Cylindrical Algebraic Decomposition

In[∘]:=
```
A = {x^2 + y^2 - 4, (x - 1) (y - 1) - 1};
```

In[∘]:=
```
Deg[p_, x_] := Exponent[p, x];
LC[p_, x_] := Coefficient[p, x, Deg[p, x]]
```

In[∘]:=
```
PFS = Union[LC[A, y], Discriminant[A, y], {Resultant[A[[1]], A[[2]], y]}] // Factor
```

Out[∘]=

$$\left\{1,\ -1+x,\ -4(-2+x)(2+x),\ -4+8x-2x^2-2x^3+x^4\right\}$$

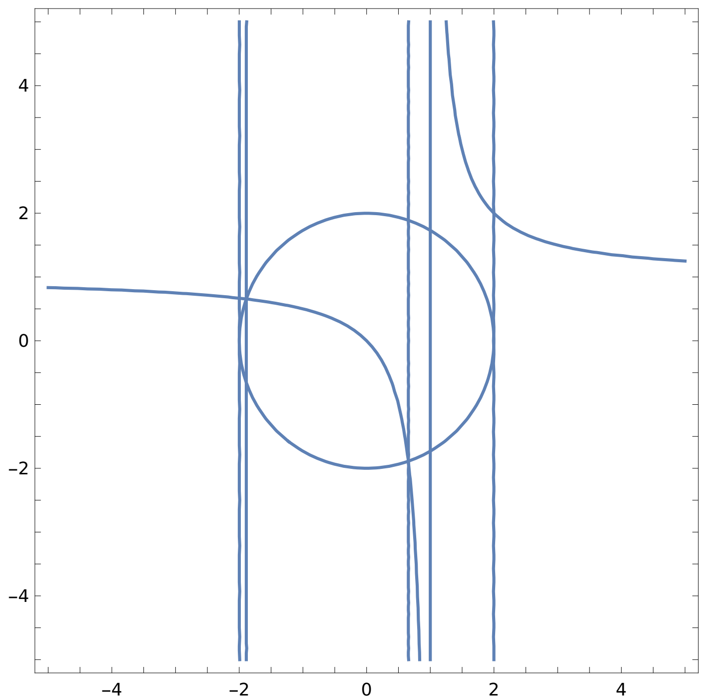
In[∘]:=
```
CAD = Union[A, PFS]
```

Out[∘]=

$$\left\{1,\ -1+x,\ -4(-2+x)(2+x),\ -4+8x-2x^2-2x^3+x^4,\ -1+(-1+x)(-1+y),\ -4+x^2+y^2\right\}$$

In[∘]:=
```
ContourPlot[CAD == 0, {x, -5, 5}, {y, -5, 5}]
```

Out[∘]=



In[∘]:=
```
PFS = Union[LC[A, x], Discriminant[A, x], {Resultant[A[[1]], A[[2]], x]}] // Factor
```

