Ordinary Differential Equations (ODEs)

In this section, we consider the following problem:

Given: a differential field K, $a_0, \dots, a_r \in K$, $a_r \neq 0$, and a differential ring $R \supseteq K$
Find: all $y \in R$ satisfying the following linear ODE,

$$a_0 y + a_1 y' + \cdots + a_r y^{(r)} = 0. \tag{1}$$

- r is called the *order* of the ODE (1).
- If the leading coefficient $a_r = 1$, then the ODE is called monic.
- If y_1, y_2 are solutions to (1) and $\alpha_1, \alpha_2 \in \mathbb{K}$, then $\alpha_1 y_1 + \alpha_2 y_2$ are solutions to (1), i.e., the set of solutions forms a vector space.

Theorem 18. *If R is (contained in) a field and V is the solution space of* (1) *in R, then* $\dim_{\mathbb{K}} V \leq r$.

Definition 19. *Any set of r $\mathbb{K}$-linear independent solutions of* (1) *is called a fundamental system of the equation.*

Example 20. *Consider the ODE $y'' - 2y + y = 0$ and its solution space V_i over $R_1 = \mathbb{K}(x, e^x), R_2 = \mathbb{K}(e^x), R_3 = \mathbb{K}(x)$. Then we have*

- $V_1 = \{e^x, xe^x\}$ *as is easily checked by plugging into the ODE;* $\dim V_1 = 2$.
- $V_2 = \{e^x\}$*;* $\dim V_2 = 1$.
- $V_3 = \{0\}$*;* $\dim V_3 = 0$.

The existence and dimension of the solution space/set depend on the differential equation and the choice of R.

Linear ODEs with constant coefficients

Linear ODEs with constant coefficients can always be solved completely in closed form over the right ring. For now, we consider (1) with coefficients $a_i \in \mathbb{K}$. We can write the ODE using operator notation as

$$L(D)y = 0, \quad \text{where} \quad L = \sum_{i=0}^{r} a_i D^i.$$

Let us assume for the time being that $\mathbb{K}$ is algebraically closed.

Note

- linear differential operators with constant coefficients commute, i.e., $L_1(D)L_2(D) = L_2(D)L_1(D)$.
- the general solution to $y' - ay = 0$ is $y = e^{ax}$.
- the fundamental system of solutions to $(D-a)^m y = 0$ is $e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax}$.
- if y_i is a solution to $L_i(D)y = 0$, then $\alpha_1 y_1 + \alpha_2 y_2$, $\alpha_1, \alpha_2 \in \mathbb{K}$, is a solution to $L_2(D)L_1(D)y = 0$.

Hence, if $p(x) = a_0 + a_1 x + \cdots + a_r x^r = a_r \prod_{j=1}^{s}(x-\alpha_j)_j^m$, $\alpha_j \in \mathbb{K}, m_j \in \mathbb{N}$, then the fundamental system of $p(D)y = 0$ is given by

$$\{x^i e^{\alpha_j x} \mid j = 1, \ldots, s, i = 0, \ldots, m_j - 1\}.$$

Note

- if $\mathbb{K} = \mathbb{Q}$ and $\alpha \in \mathbb{C}$ is a root of p of multiplicity m, then also its complex conjugate $\bar{\alpha} \in \mathbb{C}$ has to be a root; say $\alpha = u + \mathrm{i}v$, then

  $$\exp(u \pm \mathrm{i}v), x\exp(u \pm \mathrm{i}v), \ldots, x^{m-1}\exp(u \pm \mathrm{i}v)$$

  are part of the fundamental system.

- Since $\exp(u \pm \mathrm{i}v) = e^{ux}\big(\cos(vx) \pm \sin(vx)\big)$, we get the linear independent *real* solutions,

  $$e^{ux}\cos(vx), e^{ux}\sin(vx), xe^{ux}\cos(vx), xe^{ux}\sin(vx), \ldots, x^{m-1}e^{ux}\cos(vx), x^{m-1}e^{ux}\sin(vx).$$

Summarizing, the solution of the ODE (1) with constant coefficients is easy, for more general fields K it's not so clear what to do.

Polynomial solutions of linear ODEs with rational coefficients

Next, we will discuss finding polynomial solutions of ODEs with rational function coefficients, i.e., we consider (1) with $K = \mathbb{K}(x)$ and $R = \mathbb{K}[x]$. After clearing denominators, we may assume that the coefficients are all polynomial, i.e., $a_i \in \mathbb{K}[x]$.

