In[1]:= `<< RISC`fastZeil``

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
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In[2]:= `? Gosper`

Symbol

Out[2]= Gosper[function, range],

uses Gosper's algorithm to find a hypergeometric closed form for the sum

of the function over the range,

Gosper[function, k], computes the hypergeometric forward anti–difference

of function in k, if it exists,

Gosper[function, range, degree] or

Gosper[function, k, degree] use Gosper's algorithm with an

undetermined polynomial of given degree in k multiplied to the function.

Example 3.8: Gauss sum

In[3]:= `Gosper[k, {k, 0, n}]`

```
If `n' is a natural number, then:
```

Out[3]= $\left\{\text{Sum}[k, \{k, 0, n\}] == \frac{1}{2}\, n\,(1+n)\right\}$

In[4]:= `Gosper[k, k]`

Out[4]= $\left\{k == \Delta_k\left[\frac{1}{2}\,(-1+k)\,k\right]\right\}$

Example 3.9: harmonic numbers

In[5]:= `Gosper[1/k, {k, 1, n}]`

Out[5]= {}

Examples from MMA demonstration in lecture notes p. 44

In[6]:= **Gosper[k k!, {k, 0, n}]**

```
If `n' is a natural number, then:
```

Out[6]= $\{\text{Sum}[k\,k!, \{k, 0, n\}] == -1 + (1+n)\,n!\}$

In[7]:= **Gosper[k k!, k]**

Out[7]= $\{k\,k! == \Delta_k[k!]\}$

In[8]:= **Gosper[k k!, {k, 2, 2 n}]**

```
If `-2 + 2 n' is a natural number, then:
```

Out[8]= $\{\text{Sum}[k\,k!, \{k, 2, 2\,n\}] == -2 + (1+2\,n)\,(2\,n)!\}$

In[9]:= **Gosper[k k!, {k, a, b}]**

```
If `-a + b' is a natural number, then:
```

Out[9]= $\{\text{Sum}[k\,k!, \{k, a, b\}] == -a! + (1+b)\,b!\}$

In[10]:= **Gosper[(4 k - 1) / (2 k - 1)^2 16^(-k) Binomial[2 k, k]^2, {k, 0, n}]**

```
If `n' is a natural number, then:
```

Out[10]=

$$\left\{\text{Sum}\left[\frac{16^{-k}\,(-1+4\,k)\,\text{Binomial}[2\,k,\,k]^2}{(-1+2\,k)^2}, \{k, 0, n\}\right] == -16^{-n}\,\text{Binomial}[2\,n,\,n]^2\right\}$$

In[11]:= **Gosper[(4 k - 1) / (2 k - 1)^2 16^(-k) Binomial[2 k, k]^2, k]**

Out[11]=

$$\left\{\frac{16^{-k}\,(-1+4\,k)\,\text{Binomial}[2\,k,\,k]^2}{(-1+2\,k)^2} == \Delta_k\left[-\frac{4^{1-2\,k}\,k^2\,\text{Binomial}[2\,k,\,k]^2}{(-1+2\,k)^2}\right]\right\}$$

In[12]:= **Gosper[(4 k - 1) / (2 k - 1)^2 16^(-k) Binomial[2 k, k]^2, {k, 0, n}]**

```
If `n' is a natural number, then:
```

Out[12]=

$$\left\{\text{Sum}\left[\frac{16^{-k}\,(-1+4\,k)\,\text{Binomial}[2\,k,\,k]^2}{(-1+2\,k)^2}, \{k, 0, n\}\right] == -16^{-n}\,\text{Binomial}[2\,n,\,n]^2\right\}$$

In[13]:= **Prove[]**

In[14]:= **Gosper[Binomial[n, k] , k]**

Out[14]=

{}

Zeilberger's algorithm

In[15]:=
```
? Zb
```

Out[15]=

Symbol

Zb[function, range, n, order],

uses Zeilberger's algorithm to find a recurrence relation of given order in n

for the sum of the function over the range.

Zb[function, k, n, order],

uses Zeilberger's algorithm to find a recurrence relation of given order in n

for the function. This recurrence is — up to a telescoping part —

free of k.

In both calls, if the order is of the form {ord1, ord2}, Zb tries to find

a recurrence whose order is between ord1 and ord2. Omitting the order is equivalent to

specifying {0, Infinity}.