Out[∘]=

$$\left\{1,\ -1+y,\ -4(-2+y)(2+y),\ -4+8y-2y^2-2y^3+y^4\right\}$$

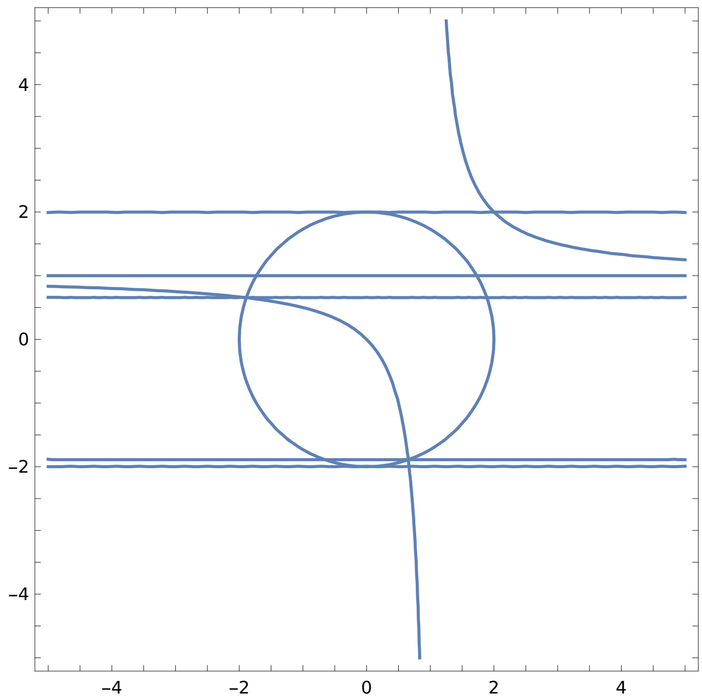
In[∘]:=
```
CADx = Union[A, PFS]
```

Out[∘]=

$$\left\{1,\ -1+(-1+x)(-1+y),\ -1+y,\ -4(-2+y)(2+y),\ -4+x^2+y^2,\ -4+8y-2y^2-2y^3+y^4\right\}$$

In[ ]:=
```
ContourPlot[CADx == 0, {x, -5, 5}, {y, -5, 5}]
```

Out[ ]=



Example : unit ball 3D

In[ ]:=
```
A3 = {x^2 + y^2 + z^2 - 1};
A2 = Union[LC[A3, z], Discriminant[A3, z]]
```

Out[ ]=

$\{1, -4(-1 + x^2 + y^2)\}$

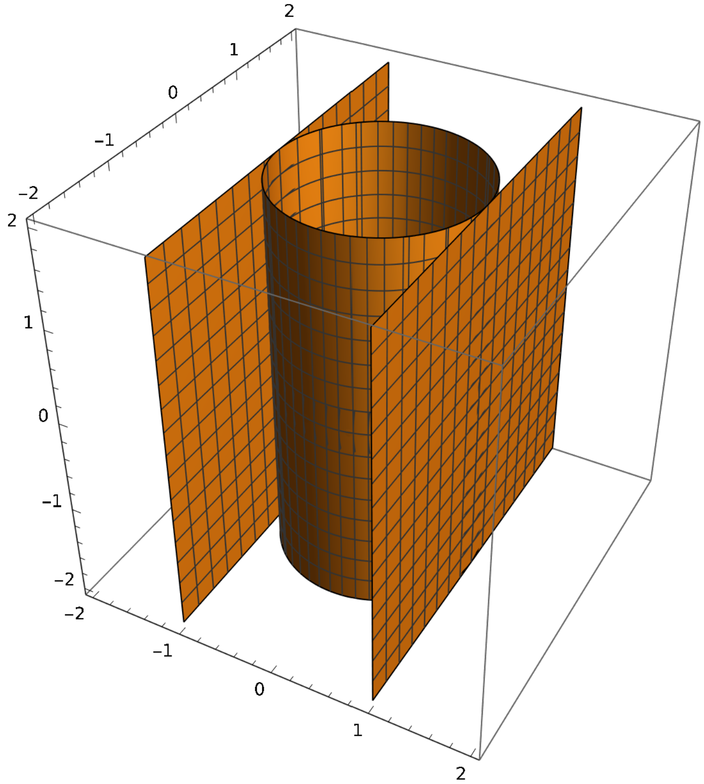
In[ ]:=
```
A1 = Union[LC[A2, y], Discriminant[A2, y]]
```

