GOSPER'S ALGORITHM

Recall: $(a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ is called $\boxed{\text{hypergeometric}}$:⟺ exists a fixed rational function r:

$$\frac{a_{n+1}}{a_n} = r(n) \qquad \boxed{\text{almost everywhere}}$$

Gosper's algorithm solves the decision problem:

Given: $(a_k)_{k\in\mathbb{N}}$ hg.

Find: $(b_k)_{k\in\mathbb{N}}$ hg. with: $\boxed{a_k = b_{k+1} - b_k}$ antidifference

OR assert that no such b_k exists

What is this used for? indefinite summation (a_k does not depend on the summation bound)

Let $\boxed{s_n = \sum_{k=0}^{n} a_k}$ with a_k hg.; if there exists a hg. antidifference b_k, then

b_k ... telescoper

$$s_n = \sum_{k=0}^{n} (b_{k+1} - b_k) = b_{n+1} - b_0$$

↳ $\boxed{\text{Closed form summation}}$

Furthermore: $\boxed{b_k = R(k)\, a_k}$

↑ rational certificate → MMA

$\boxed{\text{ZEILBERGER'S ALGORITHM}}$ a.k.a.

CREATIVE TELESCOPING

We know $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$ ← hg. in n

but the summand does not satisfy a relation of the form $\boxed{\binom{n}{k} = b_{k+1}(n) - b_k(n)}$

GIVEN: $f(n,k)$ hg. in both n and k

FIND: $g(n,k)$ hg. in both n and k AND polynomial coefficients $c_0(n), \dots, c_d(n)$, not all zero and not depending on k:

$$\boxed{c_0(n) f(n,k) + c_1(n) f(n+1,k) + \dots + c_d(n) f(n+d,k) = g(n,k+1) - g(n,k)} \quad (3.13)$$

VERBAETEN showed that for PROPER HG. summands $f(n,k)$ a k-free recurrence

$$\sum_{i=0}^{I} \sum_{j=0}^{J} a_{ij}(n) f(n+i, k+j) = 0$$

exists with explicit bounds on the orders I and J.

proper hypergeometric:

$$p(n,k) z^k \frac{\prod' (a_i n + b_i k + c_i)!}{\prod' (\alpha_j n + \beta_j k + \gamma_j)!}$$

polg. $\in \mathbb{K}$ or indet. integer linear

$\boxed{\text{WHY (3.13)?}}$ interested in the definite sum:

$$S(n) = \sum_{k=0}^{n} f(n,k)$$

Let $d=2$ and sum over both sides of (3.13): hg. in n

$$c_0(n) \underbrace{\sum_{k=0}^{n} f(n,k)}_{= S(n)} + c_1(n) \sum_{k=0}^{n} f(n+1,k) \pm f(n+1, n+1)$$

$$+ c_2(n) \sum_{k=0}^{n} f(n+2,k) = \sum_{k=0}^{n} (g(n,k+1) - g(n,k))$$

$$\pm f(n+2, n+1)$$
$$\pm f(n+2, n+2)$$

possibly inhom. rec.

$$\Rightarrow \boxed{c_0(n) S(n) + c_1(n) S(n+1) + c_2(n) S(n+2) = g(n,n+1) - g(n,0) + F(n)}$$

again: $\boxed{g(n,k) = R(n,k) f(n,k)}$, R – rational "certificate"

$\boxed{\text{MULTISUM}}$ example: product of OPs

Let $(\phi_n)_{n\geq 0}$, $(\psi_n)_{n\geq 0}$ be two families of OPs satisfying the TTRs:

$$\phi_{n+1}(x) = (a_n x + b_n)\phi_n(x) + c_n \phi_{n-1}(x)$$

$$\psi_{n+1}(x) = (A_n x + B_n)\psi_n(x) + C_n \psi_{n-1}(x)$$

and let $\boxed{m_{ij} = \int \phi_i(x)\,\psi_j(x)\,dx}$

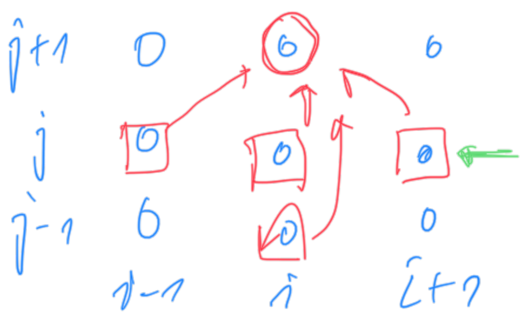
GOAL: derive a recurrence for m_{ij}

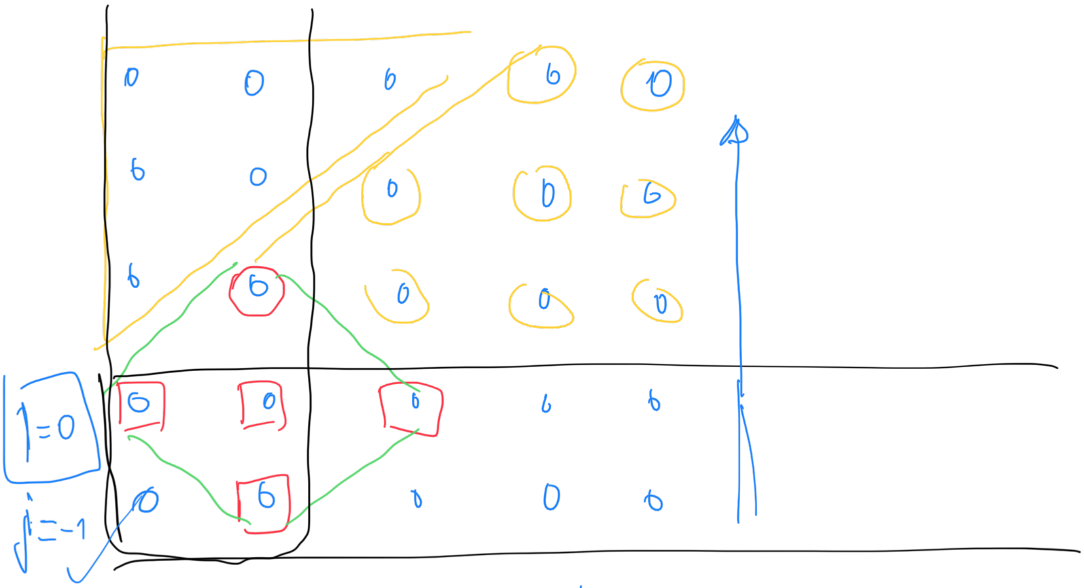
$$\phi_{i+1}(x)\,\psi_j(x) = a_i\,\phi_i(x)\boxed{\psi_j(x)\cdot x} + b_i\,\phi_i(x)\,\psi_j(x) + c_i\,\phi_{i-1}(x)\,\psi_j(x)$$

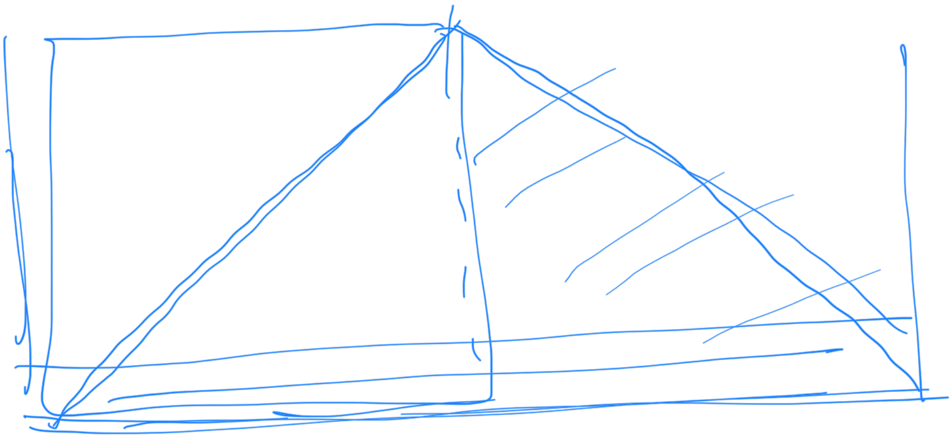
$$= \frac{a_i\phi_i(x)}{A_j}\left(\psi_{j+1}(x) - B_j\psi_j(x) - C_j\psi_{j-1}(x)\right) + \% \;//$$

$$\Rightarrow m_{i+1,j} = \frac{a_i}{A_j}\,m_{i,j+1} + \left(b_i - \frac{a_i B_j}{A_j}\right)m_{i,j} - \frac{a_i C_j}{A_j}\,m_{i,j-1} + c_i\,m_{i-1,j}$$

Support of the recurrence:







Recall:

$$\boxed{\begin{aligned} \phi_{-1}(x) &= 0 \\ \phi_0(x) &= 1 \end{aligned}}$$