \quad xy \quad y^2 - y\,)$$

and ψ is defined by the 3×2 matrix

$$B = \begin{pmatrix} y & 0 \\ -x+1 & y-1 \\ 0 & -x \end{pmatrix}. \longleftarrow \quad \text{basis for } \mathrm{Syz}(F)$$

$$\varphi : R^3 \to I \qquad r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \mapsto A \cdot r$$

$$\psi : R^2 \to R^3 \qquad r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \mapsto B \cdot r$$

For an example, consider the presentation (1.7) for

$$I = \langle x^2 - x, xy, y^2 - y\rangle$$

in $R = k[x, y]$. If

$$a_1 \begin{pmatrix} y \\ -x+1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ y-1 \\ -x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$a_i \in R$, is any syzygy on the columns of B with $a_i \in R$, then looking at the first components, we see that $ya_1 = 0$, so $a_1 = 0$. Similarly from the third components $a_2 = 0$. Hence the kernel of ψ in (1.7) is the zero submodule. An equivalent way to say this is that the columns of B are a basis for $\mathrm{Syz}(x^2 - x, xy, y^2 - y)$, so the first syzygy module is a free module. As a result, (1.7) extends to an exact sequence:

(1.10) $$0 \to R^2 \xrightarrow{\psi} R^3 \xrightarrow{\varphi} I \to 0.$$

According to Definition (1.9), this is a free resolution of length 1 for I.