Out[ ]=

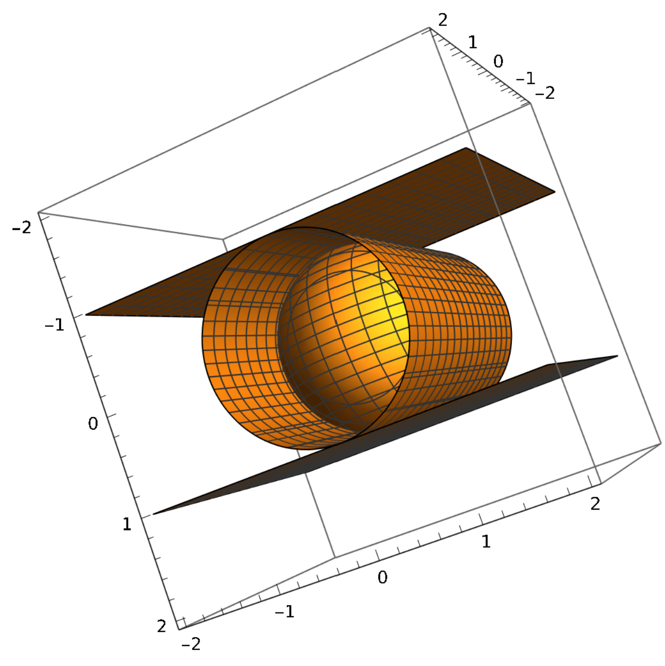
$\{-4, 1, -64(-1 + x^2)\}$

In[ ]:=
```
A = Union[A1, A2, A3];
ContourPlot3D[A == 0, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```

Out[ ]=



Out[ ]=



Example : tacnode

In[∘]:=
```
A = {2 x^4 - 3 x^2 y + y^4 - 2 y^3 + y^2}
```
Out[∘]=

$$\{2x^4 - 3x^2 y + y^2 - 2y^3 + y^4\}$$

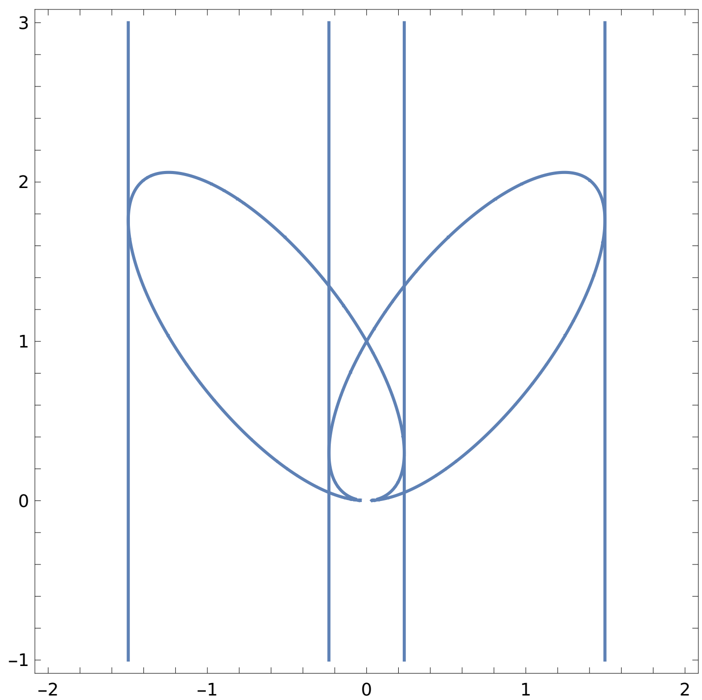
In[∘]:=
```
A1 = Union[LC[A, y], Discriminant[A, y]]
```
Out[∘]=

$$\{1, 12x^6 + 37x^8 - 4608x^{10} + 2048x^{12}\}$$

In[∘]:=
```
A2 = Union[A, A1]
```
Out[∘]=

$$\{1, 12x^6 + 37x^8 - 4608x^{10} + 2048x^{12}, 2x^4 - 3x^2 y + y^2 - 2y^3 + y^4\}$$

In[∘]:=
```
ContourPlot[A2 == 0, {x, -2, 2}, {y, -1, 3}, PlotPoints → 500]
```
Out[∘]=



In[∘]:=
```
A1x = Union[LC[A, x], Discriminant[A, x]]
```
Out[∘]=

$$\{2, 32(y^2 - 2y^3 + y^4)(-y^2 - 16y^3 + 8y^4)^2\}$$

In[∘]:=
```
A2x = Union[A, A1x] // Factor
```
Out[∘]=

$$\{2, 2x^4 - 3x^2 y + y^2 - 2y^3 + y^4, 32(-1 + y)^2 y^6 (-1 - 16y + 8y^2)^2\}$$

In[ ]:= A2x1 = $\{2,\ 2x^4 - 3x^2 y + y^2 - 2y^3 + y^4,\ (-1+y)^2,\ y^6,\ (-1-16y+8y^2)^2\}$

Out[ ]=

$\{2,\ 2x^4 - 3x^2 y + y^2 - 2y^3 + y^4,\ (-1+y)^2,\ y^6,\ (-1-16y+8y^2)^2\}$

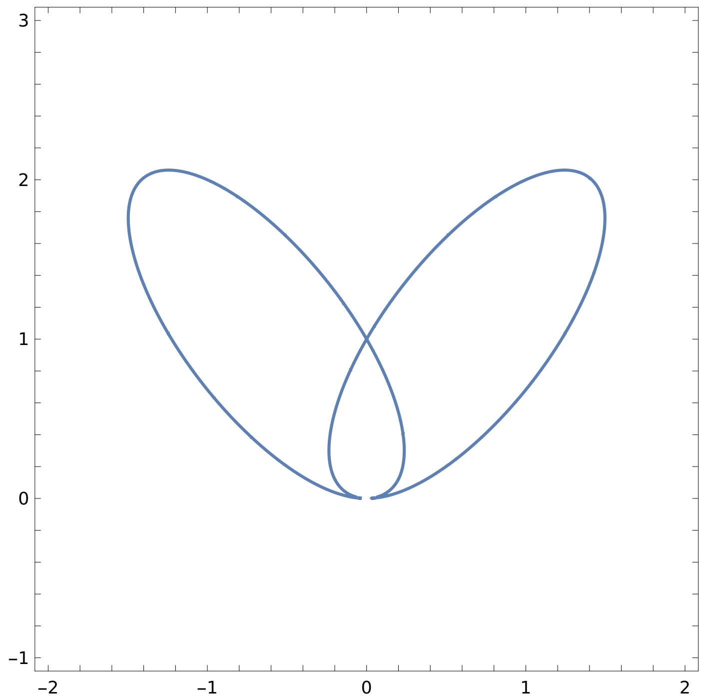
In[ ]:= $\{2,\ 2x^4 - 3x^2 y + y^2 - 2y^3 + y^4,\ (-1+y)^2,\ y^6,\ (-1-16y+8y^2)^2\}$

Out[ ]=

$\{2,\ 2x^4 - 3x^2 y + y^2 - 2y^3 + y^4,\ (-1+y)^2,\ y^6,\ (-1-16y+8y^2)^2\}$

In[ ]:= ContourPlot[A2x1 == 0, {x, -2, 2}, {y, -1, 3}, PlotPoints → 500]

Out[ ]=



In[∘]:=
```
? CylindricalDecomposition
```

Out[∘]=

Symbol

CylindricalDecomposition[*expr*, {x_1, x_2, …}] finds a decomposition of the region represented by the statement *expr* into cylindrical parts whose directions correspond to the successive x_i.