In[16]:=
```
Zb[Binomial[n, k], {k, 0, n}, n]
```

```
If `n' is a natural number, then:
```

Out[16]=

$\{2\ \mathrm{SUM}[n] - \mathrm{SUM}[1 + n] == 0\}$

In[17]:=
```
Prove[]
```

In[18]:=
```
Zb[Binomial[n, k], k, n]
```

Out[18]=

$\{2\ F[k, n] - F[k, 1 + n] == \Delta_k[F[k, n] \times R[k, n]]\}$

In[19]:=
```
Zb[Binomial[n, k], {k, 0, Infinity}, n]
```

Out[19]=

$\{2\ \mathrm{SUM}[n] - \mathrm{SUM}[1 + n] == 0\}$

In[20]:=
```
TimeConstrained[Zb[1/(n^2+k^2), k, n], 30]
```

Out[20]=

```
$Aborted
```

TTR for Legendre polynomials

In[21]:=
```
Zb[Binomial[n, k]^2 (x - 1)^(n - k) (x + 1)^k 2^(-n), {k, 0, n}, n]
```

```
If `n' is a natural number, then:
```

Out[21]=

$\{(1 + n)\ SUM[n] - (3 + 2\ n)\ x\ SUM[1 + n] + (2 + n)\ SUM[2 + n] == 0\}$

In[22]:=
```
Prove[]
```

In[23]:=
```
Zb[Binomial[n, k]^2 (x - 1)^(n - k) (x + 1)^k 2^(-n), {k, 0, Infinity}, n]
```

Out[23]=

$\{(1 + n)\ SUM[n] - (3 + 2\ n)\ x\ SUM[1 + n] + (2 + n)\ SUM[2 + n] == 0\}$

TTR for Jacobi polynomials

In[24]:=
```
Zb[Pochhammer[α + 1, n] / n! Pochhammer[-n, k]
   Pochhammer[n + α + β + 1, k] / Pochhammer[α + 1, k] / k! ((1 - x) / 2)^k, {k, 0, Infinity}, n]
```

Out[24]=

$$\{-2\ (1 + n + \alpha)\ (1 + n + \beta)\ (4 + 2\ n + \alpha + \beta)\ SUM[n] + (3 + 2\ n + \alpha + \beta)\ (8\ x + 12\ n\ x + 4\ n^2\ x + 6\ x\ \alpha + 4\ n\ x\ \alpha + \alpha^2 + x\ \alpha^2 + 6\ x\ \beta + 4\ n\ x\ \beta + 2\ x\ \alpha\ \beta - \beta^2 + x\ \beta^2)\ SUM[1 + n] - 2\ (2 + n)\ (2 + n + \alpha + \beta)\ (2 + 2\ n + \alpha + \beta)\ SUM[2 + n] == 0\}$$

TTR for Hermite polynomials

In[25]:=
```
Zb[n! / (k! (n - 2 k)!) (-1)^k (2 x)^(n - 2 k), {k, 0, Infinity}, n]
```

Out[25]=

$\{-2\ (1 + n)\ SUM[n] + 2\ x\ SUM[1 + n] - SUM[2 + n] == 0\}$

In[26]:=
```
FullForm[%]
```

Out[26]//FullForm=
```
List[Equal[Plus[Times[-2, Plus[1, n], HoldForm[SUM[n]]],
    Times[2, x, HoldForm[SUM[Plus[1, n]]]], Times[-1, HoldForm[SUM[Plus[2, n]]]]], 0]]
```

In[27]:=
```
%25 /. SUM → H
```

Out[27]=

$\{-2\ (1 + n)\ H[n] + 2\ x\ H[1 + n] - H[2 + n] == 0\}$

In[28]:=
```
FullForm[%]
```

Out[28]//FullForm=
```
List[Equal[Plus[Times[-2, Plus[1, n], HoldForm[H[n]]],
    Times[2, x, HoldForm[H[Plus[1, n]]]], Times[-1, HoldForm[H[Plus[2, n]]]]], 0]]
```

In[29]:=
```
ReleaseHold[%27]
```

Out[29]=

$\{-2\ (1 + n)\ H[n] + 2\ x\ H[1 + n] - H[2 + n] == 0\}$

In[30]:=
```
FullForm[%]
```

Out[30]//FullForm=
```
List[
 Equal[Plus[Times[-2, Plus[1, n], H[n]], Times[2, x, H[Plus[1, n]]], Times[-1, H[Plus[2, n]]]], 0]]
```

Non-minimality of Zeilberger's algorithm

In[31]:=
```
For[d = 2, d ≤ 5, d++,
 rec[d] = Zb[(-1)^k Binomial[n, k] Binomial[d k, n], {k, 0, n}, n];
 Print[rec[d]];
]
```

If `n' is a natural number, then:

{-2 (1 + n) SUM[n] + (-1 - n) SUM[1 + n] == -n Binomial[0, n]}

If `n' is a natural number, then:

{9 (1 + n) (2 + n) SUM[n] + 3 (2 + n) (7 + 5 n) SUM[1 + n] + 2 (2 + n) (3 + 2 n) SUM[2 + n] == -n (3 + 2 n) Binomial[0, n]}

If `n' is a natural number, then:

{-64 (1 + n) (2 + n) (3 + n) (7 + 3 n) SUM[n] - 16 (2 + n) (3 + n) (107 + 125 n + 33 n^2) SUM[1 + n] -
 4 (3 + n) (4 + 3 n) (218 + 180 n + 37 n^2) SUM[2 + n] - 3 (3 + n) (4 + 3 n) (7 + 3 n) (8 + 3 n) SUM[3 + n] ==
 -n (4 + 3 n) (7 + 3 n) (8 + 3 n) Binomial[0, n]}

If `n' is a natural number, then:

{625 (1 + n) (2 + n) (3 + n) (4 + n) (7 + 2 n) (9 + 4 n) (13 + 4 n) SUM[n] +
 125 (2 + n) (3 + n) (4 + n) (13 + 4 n) (1623 + 2437 n + 1098 n^2 + 152 n^3) SUM[1 + n] +
 25 (3 + n) (4 + n) (5 + 4 n) (70 302 + 100 279 n + 52 919 n^2 + 12 242 n^3 + 1048 n^4) SUM[2 + n] +
 5 (4 + n) (5 + 2 n) (5 + 4 n) (9 + 4 n) (31 + 9 n) (486 + 283 n + 41 n^2) SUM[3 + n] +
 8 (4 + n) (5 + 2 n) (7 + 2 n) (5 + 4 n) (9 + 4 n) (13 + 4 n) (15 + 4 n) SUM[4 + n] ==
 -2 n (5 + 2 n) (7 + 2 n) (5 + 4 n) (9 + 4 n) (13 + 4 n) (15 + 4 n) Binomial[0, n]}

In[32]:=
```
For[d = 2, d ≤ 5, d++,
 data[d] = Table[Sum[(-1)^k Binomial[n, k] Binomial[d k, n], {k, 0, n}], {n, 0, 10}];
 Print[d, " --> ", data[d]];
]
```

2 --> {1, -2, 4, -8, 16, -32, 64, -128, 256, -512, 1024}

3 --> {1, -3, 9, -27, 81, -243, 729, -2187, 6561, -19 683, 59 049}

4 --> {1, -4, 16, -64, 256, -1024, 4096, -16 384, 65 536, -262 144, 1 048 576}

5 --> {1, -5, 25, -125, 625, -3125, 15 625, -78 125, 390 625, -1 953 125, 9 765 625}