First, we need to determine a *degree bound* d of a potential polynomial solution to (1). Once we have d,

1. make an ansatz with undetermined coefficients: $y(x) = \sum_{j=0}^{d} y_j x^j$;
2. plug the ansatz into the ODE (1);
3. set up a linear system by equating the coefficients to zero;
4. return either the polynomial solution to (1) OR "no polynomial solution exists".

Determine a degree bound Let's denote the unknown degree of y by d, i.e., let

$$y(x) = y_d x^d + y_{d-1}x^{d-1} + \cdots + y_1 x + y_0,$$

and w.l.o.g. we assume that the ODE is monic, i.e., that $y_d = 1$. Since $D^k y = d^{\underline{k}} x^{d-k} +$ lower order terms, we have

$$\deg\left(a_k D^k(y)\right) = \deg(a_k) + d - k.$$

Define $\beta = \max_{k=0,\dots,r}(\deg(a_k) - k)$. Then we have

$$[x^{d+\beta}]a_k D^k(y) = \begin{cases} \mathrm{lc}(a_k) d^{\underline{k}} & \text{if } \beta = \deg a_k - k \\ 0 & \text{else.} \end{cases}$$

By the choice of β, this term will be non-zero for at least one $k \in \{0, \dots, r\}$. Hence

$$\varphi(d) = \sum_{k=0}^{r} [x^{d+\beta}](a_k D^k y) d^{\underline{k}} = \sum_{\deg(a_k)-k=\beta} \mathrm{lc}(a_k) d^{\underline{k}} \in \mathbb{K}[d]$$

is a non-zero polynomial called the *indicial polynomial* of the ODE. If y is a polynomial solution of (1) of degree d, then d is an integer root of φ. Thus, we end up with the *degree bound*

$$d = \max\{n \in \mathbb{N} \mid \varphi n = 0\}.$$

Example 21. *Consider the ODE*

$$(x+1)y'' + (x-1)y' - 2y = 0,$$

i.e., we have the coefficients

$$a_2(x) = x + 1, \quad a_1(x) = x - 1, \quad a_0(x) = -2,$$

and so

$$\beta = \max\{1 - 2, 1 - 1, 0 - 0\} = 0.$$

Thus

$$\varphi(d) = \sum_{\deg(a_k)-k=\beta} \mathrm{lc}(a_k) d^{\underline{k}} = 1 \cdot d^{\underline{1}} - 2 \cdot d^{\underline{0}} = d - 2 \quad \Rightarrow \quad d = 2.$$

Hence we have the degree bound $d = 2$ and the ansatz

$$\begin{aligned} y(x) &= y_2 x^2 + y_1 x + y_0 \\ y'(x) &= 2y_2 x + y_1 \\ y''(x) &= 2y_2 \end{aligned}$$

which we plug into the ODE:

$$\begin{aligned}2(x+1)y_2 + 2(x-1)xy_2 + (x-1)y_1 - 2y_2x^2 - 2y_1x - 2y_0 &= 0\\ -y_1x + 2y_2 - y_1 - 2y_0 &= 0.\end{aligned}$$

Equating coefficients to zero, gives the solution

$$y_1 = 0, y_2 = y_0,$$

hence the general polynomial solution is $y(x) = C(x^2+1)$. *(Note: the full solution is given by* $y(x) = C_1e^{-x} + C_2(x^2+1)$.*)*

Example 22. *Consider next the ODE,*

$$(x^2-1)y'' + (x-1)y' - 2y = 0.$$

Here $\beta = \max\{2-2, 1-1, 0-0\} = 0$ *and* $\varphi(d) = d^{\underline{2}} + d^{\underline{1}} - 2 = d^2 - 2$, *which has no integer roots and thus the equation does not have a polynomial solution.*

Rational solutions of linear ODEs with ~~rational~~ polynomial coeffs

By the same argument as before, we can reduce the problem to polynomial coefficients and consider (1) with $K = \mathbb{K}[x]$ and $R = \mathbb{K}(x)$.

First, we need to determine a *denominator bound*. Suppose $y = \frac{p}{q}$ is a solution and suppose we know some $Q \in \mathbb{K}[x]$ with $q|Q$. Then we can write $y = \frac{P}{Q} = \frac{p}{q}$ for some (unknown) $P \in \mathbb{K}[x]$ that is not necessarily coprime with Q.

Next, we plug y into the given ODE, clear denominators and end up with the problem of finding a polynomial solution P to an ODE with polynomial coefficients.