CylindricalDecomposition[*expr*, {x_1, x_2, …}, *op*] finds a decomposition of the result of applying the topological operation *op* to the region represented by the statement *expr*.

CylindricalDecomposition[*expr*, {x_1, x_2, …}, "Function"] represents the result as CylindricalDecompositionFunction[…][x_1, x_2, …] that can be efficiently used in further computation.

In[∘]:=
```
A = {x^2 + y^2 - 4, (x - 1) (y - 1) - 1};
```

In[∘]:=
```
CylindricalDecomposition[A[[1]] > 0 && A[[2]] > 0, {x, y}]
```

Out[∘]=

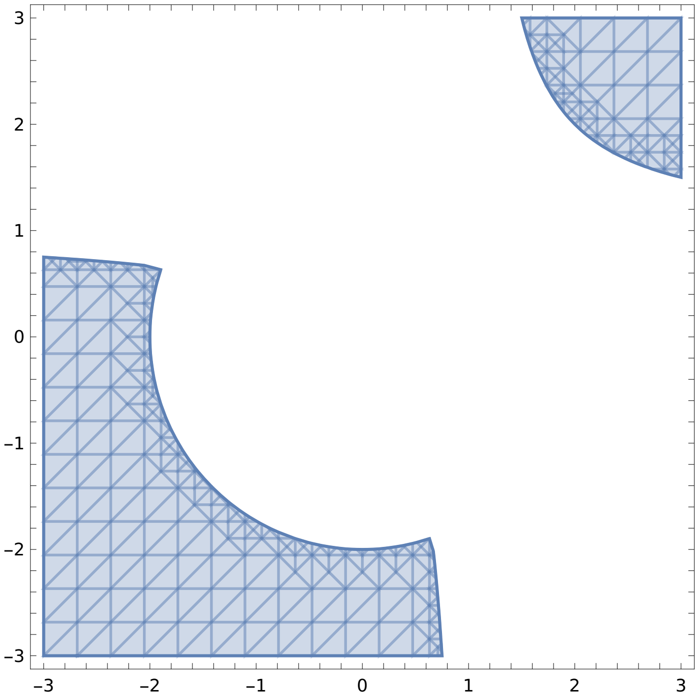
$$\left(x < -2 \,\&\&\, y < \frac{x}{-1+x}\right) \,||\, \left(-2 \le x < \boxed{-1.89\ldots} \,\&\&\, \left(y < -\sqrt{4-x^2} \,||\, \sqrt{4-x^2} < y < \frac{x}{-1+x}\right)\right) \,||$$
$$\left(\boxed{-1.89\ldots} \le x < \boxed{0.654\ldots} \,\&\&\, y < -\sqrt{4-x^2}\right) \,||$$
$$\left(\boxed{0.654\ldots} \le x < 1 \,\&\&\, y < \frac{x}{-1+x}\right) \,||\, \left(x > 1 \,\&\&\, y > \frac{x}{-1+x}\right)$$

In[∘]:=
```
CylindricalDecomposition[A[[1]] > 0 && A[[2]] > 0, {y, x}]
```

Out[∘]=

$$\left(y < -2 \,\&\&\, x < \frac{y}{-1+y}\right) \,||\, \left(-2 \le y < \boxed{-1.89\ldots} \,\&\&\, \left(x < -\sqrt{4-y^2} \,||\, \sqrt{4-y^2} < x < \frac{y}{-1+y}\right)\right) \,||$$
$$\left(\boxed{-1.89\ldots} \le y < \boxed{0.654\ldots} \,\&\&\, x < -\sqrt{4-y^2}\right) \,||$$
$$\left(\boxed{0.654\ldots} \le y < 1 \,\&\&\, x < \frac{y}{-1+y}\right) \,||\, \left(y > 1 \,\&\&\, x > \frac{y}{-1+y}\right)$$

In[∘]:=
```
RegionPlot[%30, {x, -3, 3}, {y, -3, 3}]
```
Out[∘]=



In[∘]:=
```
CylindricalDecomposition[(A⟦1⟧ > 0 && A⟦2⟧ > 0) || (A⟦1⟧ ≤ 0 && A⟦2⟧ ≤ 0), {x, y}]
```
Out[∘]=

$$\left(x < -2 \;\&\&\; y < \frac{x}{-1+x}\right) \,||\, \left(-2 \le x < \boxed{-1.89\ldots} \;\&\&\; \left(y < -\sqrt{4-x^2} \,||\, \sqrt{4-x^2} < y < \frac{x}{-1+x}\right)\right) \,||$$

