Introduction to Theory of Computability

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Outline

Why Theory of Computability?

I have written a program P, which can decide for any program Q on any input x whether $Q[x]$ terminates.

You do not believe me?

You do not believe your professor?

This is a catastrophe!

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Introduction

Various notions of computation developed by Gödel, Church, Turing and Kleene

The three computational models (recursion, λ-calculus, and Turing machine) were shown to be equivalent (1934).

Church-Turing thesis

Any real-world computation can be translated into an equivalent computation involving a Turing machine (or a program in any *reasonable* programming language).

The intuitive notion of effective computability for functions and algorithms is formally expressed by Turing machines or the lambda calculus.

A function is computable, in the intuitive sense, if and only if it is Turing-computable.

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Mathematical Preliminaries

Natural Numbers

$\mathbb{N} = \{0, 1, \ldots\}$

Sets

$\{a_1, a_2, \ldots, a_n\}$ the order of the elements is irrelevant

n-tuples

$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$
iff
$a_1 = b_1, \ldots, a_n = b_n$

Operations on Sets

$A \cup B = \{\overline{a} \mid \overline{a} \in A \text{ or } \overline{a} \in B\}$
$A \cap B = \{\overline{a} \mid \overline{a} \in A \text{ and } \overline{a} \in B\}$
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Domain of a function

$Dom[f] = \{\overline{x} \mid f[\overline{x}] \text{ is defined}\}$

Range of a function

$Ran[f] = \{y \mid \exists \overline{x} \in Dom[f] \wedge f[\overline{x}] = y\}$

Graph of a function

$Graph[f] = \{(\overline{x}, y) \mid \exists \overline{x} \in Dom[f] \wedge f[\overline{x}] = y\}$

Partial equality

$f[\overline{x}] \simeq y \quad \Leftrightarrow \quad (\overline{x}, y) \in Graph[f]$

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f is computable

f is computable function iff
there exists a program P which computes it

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Superposition

h is a superposition of $f, g_1, \ldots, g_k$

$h[\overline{x}] \simeq f[g_1[\overline{x}], \ldots, g_k[\overline{x}]]$

Theorem

Given the computable functions $f, g_1, \ldots, g_k$, then $h[\overline{x}] \simeq f[g_1[\overline{x}], \ldots, g_k[\overline{x}]]$ is computable function.

Superposition preserves computability.

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Superposition preserves computability.

Primitive Recursion

h is obtained by *weak* primitive recursion from g and a

$$h[x] \simeq \begin{cases} a & \Leftarrow\ x = 0 \\ g[x, h[x-1]] & \Leftarrow\ \text{o.w.} \end{cases}$$

h is obtained by primitive recursion from f and g

$$h[\overline{x}, y] \simeq \begin{cases} f[\overline{x}] & \Leftarrow\ y = 0 \\ g[\overline{x}, y, h[\overline{x}, y-1]] & \Leftarrow\ \text{o.w.} \end{cases}$$

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Primitive Recursive Functions

The basic functions are primitive recursive

$O[x] \simeq 0$

$S[x] \simeq x + 1$

$I_i^n[\overline{x}] \simeq x_i$

The superposition is primitive recursive

If $f, g_1, \ldots, g_k$ are primitive recursive, then

$h[\overline{x}] \simeq f[g_1[\overline{x}], \ldots, g_k[\overline{x}]]$

is primitive recursive.

The primitive recursion is primitive recursive

If f and g are primitive recursive, then

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All the primitive recursive functions are computable.

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Examples

Addition is primitive recursive

$f_1[x, y] \simeq x + y$

$$f_1[x, y] \simeq \begin{cases} x & \Leftarrow y = 0 \\ S[f_1[x, y-1]] & \Leftarrow \text{o.w.} \end{cases}$$

Multiplication is primitive recursive

$f_2[x, y] \simeq x.y$

$$f_2[x, y] \simeq \begin{cases} 0 & \Leftarrow y = 0 \\ x + f_2[x, y-1] & \Leftarrow \text{o.w.} \end{cases}$$

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Power is primitive recursive

$f_3[x, y] \simeq x^y$

$$f_3[x, y] \simeq \begin{cases} 1 & \Leftarrow y = 0 \\ x.f_3[x, y-1]] & \Leftarrow \text{o.w.} \end{cases}$$