In[33]:=
```
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan

```
--> Type  ?HolonomicFunctions  for help.
```

Package GeneratingFunctions version 0.9 written by Christian Mallinger

Guess Package version 0.52
written by Manuel Kauers

In[34]:=
```
Table[GuessMinRE[data[d], a[n]], {d, 2, 5}]
```

Out[34]=

{2 a[n] + a[1 + n], 3 a[n] + a[1 + n], 4 a[n] + a[1 + n], 5 a[n] + a[1 + n]}

In[35]:=
```
% // TableForm
```

Out[35]//TableForm=

2 a[n] + a[1 + n]
3 a[n] + a[1 + n]
4 a[n] + a[1 + n]
5 a[n] + a[1 + n]

In[36]:=
```
<< RISC`MultiSum`
```

Package MultiSum version 2.3 written by Kurt Wegschaider
enhanced by Axel Riese and Burkhard Zimmermann

In[37]:=
```
? FindRecurrence
```

Out[37]=

Symbol

FindRecurrence[Summand, MainVars, SumVars, DegreeBound:0] gives a list of linear homogeneous polynomial certificate recurrences for a proper hypergeometric term Summand in MainVars and SumVars (MainVars and SumVars are either variables or lists of variables), s.t.the polynomial coefficients of the principal part are free of SumVars. DegreeBound specifies the maximal degree in the SumVars in the delta parts of the certificate recurrence. DegreeBound is optional with default value 0.

FindRecurrence[Summand, MainVars, MainOrders, SumVars, SumOrders, DegreeBound:0] tries to find a list of certificate recurrences for Summand on the structureset (or the Verbaeten completion of it) specified by MainOrders and SumOrders (which are integers, one for every variable)

FindRecurrence[Summand, MainVars, SumVars, StructureSet, DegreeBound:0] tries to find a list of certificate recurrences for Summand on (the Verbaeten completion of) the StructureSet (which is a list of integer tuples).

trinomial sum

In[38]:=
```
frec = FindRecurrence[Binomial[n, j] Binomial[j, i] x^i y^(j - i) z^(n - j), {n}, {i, j}]
```

Out[38]=
```
{(x + y + z) F[-1 + n, -1 + i, -1 + j] - F[n, -1 + i, -1 + j] ==
  Δi[-y F[-1 + n, -1 + i, -1 + j] - z F[-1 + n, -1 + i, j] + F[n, -1 + i, j]] +
   Δj[-z F[-1 + n, -1 + i, -1 + j] + F[n, -1 + i, -1 + j]]}
```

In[39]:=
```
SumCertificate[frec]
```

Out[39]=
```
{(x + y + z) SUM[-1 + n] - SUM[n] == 0}
```

Product of Legendre polynomials

summand for Legendre polynomials:

In[40]:=
```
Binomial[n, k]^2 (x - 1)^(n - k) (x + 1)^k 2^(-n) /. n → m /. k → j
```
Out[40]=

$2^{-m} (-1+x)^{-j+m} (1+x)^j \text{Binomial}[m, j]^2$

In[41]:=
```
frec = FindRecurrence[Binomial[n, k]^2 (x - 1)^(n - k)
    (x + 1)^k 2^(-n) × 2^-m (-1 + x)^(-j+m) (1 + x)^j Binomial[m, j]^2, {m, n}, {j, k}]
```
Out[41]=

$\{4(-1+n) F[m, -2+n, j, -2+k] - 4(-1+2n) x F[m, -1+n, j, -2+k] + 4n F[m, n, j, -2+k] ==$
$\Delta_j[0] + \Delta_k[(-1+n)(-1+x)(3+x) F[m, -2+n, j, -2+k] -$
$(-1+n)(-1+x)^2 F[m, -2+n, j, -1+k] + 4(-1+2n) x F[m, -1+n, j, -2+k] +$
$2(-1+2n)(-1+x) F[m, -1+n, j, -1+k] - 4n F[m, n, j, -2+k] - 4n F[m, n, j, -1+k]],$
$4(-1+m) F[-2+m, n, -2+j, k] - 4(-1+2m) x F[-1+m, n, -2+j, k] + 4m F[m, n, -2+j, k] ==$
$\Delta_j[(-1+m)(-1+x)(3+x) F[-2+m, n, -2+j, k] - (-1+m)(-1+x)^2 F[-2+m, n, -1+j, k] +$
$4(-1+2m) x F[-1+m, n, -2+j, k] + 2(-1+2m)(-1+x) F[-1+m, n, -1+j, k] -$
$4m F[m, n, -2+j, k] - 4m F[m, n, -1+j, k]] + \Delta_k[0]\}$

simplify the TTR recurrence for P[n,x], P[m,x]

In[42]:=
```
SumCertificate[frec]
```
Out[42]=

$\{(-1+n) \text{SUM}[m, -2+n] - (-1+2n) x \text{SUM}[m, -1+n] + n \text{SUM}[m, n] == 0,$
$(-1+m) \text{SUM}[-2+m, n] - (-1+2m) x \text{SUM}[-1+m, n] + m \text{SUM}[m, n] == 0\}$

find a recurrence that does not contain x

In[ ]:=
```
? TimeConstrained
```
In[43]:=
```
TimeConstrained[
 frec = FindRecurrence[Binomial[n, k]^2 (x - 1)^(n - k) (x + 1)^k 2^(-n) × 2^-m (-1 + x)^(-j+m)
    (1 + x)^j Binomial[m, j]^2, {m, n}, {2, 2}, {j, k}, {2, 2}, FreeOf → {x}], 30]
```
Out[43]=
```
$Aborted
```
In[44]:=
```
<< RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

In[46]:=
```
annLeg = Annihilator[LegendreP[n, x], {Der[x], S[n]}] // Factor
```
Out[46]=

$$\left\{-(-1+x)(1+x)D_x+(1+n)S_n-(1+n)x,\ (2+n)S_n^2-(3+2n)xS_n+(1+n)\right\}$$

In[47]:=
```
annLeg = Annihilator[LegendreP[n, x], {S[n], Der[x]}] // Factor
```
Out[47]=

$$\left\{(1+n)S_n-(-1+x)(1+x)D_x-(1+n)x,\ (-1+x)(1+x)D_x^2+2xD_x-n(1+n)\right\}$$

In[48]:=
```
ApplyOreOperator[annLeg, Subscript[P, n][x]]
```
Out[48]=

$$\left\{-\left((1+n)xP_n[x]\right)+(1+n)P_{1+n}[x]-(-1+x)(1+x)P_n'[x],\ -n(1+n)P_n[x]+2xP_n'[x]+(-1+x)(1+x)P_n''[x]\right\}$$

