Automatic Theorem Proving
First-Order Predicate Logic

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Outline

Syntax

Semantics

Motivation

Example

Everything greater than 0 has a square root.

This statement can be interpreted, for example, in arithmetic on $\mathbb{R}$, the set of real numbers, where it is true.

It can be also interpreted, for example, in arithmetic on $\mathbb{Q}$, the set of rational numbers, where it is false.

Formalization

For all x ($0 < x \implies$ there is some y such that $x = y \cdot y$).

Better Formalization

$\forall x\ (0 < x \implies \exists y\ (x = y \cdot y))$

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Terms and Formulae

Terms

- Constants: 0, 1, 230, *John*, $\mathbb{N}$, … ;
- Variables: $x, y, z, \ldots$;
- Function symbols: $+$, $-$, *Son_of*, … .

Formulae

- Predicates: $=$, $>$, $\nleq$, *Is_Son_of*, … ;
- Connectives: $\wedge$, $\vee$, $\neg$, $\Longrightarrow$, $\Longleftrightarrow$;
- Quantifiers: $\forall$, $\exists$.

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- Constants are terms.
- Variables are terms.
- If f is a function symbol of arity n and $t_1, \ldots, t_n$ are terms, then $f[t_1, \ldots, t_n]$ is a term.
- All terms are generated by applying the above rules.

Examples

- x, $Sum[z, 3]$, $f[7, a, Prod[x, Sum[y, 3]]]$ are terms which in practical mathematics are presented in the form x, $z + 3$, $f[7, a, x(y + 3)]$, respectively.

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Examples

- $\neg LessEqual[Sum[x, 5], 7]$, $\forall x\, \exists y\, Equal[x, y]$ are formulae, which in practical mathematics are represented as: $x + 5 \not\leq 7$, $\forall x\, \exists y\, (x = y)$, respectively.

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Example

In the formula:

$$\forall x \; (Less[0, x] \implies \exists y \; Equal[x, Prod[y, y]]$$

$$\forall x \; (0 < x \implies \exists y \; (x = y \cdot y))$$

- What are the constants?
- What are the variables?
- What are the function symbols?
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Domain and Interpretation

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Any non-empty set D can be domain.
Intuitively: it is the place where everything happens.
All constant symbols from the language will be *interpreted* as some elements from D.
All variables from the language will range over D, etc.

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Intuitively: it is the one that gives meaning to the symbols from the languages.
Constant symbols become constants from D.
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Let D be non-empty set.
Interpretation I over D is a function defined in the following way:

- for any constant symbol c, $c_I \in D$;
- for any function symbol f of arity n, $f_I : D^n \rightarrow D$;
- for any predicate symbol P of arity n, $P_I : D^n \rightarrow \{\mathbb{T}, \mathbb{F}\}$.

Truth Evaluation of Formulae Under Interpretation

Definition

Let I be an interpretation and σ be an assignment. Then:

- If φ is a formula, then $\langle\varphi\rangle_I = \langle\varphi\rangle^I_{\{\}}$.
- $\langle\mathbb{T}\rangle^I_\sigma = \mathbb{T}$ and $\langle\mathbb{F}\rangle^I_\sigma = \mathbb{F}$.
- If P is a predicate symbol of arity n and $t_1, \ldots, t_n$ are terms, then $\langle P[t_1, \ldots, t_n]\rangle^I_\sigma = P_I[\langle t_1\rangle^I_\sigma, \ldots, \langle t_n\rangle^I_\sigma]$.
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- If φ is a formula and x is a variable, then $\langle\forall x\varphi\rangle^I_\sigma = \mathbb{T}$ iff for all $d \in D$, $\langle\varphi\rangle^I_{\sigma\sqcup\{x\leftarrow d\}} = \mathbb{T}$.
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Truth Evaluation of Formulae Under Interpretation

Definition

Let I be an interpretation and σ be an assignment. Then:

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- $\langle\mathbb{T}\rangle^I_\sigma = \mathbb{T}$ and $\langle\mathbb{F}\rangle^I_\sigma = \mathbb{F}$.
- If P is a predicate symbol of arity n and $t_1, \ldots, t_n$ are terms, then $\langle P[t_1, \ldots, t_n]\rangle^I_\sigma = P_I[\langle t_1\rangle^I_\sigma, \ldots, \langle t_n\rangle^I_\sigma]$.
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Let I be an interpretation and σ be an assignment. Then:

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- If x is a variable, then $\langle x \rangle^I_{\{x \leftarrow d, \ldots\}} = d$.
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Term Evaluation Under Interpretation

Example

Consider the formula $\forall x\ \exists y(x \leqq y)$.

Let $D = \{0, 1\}$ and I be an interpretation over D, such that $\langle \leqq \rangle^I = \leq$.

$\langle \forall x\ \exists y(x \leqq y) \rangle^I = \mathbb{T}$ iff for all $d \in D$, $\langle \exists y(d \leqq y) \rangle^I = \mathbb{T}$

- case 1: $d = 0$
  $\langle \exists y(0 \leqq y) \rangle^I = \mathbb{T}$ iff for some $e \in D$, $\langle (0 \leqq e) \rangle^I = \mathbb{T}$.
  Let $e = 0$.
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Example

Consider again the formula $\forall x\,(0 < x \implies \exists y\,(x = y \cdot y))$ on two different domains $\mathbb{Q}$ and $\mathbb{R}$ under *standard* interpretation.

An interpretation I is normally called *standard* for a domain when it interprets the constants, the function symbols and predicate symbols with their standard meaning, e.g.,
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$\langle \forall x\ (0 < x \implies \exists y\ (x = y \cdot y)) \rangle^I = \mathbb{T}$ iff
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Let r_1 be arbitrary but fixed real, such that $r_1 > 0$.
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Let $r_2 = \sqrt{r_1}$.
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Does the formula hold? May we use the same proof as in the other domain to show it holds?

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