Subtraction-dot-one is primitive recursive

$$f_4[x] \simeq x \dot{-} 1 \simeq \begin{cases} 0 & \Leftarrow x = 0 \\ x - 1 & \Leftarrow \text{o.w.} \end{cases}$$

$$f_4[x] \simeq \begin{cases} 0 & \Leftarrow x = 0 \\ f_4[x-1] + 1 & \Leftarrow \text{o.w.} \end{cases}$$

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Subtraction-dot is primitive recursive

$$f_5[x, y] \simeq x \dot{-} y \simeq \begin{cases} 0 & \Leftarrow x < y \\ x - y & \Leftarrow \text{o.w.} \end{cases}$$

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Factorial is primitive recursive

$f_6[x] \simeq x!$

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Examples

Sign is primitive recursive

$$sg[x] \simeq \begin{cases} 0 & \Leftarrow x = 0 \\ 1 & \Leftarrow \text{o.w.} \end{cases}$$

$$sg[x] \simeq \begin{cases} 0 & \Leftarrow x = 0 \\ O[sg[x-1]] + 1 & \Leftarrow \text{o.w.} \end{cases}$$

Opposite-sign is primitive recursive

$$\overline{sg}[x] \simeq \begin{cases} 1 & \Leftarrow x = 0 \\ 0 & \Leftarrow \text{o.w.} \end{cases}$$

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Examples

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$$sg[x] \simeq \begin{cases} 0 & \Leftarrow x = 0 \\ 1 & \Leftarrow \text{o.w.} \end{cases}$$

$$sg[x] \simeq \begin{cases} 0 & \Leftarrow x = 0 \\ O[sg[x-1]] + 1 & \Leftarrow \text{o.w.} \end{cases}$$

Opposite-sign is primitive recursive

$$\overline{sg}[x] \simeq \begin{cases} 1 & \Leftarrow x = 0 \\ 0 & \Leftarrow \text{o.w.} \end{cases}$$

$$\overline{sg}[x] \simeq \begin{cases} 1 & \Leftarrow x = 0 \\ O[\overline{sg}[x-1]] & \Leftarrow \text{o.w.} \end{cases}$$

Examples

Absolute value is primitive recursive

$mod[x, y] \simeq |x - y|$

$mod[x, y] \simeq (x \dot{-} y) + (y \dot{-} x)$

Minimum is primitive recursive

$min[x, y]$

$min[x, y] \simeq x \dot{-} (x \dot{-} y)$

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Primitive recursion. Properties

Theorem *If then else*
Let f_0, f_1, g be primitive recursive.
Then

$$h[\overline{x}] \simeq \begin{cases} f_0[\overline{x}] & \Leftarrow g[\overline{x}] = 0 \\ f_1[\overline{x}] & \Leftarrow \text{o.w.} \end{cases}$$

is primitive recursive.

proof:

$h[\overline{x}] \simeq \overline{sg}[g[\overline{x}]].f_0[x] + sg[g[\overline{x}]].f_1[x]$

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Primitive recursion. Properties

Theorem *If $then_1$ ... $then_k$ else*

Let $f_0, \ldots, f_k, g_0, \ldots, g_{k-1}$ be primitive recursive.
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$$h[\overline{x}] \simeq \begin{cases} f_0[\overline{x}] & \Leftarrow \; g_0[\overline{x}] = 0 \\ f_1[\overline{x}] & \Leftarrow \; g_0[\overline{x}] \neq 0 \wedge g_1[\overline{x}] = 0 \\ \ldots & \\ \ldots & \\ f_k[\overline{x}] & \Leftarrow \; \text{o.w.} \end{cases}$$

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Partial Functions

While loop

$input[x]$
$y := 0$
while $f[x, y] > 0$ *do* $y := y + 1$
return$[y]$

g is obtained by minimization from f

$g[\overline{x}] \simeq y$
iff
$\forall z < y(f[\overline{x}, z] \downarrow \ \wedge f[\overline{x}, z] > 0)$
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Partial Functions

The basic functions are partial

$O[x] \simeq 0$

$S[x] \simeq x + 1$

$I_i^n[\overline{x}] \simeq x_i$

The superposition is partial

If $f, g_1, \ldots, g_k$ are partial, then

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If f and g are partial, then

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All the partial functions are computable.

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Partial functions = Computable functions.

