

Each task is worth 4P and 36P are considered 100%, passing level is 18P. Show all your work. The use of computer algebra for polynomial arithmetic and linear algebra subtasks is recommended (e.g., resultants, solving systems, etc.), but no built-in functions or add-on packages for handling integrals or differential equations are allowed. For programming tasks and subtasks you may choose your favourite computer algebra system. If you choose to do one of the “Implement...” tasks test your program on different examples and also hand in an executable version of your code (e.g., Mathematica Notebook, Sage Worksheet, etc.). Submission deadline is Friday, July 21, 2023. Good luck!

Task 1. Consider the differential field $K = \mathbb{Q}(x, y)$ with $D(x) = 1$ and $D(y) = \frac{1}{x}$. Prove that $\text{const}(K) = \mathbb{Q}$.

Task 2. Show, that there is no linear differential equation *with polynomial coefficients*, which has $\exp(\exp(x))$ among its solutions. You may use without proof that $\exp(x)$ is not algebraic.

Task 3. Prove Lemma 10, i.e., show that the coefficient lifting κ_D (see Definition 9) is a derivation on $R[x]$, where (R, D) is a differential ring and x an indeterminate over R .

Task 4. Use Hermite reduction on $f(x) = \frac{x^5 - x^4 + 4x^3 + x^2 - x + 5}{(x^2 - x + 2)^2}$ to compute g, h .

Task 5. Use Rothstein-Trager on the function $h(x)$ computed in the previous Task.

Task 6. Consider the integrand $f(x) = \log(x)\log(x+1)\log(2x^2+2x)$ and the tower of extensions $\mathbb{Q}(x, t_1, t_2, t_3)$ with

$$Dx = 1, \quad Dt_1 = \frac{1}{x}, \quad Dt_2 = \frac{1}{x+1}, \quad Dt_3 = \frac{2x+1}{x^2+x}.$$

Check for each extension, which type it is and whether the constants are extended.

Task 7. Use the algorithm Hermite Reduction II to integrate $f = \frac{\log(x) - 1}{\log(x)^2}$.

Task 8. Find all rational solutions to

$$2(2x+1)(x+3)^2 y''(x) - (2x^2 - 17x - 29)(x+3)y'(x) - 2(x-2)(3x+7)y(x) = 0.$$

Task 9. Compute all generalized series solutions to

$$(3x^2 + 1)y''(x) - (9x^4 + 6x - 1)y'(x) - 3x(3x^3 + x + 2)y(x) = 0.$$

Task 10. Compute $\text{ann}(f+g)$, where

$$\text{ann}(f) = \{D_x - 1, D_y - 1\} \quad \text{and} \quad \text{ann}(g) = \{2yD_y - 1, 2xD_x - 1\}.$$

Task 11. Implement the Extended Euclidean Algorithm for solving diophantine equations of the form: given $a, b, c \in \mathbb{K}[x]$, find $s, t \in \mathbb{K}[x]$ with $c = sa + tb$ and $\deg(s) < \deg(b)$.

Task 12. Implement Hermite Reduction (Algorithm 1) for rational input of the form $f = u/v$ with $\deg(u) < \deg(v)$.