Non-minimality of Zeilberger's algorithm

In[49]:=
```
d = 3;
Annihilator[Sum[(-1)^k Binomial[n, k] Binomial[d k, n], {k, 0, n}], S[n]] // Factor
```
Out[50]=

$$\left\{-2(3+n)(3+2n)(5+2n)S_n^3-(3+2n)\left(128+99n+19n^2\right)S_n^2-6(2+n)\left(31+33n+8n^2\right)S_n-9(1+n)(2+n)(5+2n)\right\}$$

compare to the result obtained using Zb

In[51]:=
```
rec[3]
```
Out[51]=

$$\{9(1+n)(2+n)\,\mathrm{SUM}[n]+3(2+n)(7+5n)\,\mathrm{SUM}[1+n]+2(2+n)(3+2n)\,\mathrm{SUM}[2+n]==-n(3+2n)\,\mathrm{Binomial}[0,n]\}$$

In[ ]:=
```
? CreativeTelescoping
```

Dixon's identity

In[ ]:=

$$\sum_{k=0}^{\infty}(-1)^k\binom{2n}{k}^3=\frac{(-1)^n(3n)!}{(n!)^3}$$

In[52]:=
```
f[n_, k_] := (-1)^k Binomial[2 n, k]^3;
```

In[53]:=
```
{ann, cert} = CreativeTelescoping[(-1)^k Binomial[2 n, k]^3, S[k] - 1, S[n]]
```
Out[53]=

$$\left\{\left\{\left(-1-2n-n^2\right)S_n+\left(-6-27n-27n^2\right)\right\},\ \left\{\frac{1}{2\left(2-3k+k^2+6n-4kn+4n^2\right)^3}\left(-116k^3+207k^4-147k^5+48k^6-6k^7-784k^3n+1113k^4n-594k^5n+132k^6n-9k^7n-2084k^3n^2+2214k^4n^2-792k^5n^2+90k^6n^2-2728k^3n^3+1932k^4n^3-348k^5n^3-1760k^3n^4+624k^4n^4-448k^3n^5\right)\right\}\right\}$$

In[54]:=
```
r[n_, k_] := (-116 k  + 207 k  - 147 k  + 48 k  - 6 k  - 784 k  n + 1113 k  n - 594 k  n + 132 k  n -
    9 k  n - 2084 k  n  + 2214 k  n  - 792 k  n  + 90 k  n  - 2728 k  n  + 1932 k  n  -
    348 k  n  - 1760 k  n  + 624 k  n  - 448 k  n  ) / (2 (2 - 3 k + k  + 6 n - 4 k n + 4 n  ) );
g[n_, k_] := r[n, k] × f[n, k];
```

In[56]:=
```
((ApplyOreOperator[ann[[1]], h[n, k]] /. h → f) + (g[n, k + 1] - g[n, k])) / f[n, k] // FunctionExpand //
  Factor
```

Out[56]=

0

In[57]:=
```
f2[n_] := (-1)^n (3 n)! / (n!)^3;
```

In[58]:=
```
Table[Sum[f[n, k], {k, 0, 2 n}] - f2[n], {n, 0, 10}]
```

Out[58]=

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[59]:=
```
ApplyOreOperator[ann[[1]], f2[n]] // FullSimplify
```

Out[59]=

0

x-free product recurrence for Legendre polynomials

In[60]:=
```
annProd = Annihilator[LegendreP[i, x] LegendreP[j, x], {S[i], S[j]}]
```

Out[60]=

$$\left\{(2+j)\,S_j^2 + (-3\,x - 2\,j\,x)\,S_j + (1+j),\ (2+i)\,S_i^2 + (-3\,x - 2\,i\,x)\,S_i + (1+i)\right\}$$

In[61]:=
```
FindRelation[annProd, Eliminate → {x}]
```

Out[61]=

$$\left\{(-6-3\,i-4\,j-2\,i\,j)\,S_i^2\,S_j + (6+4\,i+3\,j+2\,i\,j)\,S_i\,S_j^2 + (3+2\,i+3\,j+2\,i\,j)\,S_i + (-3-3\,i-2\,j-2\,i\,j)\,S_j\right\}$$

In[62]:=
```
Factor[%]
```

Out[62]=

$$\left\{-(2+i)\,(3+2\,j)\,S_i^2\,S_j + (3+2\,i)\,(2+j)\,S_i\,S_j^2 + (3+2\,i)\,(1+j)\,S_i - (1+i)\,(3+2\,j)\,S_j\right\}$$