$$\left(x == \boxed{-1.89\ldots} \;\&\&\; \left(y < -\sqrt{4-x^2} \,||\, y == \frac{x}{-1+x}\right)\right) \,||$$

$$\left(\boxed{-1.89\ldots} < x < \boxed{0.654\ldots} \;\&\&\; \left(y < -\sqrt{4-x^2} \,||\, \frac{x}{-1+x} \le y \le \sqrt{4-x^2}\right)\right) \,||$$

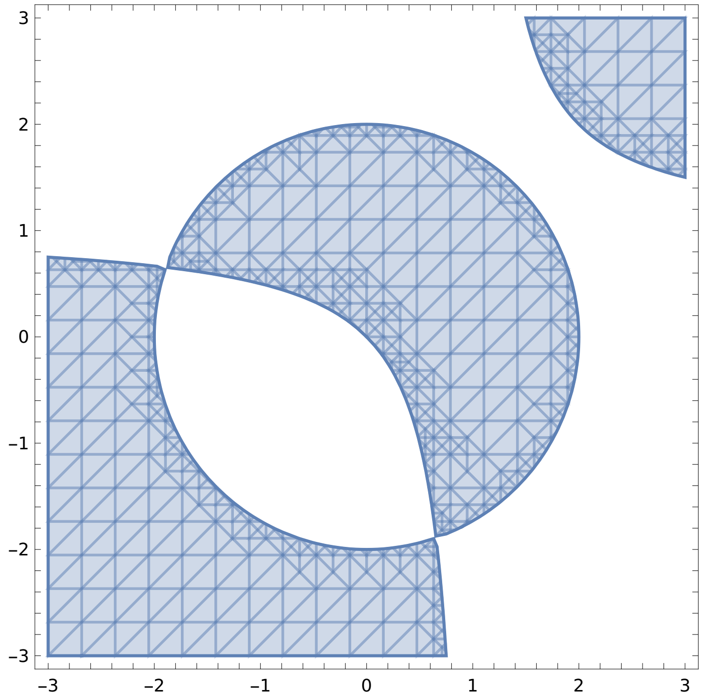
$$\left(x == \boxed{0.654\ldots} \;\&\&\; y \le \sqrt{4-x^2}\right) \,||\, \left(\boxed{0.654\ldots} < x < 1 \;\&\&\; \left(y < \frac{x}{-1+x} \,||\, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\right)\right) \,||$$

$$\left(x == 1 \;\&\&\; -\sqrt{3} \le y \le \sqrt{3}\right) \,||\, \left(1 < x < 2 \;\&\&\; \left(-\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \,||\, y > \frac{x}{-1+x}\right)\right) \,||$$

$$(x == 2 \;\&\&\; (y == 0 \,||\, y > 2)) \,||\, \left(x > 2 \;\&\&\; y > \frac{x}{-1+x}\right)$$

In[ ]:=
```
RegionPlot[%32, {x, -3, 3}, {y, -3, 3}]
```

Out[ ]=



In[ ]:=
```
CylindricalDecomposition[
 ForAll[x, Exists[y, (A[[1]] > 0 && A[[2]] > 0) || (A[[1]] ≤ 0 && A[[2]] ≤ 0)]], {}]
```

Out[ ]=
```
True
```

In[ ]:=
```
? Reduce
```

Out[ ]=

Symbol

Reduce[*expr*, *vars*] reduces the statement *expr* by solving equations or inequalities for *vars* and eliminating quantifiers.

Reduce[*expr*, *vars*, *dom*] does the reduction over the domain *dom*. Common choices of *dom* are Reals, Integers, and Complexes.

In[ ]:=
```
? Resolve
```
Out[ ]=

Symbol

Resolve[*expr*] attempts to resolve *expr* into a form that eliminates ForAll and Exists quantifiers.

Resolve[*expr*, *dom*] works over the domain *dom*.

Common choices of *dom* are Complexes, Reals, and Booleans.

In[ ]:=
```
Resolve[ForAll[x, Exists[y, (A[[1]] > 0 && A[[2]] > 0) || (A[[1]] ≤ 0 && A[[2]] ≤ 0)]], {}]
```
Out[ ]=
```
True
```

In[ ]:=
```
Resolve[Exists[y, (A[[1]] > 0 && A[[2]] > 0) || (A[[1]] ≤ 0 && A[[2]] ≤ 0)], {x}]
