Extended Euclidean Algorithm (following lecture notes Lineare Algebra, Manuel Kauers):
Input: polynomials a,b, polynomial variable x
Output: PolynomialGCD g of a and b and Bezout Co-factors u,v s.t. g = u a + v b

In[1]:=
```
EEA[a_, b_, x_] := Module[{r0, u0, v0, r1, u1, v1, q, lcr0},
    If[Factor[a] == 0 && Factor[b] == 0, Return[{0, 0, 0}]]; (*Return[{g,u,v}]*)
    {r0, u0, v0, r1, u1, v1} = {a, 1, 0, b, 0, 1};
    While[r1 =!= 0,

      q = PolynomialQuotient[r0, r1, x] // Factor;
      {r0, u0, v0, r1, u1, v1} = Factor[{r1, u1, v1, r0 - q r1, u0 - q u1, v0 - q v1}];
    ];
    lcr0 = Coefficient[r0, x, Exponent[r0, x]];
    Return[{r0 / lcr0, u0 / lcr0, v0 / lcr0}];
  ];
```

In[2]:=
```
u = x^2; v = x^2 + 2;
EEA[u D[v, x], v, x]
```

Extended Euclidean Algorithm for solving the diophatine equation c = s a + t b with deg (s) < deg (b)
Input : polynomials a, b, c polynomial variable x
Output : If the PolynomialGCD g of a and b divides c, a solution {s, t} to c = s a + t b with deg (s) < deg (b)
OR { }, if no solution exists

In[4]:=
```
EEA[a_, b_, c_, x_] := Module[{g, s, t, q, r},
    {g, s, t} = EEA[a, b, x];
    q = PolynomialQuotient[c, g, x]; r = PolynomialRemainder[c, g, x];
    If[r =!= 0, Return[{}]];
    s *= q; t *= q;
    If[Exponent[s, x] < Exponent[b, x], Return[Factor[{s, t}]]];
    q = PolynomialQuotient[s, b, x]; r = PolynomialRemainder[s, b, x];
    Return[Factor[{r, t + q a}]];
  ];
```

In[6]:=
```
a = x^7 - 24 x^4 - 4 x^2 + 8 x - 8;
```

In[7]:=
```
EEA[u D[v, x], v, -a / 2, x]
```

Out[7]= $\left\{6\,x,\ \frac{1}{2}\left(4 - 4\,x + 2\,x^{3} - x^{5}\right)\right\}$

In[8]:=
```
? FactorSquareFree
```

Out[8]=

Symbol

FactorSquareFree[*poly*] pulls out any multiple factors in a polynomial.

In[9]:=
```
FactorSquareFree[x^8 + 6 x^6 + 12 x^4 + 8 x^2]
```

Out[9]= $x^2 (2 + x^2)^3$

In[10]:=
```
? FactorSquareFreeList
```

Out[10]=

Symbol

FactorSquareFreeList[*poly*] gives a list of square-free factors of a polynomial, together with their exponents.

In[11]:=
```
FactorSquareFreeList[x^8 + 6 x^6 + 12 x^4 + 8 x^2]
```

Out[11]=

$\{\{1, 1\}, \{x, 2\}, \{2 + x^2, 3\}\}$

In[12]:=
```
FactorSquareFreeList[x^5 + 6 x^4 + 11 x^3 + 2 x^2 - 12 x - 8]
```

Out[12]=

$\{\{1, 1\}, \{2 + x, 3\}, \{-1 + x^2, 1\}\}$

In[13]:=
```
? Apart
```

Out[13]=

Symbol

Apart[*expr*] rewrites a rational expression as a sum of terms with minimal denominators.

Apart[*expr*, *var*] treats all variables other than *var* as constants.

In[14]:=
```
Apart[a / (u v^3)]
```

Out[14]=

$$-\frac{1}{x^2} + \frac{1}{x} + \frac{48}{(2 + x^2)^3} - \frac{2 (11 + 3 x)}{(2 + x^2)^2} + \frac{1}{2 + x^2}$$

In[15]:=
```
p = 4 x^8 - 3 x^7 + 25 x^6 - 11 x^5 + 18 x^4 - 9 x^3 + 8 x^2 - 3 x + 1;
q = 3 x^9 - 2 x^8 + 7 x^7 - 4 x^6 + 5 x^5 - 2 x^4 + x^3;
```

In[17]:=
```
? SortBy
```

Out[17]=

Symbol

SortBy[*list*, *f*] sorts the elements of *list* in the order defined by applying *f* to each of them.

SortBy[*list*, {f_1, f_2, …}] breaks ties by successively using the values obtained from the $f_{\text{Â}}$.

SortBy[*list*, *f*, *p*] sorts the elements of *list* using the

function *p* to compare the results of applying *f* to each element.

SortBy[*f*] represents an operator form of SortBy that can be applied to an expression.

