Logic 1, WS 2004. Homework 4, given Oct 28, due Nov 04

1. Consider the sequent calculus definded by the axioms:

$$\mathcal{A} = \{\Phi \vdash \Psi \mid \Phi \cap \Psi\neg = \emptyset\},$$

and by the rules:

$$\frac{\Phi, \varphi_1, \varphi_2 \vdash \Psi}{\Phi, \varphi_1 \wedge \varphi_2 \vdash \Psi} (\wedge \vdash) \qquad \frac{\Phi \vdash \Psi, \varphi_1 \quad \Phi \vdash \Psi, \varphi_2}{\Phi \vdash \Psi, \varphi_1 \wedge \varphi_2} (\vdash \wedge)$$

$$\frac{\Phi \vdash \Psi, \varphi}{\Phi, \neg\varphi \vdash \Psi} (\neg \vdash) \qquad \frac{\Phi, \psi \vdash \Psi}{\Phi \vdash \Psi, \neg\psi} (\vdash \neg)$$

$$\frac{\Phi, \varphi' \vdash \Psi}{\Phi, \varphi \vdash \Psi} \text{ if } \varphi \equiv \varphi' \ (\equiv\vdash) \qquad \frac{\Phi \vdash \Psi, \psi'}{\Phi \vdash \Psi, \psi} \text{ if } \psi \equiv \psi' \ (\vdash\equiv)$$

Using this calculus, prove the correctness of the following sequent rules (that is "eliminate" them, by showing how they can be simulated by the rules of the above calculus):

$$(a) \quad \frac{\Phi, \psi_1 \vdash \Psi, \psi_2}{\Phi \vdash \Psi, \psi_1 \Rightarrow \psi_2}$$

$$(b) \quad \frac{\Phi, \psi_1 \vdash \Psi, \psi_2 \qquad \Phi, \psi_2 \vdash \Psi, \psi_1}{\Phi \vdash \Psi, \psi_1 \Leftrightarrow \psi_2}$$

You may use the rule for "$\Rightarrow$" when eliminating the rule for "$\Leftrightarrow$".

2. Using the same calculus as above, eliminate the rules:

$$(c) \quad \frac{\Phi, \varphi_1, \varphi_2 \vdash \Psi}{\Phi, \varphi_1, (\neg\varphi_1) \vee \varphi_2 \vdash \Psi}$$

$$(d) \quad \frac{\Phi, \neg\psi_1 \ \vdash \ \Psi, \psi_2}{\Phi \ \vdash \ \Psi, \psi_1 \vee \psi_2}$$

3. Write the following definition in the standard prefix syntax of predicate logic:
   A function f is continuous if and only if:

$$\forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_y\ (|y - x| < \delta \ \Rightarrow \ |f(y) - f(x)| < \epsilon)$$

4. Construct an interpretation for the formula:

$$((\forall_x P(x) \Rightarrow P(f(x))) \wedge P(a)) \Rightarrow P(f(f(a))).$$

5. Find an example of subformulae ϕ and ψ for which the following equivalence does not hold:

$$\forall_x(\phi \vee \psi) \ \equiv \ (\forall_x\phi) \vee (\forall_x\psi).$$