```
Out[ ]=

$x \in \mathbb{R}$

In[ ]:=
```
? GenericCylindricalDecomposition
```
Out[ ]=

Symbol

GenericCylindricalDecomposition[*ineqs*, {x_1, x_2, …}] finds the full–dimensional part of the decomposition of the region represented by the inequalities *ineqs* into cylindrical parts whose directions correspond to the successive x_i, together with any hypersurfaces containing the rest of the region.

In[ ]:=
```
{cad, hypersurfaces} =
  GenericCylindricalDecomposition[(A[[1]] > 0 && A[[2]] > 0) || (A[[1]] ≤ 0 && A[[2]] ≤ 0), {x, y}];
```

In[ ]:=
```
cad
```
Out[ ]=

$$\left(x < -2 \;\&\&\; y < \frac{x}{-1+x}\right) \,||\, \left(-2 < x < \boxed{-1.89\ldots} \;\&\&\; \left(y < -\sqrt{4-x^2} \,||\, \sqrt{4-x^2} < y < \frac{x}{-1+x}\right)\right) \,||$$
$$\left(\boxed{-1.89\ldots} < x < \boxed{0.654\ldots} \;\&\&\; \left(y < -\sqrt{4-x^2} \,||\, \frac{x}{-1+x} < y < \sqrt{4-x^2}\right)\right) \,||$$
$$\left(\boxed{0.654\ldots} < x < 1 \;\&\&\; \left(y < \frac{x}{-1+x} \,||\, -\sqrt{4-x^2} < y < \sqrt{4-x^2}\right)\right) \,||$$
$$\left(1 < x < 2 \;\&\&\; \left(-\sqrt{4-x^2} < y < \sqrt{4-x^2} \,||\, y > \frac{x}{-1+x}\right)\right) \,||\, \left(x > 2 \;\&\&\; y > \frac{x}{-1+x}\right)$$

In[ ]:=
```
hypersurfaces
```
Out[ ]=

$-2+x==0 \,||\, -1+x==0 \,||\, 2+x==0 \,||\, -4+8x-2x^2-2x^3+x^4==0 \,||\, -x-y+xy==0 \,||\, -4+x^2+y^2==0$

In[ ]:=
```
<< RISC`SumCracker`
```

SumCracker Package version 0.11
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[ ]:=
```
ProveInequality[LegendreP[n + 1, x]^2 - LegendreP[n, x] LegendreP[n + 2, x] ≥ 0,
 Using → {-1 ≤ x ≤ 1}, Variable → n]
```
Out[ ]=
```
True
```

In[ ]:=
```
ProveInequality[LegendreP[n + 1, x]^2 - LegendreP[n, x] LegendreP[n + 2, x] ≥ 0,
 Using → {-1 ≤ x ≤ 1}, Variable → n, Infolevel → 3]
```