Finding a denominator bound Let $y = p/q$ be a solution of (1) and $q = q_1^{m_1} \cdots q_s^{m_s}$ the factorization of q over $\mathbb{K}[x]$. Then

$$D^k(p/q) = \frac{\text{poly}}{q_1^{m_1+k} \cdots q_s^{m_s+k}}$$

with <u>no</u> possibility of further cancelation, since we have for squarefree q and pairwise relatively prime $u, q, v \in \mathbb{K}[x]$,

$$D\left(\frac{u}{q^l v}\right) = \frac{u'q^l u - u(lvq^{l-1}q' + q^l v')}{q^{2l}v^2} = \frac{u'qv - luvq' - quv'}{q^{l+1}v^2}.$$

If q divides the numerator if and only if $q|luvq'$, but $l > 0$ and $1 = \gcd(q,u) = \gcd(q,v) = \gcd(q,q')$.

Plugging $y = p/q$ into the ODE (1) in the form

$$a_r y^{(r)} = a_{r-1}y^{(r-1)} - \cdots - a_1 y' - a_0 y,$$

gives something of the form

$$\frac{p_r}{q_1^{m_1+r}\cdots q_s^{m_s+r}} = -\frac{p_{r-1}}{q_1^{m_1+r-1}\cdots q_s^{m_s+r-1}} - \cdots - \frac{p_0}{q_1^{m_1}\cdots q_s^{m_s}},$$

for some polynomials p_i. Hence something has to cancel on the LHS of the equation. Since there is no common factor with the numerator of $y^{(r)}$, we have that it has to be with the leading coefficient a_r,

$$q_1\cdots q_s | a_r.$$

This means that the factors that can occur in the denominator appear among the factors of the leading coefficient.

The next step is to find a bound on the multiplicities of the factors. The idea is as follows:

- Let q be a factor of a_r and $y = \frac{u}{vq^l}$ be a solution of (1);
- w.l.o.g., we may assume that $\gcd(u,q) = \gcd(v,q) = 1$;
- let $q = x - \alpha$ for some $\alpha \in \overline{\mathbb{K}}$; w.l.o.g. assume $\alpha = 0$;
- expand

$$\frac{u}{v} = c_0 + c_1 x + c_2 x^2 + \ldots \quad \Rightarrow \quad y(x) = c_0 x^{-l} + \ldots, \quad y'(x) = -c_0 l y^{-l-1} + \ldots$$

- plug $y = x^{-l}$ into the ODE (1)
- the *trailing coefficient* is a polynomial in l: $\varphi(l)$, called the *indicial polynomial*;
- if $y = \frac{u}{vx^l}$ is a rational solution, then $-l$ is an integer root of φ.

Example 23. *We consider the second order linear ODE*

$$x(x+2)y'' + (6 - x^2)y' - 2(x+3)y = 0.$$

Candidates for the denominator bound are $q_1(x) = x, q_2(x) = x + 2$.

Next, we need to determine the multiplicities:

- $q_1 = x$*: write* $y = \frac{u}{vq_1^l}$ *and expand* $u/v = c_0 + c_1 x + c_2 x^2 + \ldots$*. Then*

$$\begin{aligned} y &= c_0 x^{-l} + c_1 x^{-l+1} + \ldots \\ y' &= -lc_0 x^{-l-1} - lc_1 x^{-l} + \ldots \\ y'' &= -l(l+1)c_0 x^{-l-2} + l(l-1)c_1 x^{-l-1} + \ldots \end{aligned}$$

Plugging into the given ODE yields

$$0 = x(x+2)c_0 l(l+1)x^{-l-2} + \cdots - (6 - x^2)lc_0 x^{-l-1} - \cdots - 2(x+3)c_0 x^{-l} + \ldots$$

The trailing coefficient is the coefficient of x^{-l-1} *and so we obtain the indicial polynomial*

$$\varphi(l) = 2l(l+1) - 6l = 2l(l-2) \quad \Rightarrow \quad l = 2.$$

Thus the first factor of the denominator bound is x^2.

- $q_2 = x + 2$: $y(x-2)$ *is a solution to*

  $$x(x-2)y'' - (x^2 - 4x - 2)y' - 2(x+1)y = 0.$$

  By shifting the argument, we can consider now again $\tilde{q}_2 = x$ *and the same procedure as before now yields* $\varphi(l) = -2l(l+3)$, *i.e.,* $l = -3$, *which is not a valid bound.*

Summarizing, we plug in the ansatz $y(x) = u(x)/x^2$ *into the original ODE and look for the polynomial solution* $u(x)$ *with the previous method.*