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Subtraction is partial

$$f[x,y] \simeq \begin{cases} x-y & \Leftarrow\ x \geq y \\ \uparrow & \Leftarrow\ \text{o.w.} \end{cases}$$

$$f[x,y] \simeq \mu z[x+y=z]$$

Division is partial

$$g[x,y] \simeq \begin{cases} x/y & \Leftarrow\ \exists k(y.k=x) \\ \uparrow & \Leftarrow\ \text{o.w.} \end{cases}$$

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Enumeration of the computable functions

Enumeration = Encoding = Effective codding

- Uniqueness: each object has a unique code
- Totality: each natural number is a code of an object
- Effectiveness: For each object one can find algorithmically its code and for each code (number) one can find its object.

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- Let $P_0, P_1, \ldots, P_n, \ldots$
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- Each program corresponds to a computable function φ
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Example

Total function which is not computable

$$f[x] \simeq \begin{cases} \varphi_x[x] + 1 & \Leftarrow \varphi_x[x] \downarrow \\ 0 & \Leftarrow \text{o.w.} \end{cases}$$

Assume f is computable. Then $f = \varphi_a$ for some a.
If $a \in Dom[\varphi_a]$ then $\varphi_a[a] \downarrow$. Hence, $f[a] = \varphi_a[a] = \varphi_a[a] + 1$
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Kleene's S-m-n Theorem

S-m-n Theorem

For any n, m exists a primitive recursive function S_n^m, such that for any $a, \overline{x}, \overline{y}$

$$\varphi_a^{(m+n)}[\overline{x}, \overline{y}] \simeq \varphi^{(n)}_{S_n^m[a,\overline{x}]}[\overline{y}]$$

Property

Let F be a computable function. Then there exists a number e, such that,

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Universal Function

Universal Function Theorem

The universal function

$\Phi_n[a, \overline{x}] \simeq \varphi_a^{(n)}[\overline{x}]$

is computable.

Property

The class of all the total functions on n-variables does not have a computable universal function.

proof

Assume $\Phi[a, \overline{x}]$ is an universal function for the class of all the total functions on one variable.

Let $\varphi[x] \simeq \Phi[x, x] + 1$.

Since Φ is total, φ is also total and hence, there exists a, such that

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Decidable and Semidecidable Sets $A \subseteq \mathbb{N}^n$

Characteristic function of a set χ_A

$$\chi_A[\overline{x}] \simeq \begin{cases} 1 & \Leftarrow \overline{x} \in A \\ 0 & \Leftarrow \text{o.w.} \end{cases}$$

Decidable Set

A set A is decidable iff χ_A is computable.

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If A is decidable then it is also semidecidable.

Theorem
If A is decidable then $\overline{A}$ is also decidable.

Theorem
If A and B are decidable then
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The set $\mathbb{K} = \{x \mid \varphi_x[x] \downarrow\}$ is called Kleene set.

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$\mathbb{K}$ is semidecidable but not decidable.

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Outline

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Halting Problem

$$P[Q, x] \simeq \begin{cases} 1 & \Leftarrow Q[x] \downarrow \\ 0 & \Leftarrow \text{o.w.} \end{cases}$$

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$$a \in \mathbb{K} \quad \Leftrightarrow \quad \varphi_a[a] \downarrow \Leftrightarrow \quad P[a, a] = 1$$

$$a \in \overline{\mathbb{K}} \quad \Leftrightarrow \quad \varphi_a[a] \uparrow \Leftrightarrow \quad P[a, a] = 0$$

Thus $\overline{\mathbb{K}}$ is decidable.

Conclusions and Discussion

Halting Problem

$$P[Q,x] \simeq \begin{cases} 1 & \Leftarrow Q[x]\downarrow \\ 0 & \Leftarrow \text{o.w.} \end{cases}$$

$$P[a,x] \simeq \begin{cases} 1 & \Leftarrow \varphi_a[x]\downarrow \\ 0 & \Leftarrow \text{o.w.} \end{cases}$$

$$a \in \mathbb{K} \quad \Leftrightarrow \quad \varphi_a[a]\downarrow \Leftrightarrow \quad P[a,a] = 1$$

$$a \in \overline{\mathbb{K}} \quad \Leftrightarrow \quad \varphi_a[a]\uparrow \Leftrightarrow \quad P[a,a] = 0$$

Thus $\overline{\mathbb{K}}$ is decidable.