Automatic Theorem Proving

Normal Forms. Logical Consequence

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Outline

Normal Forms

Logical Consequence

Equivalence

Definition

Two formulae F and G are *equivalent*, denoted as $F = G$, iff their truth values are the same under every interpretation.

Example

$P \Rightarrow Q$ is equivalent to $(\neg P \vee Q)$

True and False

Henceforth, by $\blacksquare$ we denote the formula that is always true and by $\square$ the one that is always false.

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Laws for transformation on formulae

$$F \Leftrightarrow G = (F \Rightarrow G) \land (G \Rightarrow F)$$
$$F \Rightarrow G = \neg F \lor G$$
$$F \lor G = G \lor F$$
$$F \land G = G \land F$$
$$(F \lor G) \lor H = F \lor (G \lor H)$$
$$(F \land G) \land H = F \land (G \land H)$$
$$(F \lor G) \land H = (F \land H) \lor (G \land H)$$
$$(F \land G) \lor H = (F \lor H) \land (G \lor H)$$
$$G \land \square = \square$$
$$G \land \blacksquare = G$$
$$G \lor \square = G$$
$$G \lor \blacksquare = \blacksquare$$
$$G \land \neg G = \square$$
$$G \lor \neg G = \blacksquare$$
$$\neg\neg G = G$$
$$\neg(F \lor G) = \neg F \land \neg G$$
$$\neg(F \land G) = \neg F \lor \neg G$$

Normal Forms

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A *literal* is an atom or the negation of an atom.

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A formula F is in a *conjunctive normal form* iff $F \stackrel{\circ}{=} F_1 \wedge \cdots \wedge F_n$, for some $n \geq 1$, where each of $F_1, \ldots, F_n$ is a disjunction of literals.

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A formula F is in a *disjunctive normal form* iff $F \stackrel{\circ}{=} F_1 \vee \cdots \vee F_n$, for some $n \geq 1$, where each of $F_1, \ldots, F_n$ is a conjunction of literals.

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Normal Forms

Example

$F_1 \doteq (P \vee \neg Q \vee R) \wedge (\neg P \vee Q)$

Example

$F_2 \doteq (P \wedge R) \vee (\neg Q \wedge R) \vee (P \wedge Q \vee \wedge R)$

Example

$F_3 \doteq (P \wedge R) \vee (Q \wedge R) \vee (P \wedge \neg\neg Q)$

Normal Forms

Example

$F_1 \triangleq (P \lor \neg Q \lor R) \land (\neg P \lor Q)$

Example

$F_2 \triangleq (P \land R) \lor (\neg Q \land R) \lor (P \land Q \lor \land R)$

Example

$F_3 \triangleq (P \land R) \lor (Q \land R) \lor (P \land \neg\neg Q)$

Normal Forms

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$$F_1 \triangleq (P \vee \neg Q \vee R) \wedge (\neg P \vee Q)$$

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$$F_2 \triangleq (P \wedge R) \vee (\neg Q \wedge R) \vee (P \wedge Q \vee \wedge R)$$

Example

$$F_3 \triangleq (P \wedge R) \vee (Q \wedge R) \vee (P \wedge \neg\neg Q)$$

Transformation into Normal Forms

Step 1 **Use the laws:**

$$F \Leftrightarrow G = (F \Rightarrow G) \wedge (G \Rightarrow F)$$
$$F \Rightarrow G = \neg F \vee G$$

Step 2 **Use the laws:**

$$\neg(\neg G) = G$$
$$\neg(F \vee G) = \neg F \wedge \neg G$$
$$\neg(F \wedge G) = \neg F \vee \neg G$$

Step 3 **Use the laws:**

$$(F \vee G) \wedge H = (F \wedge G) \vee (G \wedge H)$$
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Transformation into Normal Forms

Example

Obtain *dnf* for the formula $F_1 \ \hat{=}\ (P \vee \neg Q) \Rightarrow R$

Example

Obtain *cnf* for the formula $F_1 \ \hat{=}\ (P \vee \neg Q) \Rightarrow R$

Example

Obtain *dnf* for the formula $F_2 \ \hat{=}\ (P \wedge (Q \Rightarrow R)) \Rightarrow S$

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Obtain *cnf* for the formula $F_2 \ \hat{=}\ (P \wedge (Q \Rightarrow R)) \Rightarrow S$

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Transformation into Normal Forms

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Obtain *dnf* for the formula $F_1 \;\hat{=}\; (P \lor \neg Q) \Rightarrow R$

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Obtain *dnf* for the formula $F_2 \;\hat{=}\; (P \land (Q \Rightarrow R)) \Rightarrow S$

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Outline

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Example

Suppose the stock prices go down if the prime interest rate goes up. Suppose also that most people are unhappy when stock prices go down. Assume that prime interest rate does go up. Show that we can conclude that most people are unhappy.

$P \triangleq$ *Prime interest goes up.*
$S \triangleq$ *Stock prices go down.*
$U \triangleq$ *Most people are unhappy.*

If Prime interest goes up, stock prices go down.
If Stock prices go down, most people are unhappy.
Prime interest goes up.
Most people are unhappy.

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If Stock prices go down, most people are unhappy.
Prime interest goes up.
Most people are unhappy.

Logical Consequence

$P \Rightarrow S$
$S \Rightarrow U$
P
U

Show that U is a logical consequence of the other three formulae.

$(P \Rightarrow S) \wedge (S \Rightarrow U) \wedge P \ = \ldots$

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$S \Rightarrow U$
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Show that U is a logical consequence of the other three formulae.

$$(P \Rightarrow S) \wedge (S \Rightarrow U) \wedge P \;= \ldots$$

Logical Consequence

Definition

Given formulae $F_1, \ldots, F_n$ and G, G is a *logical consequence* of $F_1, \ldots, F_n$, iff for any interpretation I in which $F_1 \wedge \cdots \wedge F_n$ is true, G is also true.

$F_1, \ldots, F_n$ are called *axioms* (*postulates*, *premises*) of G.

Theorem (Deduction Theorem)

Given formulae $F_1, \ldots, F_n$ and G, G is a *logical consequence* of $F_1, \ldots, F_n$, iff the formula $(F_1 \wedge \cdots \wedge F_n) \Rightarrow G$ is valid.

Theorem

Given formulae $F_1, \ldots, F_n$ and G, G is a *logical consequence* of $F_1, \ldots, F_n$, iff the formula $(F_1 \wedge \cdots \wedge F_n \wedge \neg G)$ is inconsistent.

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Logical Consequence

Example

Consider the formulae:

$F_1 \stackrel{\circ}{=} P \Rightarrow Q,$

$F_2 \stackrel{\circ}{=} \neg Q,$

$G \stackrel{\circ}{=} \neg P.$

Show that G is a logical consequence of F_1 and F_2.

Method 1

Construct the truth table of $(P \Rightarrow Q) \wedge \neg Q$ and $\neg P$ and show that any time the earlier one is true, the latter one is true as well.

Method 2

Use the deduction theorem, i.e., construct the truth table of $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$ and show that it is true under any interpretation.

Method 3

Use the second theorem, i.e., prove that $((P \Rightarrow Q) \wedge \neg Q) \wedge (\neg(\neg P))$ is inconsistent.

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