```
Collecting terms from given inequality...
Creating difference ring and homomorphism...
  Normalizing exponentials...
  Cracking expression into difference system...
    Determining bounded variables...
    Traversing expression tree...
    Creating expression transformators...
    Expression successfully traversed.
  Creating difference ring...
    Shifting back the system as far as possible...
    Autoreducing the system...
    Bringing system into standard shape...
    Translating expressions into difference variables...
    Adjoining denominators...
    Defining methods...
    Difference ring construction completed.
  Determining startpoint and building evaluator...
    Taking startpoint 0
    Collecting initial values, with pre-values starting from -2...
```

```
  Defining methods...
    GetDifferenceRing
    GetVariable
    Expr2Poly
    Declaring relations...
    Poly2Expr
    GetStartpoint
    Evaluate
    ZeroQ
    GetGroundfield
  Exiting phi constructor.
Adding positivity of natural numbers to facts...
Converting terms to difference polynomials...
Shift adjustment of facts and claim...
searching for induction step...
  checking initial value...
  extending induction hypothesis...
  selecting and sorting variables...
  checking induction step...
    Probabilistic check yields "True". Validating...
True.
```

Out[∘]=

```
True
```

ProveInequality[..., Where → {recursive definition}, Using → {-1 < x < 1}, ...

In[∘]:= **ProveInequality[3 ^ n ≥ 2 ^ n, Infolevel → 3]**

```
Collecting terms from given inequality...
Creating difference ring and homomorphism...
  Extracting dependent variable...
    Taking n.
  Normalizing exponentials...
  Cracking expression into difference system...
    Determining bounded variables...
    Traversing expression tree...
```

```
    Creating expression transformators...
    Expression successfully traversed.
  Creating difference ring...
    Shifting back the system as far as possible...
    Autoreducing the system...
    Bringing system into standard shape...
    Translating expressions into difference variables...
    Adjoining denominators...
    Defining methods...
    Difference ring construction completed.
  Determining startpoint and building evaluator...
    Taking startpoint 0
    Collecting initial values, with pre-values starting from -1...
  Defining methods...
    GetDifferenceRing
    GetVariable
    Expr2Poly
    Declaring relations...
    Poly2Expr
    GetStartpoint
    Evaluate
    ZeroQ
    GetGroundfield
  Exiting phi constructor.
Adding positivity of natural numbers to facts...
Converting terms to difference polynomials...
Shift adjustment of facts and claim...
searching for induction step...
  checking initial value...
  extending induction hypothesis...
  selecting and sorting variables...
  checking induction step...
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Out[ ]=

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$Aborted
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In[ ]:=

```
ProveInequality[LegendreP[n + 1, x]^2 - x LegendreP[n, x] LegendreP[n + 2, x] ≥ 0,
  Using → {0 < x < 1}, Variable → n, From → 1, Infolevel → 2]
```

```
Collecting terms from given inequality...
Creating difference ring and homomorphism...
  Normalizing exponentials...
  Cracking expression into difference system...
  Creating difference ring...
  Determining startpoint and building evaluator...
  Defining methods...
  Exiting phi constructor.
Adding positivity of natural numbers to facts...
Converting terms to difference polynomials...
Shift adjustment of facts and claim...
searching for induction step...
  checking initial value...
  extending induction hypothesis...
  selecting and sorting variables...
  checking induction step...
True.
```

Out[ ]=

```
True
```