Congruence-simple subsemirings of $\mathbb{Q}^+$

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TSW
RISC, December 2009

1. Introduction
   - Definition
   - Classification of cong.-simp. semirings

2. Results
   - Constructive approach
   - New class - saturated cong.-simp. subsemirings of $\mathbb{Q}^+$
   - Maximal cong.-simp. subsemirings of $\mathbb{Q}^+$
   - Open problems

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A semiring S is *congruence-simple* if it has just two congruences.
(This means that every non-trivial homomorphism from S is a monomorphism.)

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- (Cong.-simp.) subsemirings of the semiring $\mathbb{R}^+$.

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(In case of $S \subseteq \mathbb{Q}^+$: conical $\Rightarrow$ archimedean.)

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- Classify all (proper) maximal cong.-simp. subsemirings of $\mathbb{Q}^+$.

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Note, that $S \subseteq \mathbb{V}^{\circ}(p, \boldsymbol{u}(S,p))$ for S a cong.-simp. subsemiring of $\mathbb{Q}^+$.

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Moreover, if $p_1, \dots, p_n$ *are pairwise distinct and* $p \in \mathbb{P}$, *then* $\boldsymbol{u}(S, p) = \boldsymbol{r}_i$ *if* $p = p_i$, *and* $\boldsymbol{u}(S, p) = 0$ *otherwise.*

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Saturation in this sense means that no element can be added to the semiring S without changing some of the characteristic sequences.

Inspiration for finding of the maximal cong.-simp. ones:

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Theorem (Korbelář et al., A. Univ. Carol. (2009))

The semirings

$$\mathbb{Q}_1^+ = \{x \in \mathbb{Q}^+ \mid 1 \leq x\},$$

$$\mathbb{S}_p = \{x \in \mathbb{Q}^+ \mid \mathrm{v}_p(x) \geq 0\}$$

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These semirings are saturated and pairwise distinct (and hence non-isomorphic).
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- Is a finite intersection of cong.-simp. subsemirings of $\mathbb{Q}^+$ again cong.-simp.?
- Are there any maximal subsemirings in $\mathbb{R}^+$?

Thank you for your attention!