In[18]:=
```
FactorSquareFreeList[q]
```

Out[18]=

$\{\{1, 1\}, \{x, 3\}, \{1 + x\ , 2\}, \{1 - 2x + 3x\ , 1\}\}$

In[19]:=
```
SortBy[FactorSquareFreeList[q], Last]
```

Out[19]=

$\{\{1, 1\}, \{1 - 2x + 3x\ , 1\}, \{1 + x\ , 2\}, \{x, 3\}\}$

In[20]:=
```
Apart[p / q]
```

Out[20]=

$$\frac{1}{x} - \frac{1}{x} + \frac{1}{x} + \frac{2 + 3x}{(1 + x\ )} - \frac{x}{1 + x} + \frac{4x}{1 - 2x + 3x}$$

Algorithmus Hermite Reduction:

Input: rational function f in the argument x

Output: {g,h} with f = D(g) + h, g,h rational, h with squarefree denominator

In[21]:=
```
HermiteReduction[f_, x_] := Module[{num, den, p, a, dlist, u, v, m, g, b, c},
    num = Numerator[f]; den = Denominator[f];
    p = PolynomialQuotient[num, den, x];
    a = PolynomialRemainder[num, den, x];
    dlist = SortBy[FactorSquareFreeList[den], Last];
    m = Last[dlist][[2]];
    v = Last[dlist][[1]];
    u = Factor[den / v ^ m];
    g = Integrate[p, x];
    dlist = Most[dlist];
    While[m ≥ 2,
     {b, c} = EEA[u D[v, x], v, a / (1 - m), x];
     g += b / v ^ (m - 1);
     a = (1 - m) c - u D[b, x];
     If[Last[dlist][[2]] == m - 1,
      u = Factor[ u / Last[dlist][[1]] ^ Last[dlist][[2]]];
      v *= Last[dlist][[1]];
      dlist = Most[dlist];
     ];
     m--;
    ];
    Return[{g, Factor[f - D[g, x]]}];
  ];
```

In[22]:=
```
{g, h} = HermiteReduction[a / (u v ^ 3), x]
```

Out[22]=

$$\left\{\frac{6\,x}{(2+x\ )}+\frac{2+3\,x}{x\,(2+x\ )},\ \frac{1}{x}\right\}$$

In[23]:=
```
Integrate[a / (u v ^ 3), x]
```

Out[23]=

$$\frac{1}{x}+\frac{6\,x}{(2+x\ )}+\frac{3-x}{2+x}+\mathrm{Log}[x]$$

In[24]:=
```
g // Apart
```

Out[24]=

$$\frac{1}{x}+\frac{6\,x}{(2+x\ )}+\frac{3-x}{2+x}$$

Example discussed in the lecture right after Algorithm Hermite Reduction

In[25]:=
```
HermiteReduction[2 (x - 1) (x + 2) / (2 x + 1) ^ 2, x]
```

Out[25]=

$$\left\{\frac{x}{2} + \frac{9}{4\left(1+2\,x\right)},\ 0\right\}$$

In[26]:=
```
Integrate[2 (x - 1) (x + 2) / (2 x + 1) ^ 2, x] // Apart
```

Out[26]=

$$\frac{1}{4} + \frac{x}{2} + \frac{9}{4\left(1+2\,x\right)}$$

In[27]:=
```
{g, h} = HermiteReduction[(x ^ 2 - 1) / (2 x + 1) ^ 2, x]
```

Out[27]=

$$\left\{\frac{x}{4} + \frac{3}{8\left(1+2\,x\right)},\ -\frac{1}{2\left(1+2\,x\right)}\right\}$$

In[28]:=
```
sol2 = Integrate[(x ^ 2 - 1) / (2 x + 1) ^ 2, x]
```

Out[28]=

$$\frac{3}{8\left(1+2\,x\right)} + \frac{1}{8}\left(1+2\,x\right) - \frac{1}{4}\,\mathrm{Log}\left[1+2\,x\right]$$

Rothstein – Trager

Example 2.4 .1 (Bronstein, Symbolic Integration - Transcendental Functions)
(algorithm executed step by step)

In[29]:=
```
p = x ^ 4 - 3 x ^ 2 + 6
q = x ^ 6 - 5 x ^ 4 + 5 x ^ 2 + 4
```

Out[29]=

$6 - 3\,x\ \ + x$

Out[30]=

$4 + 5\,x\ \ - 5\,x\ \ + x$

In[31]:=
```
FactorSquareFree[q]
```

Out[31]=

$4 + 5\,x\ \ - 5\,x\ \ + x$

In[32]:=
```
R = Resultant[q, p - z D[q, x], x] // Factor
```

Out[32]=

$45\,796\left(1 + 4\,z\ \right)$

In[33]:=
```
roots = Union[ z /. Solve[R == 0]]
```
Out[33]=

$$\left\{-\frac{i}{2}, \frac{i}{2}\right\}$$

In[34]:=
```
factors = Table[PolynomialGCD[q, (p - z D[q, x]) /. {z → roots[[ii]]}], {ii, Length[roots]}]
```
Out[34]=

$$\{2 i - 3 x - i x + x, -2 i - 3 x + i x + x\}$$

In[35]:=
```
Sum[roots[[ii]] Log[factors[[ii]]], {ii, Length[roots]}]
```
Out[35]=

$$-\frac{1}{2} i \operatorname{Log}[2 i - 3 x - i x + x] + \frac{1}{2} i \operatorname{Log}[-2 i - 3 x + i x + x]$$

In[36]:=
```
Integrate[p / q, x]
```
Out[36]=

$$-\frac{1}{2} \operatorname{ArcTan}\left[\frac{x(-3 + x)}{2 - x}\right] + \frac{1}{2} \operatorname{ArcTan}\left[\frac{x(-3 + x)}{-2 + x}\right]$$

Example : $h = \frac{1}{x^3 + x}$

In[37]:=
```
p = 1;
q = x^3 + x;
```

In[39]:=
```
FactorSquareFree[q]
```
Out[39]=

$$x(1 + x)$$

In[40]:=
```
R = Resultant[q, p - z D[q, x], x]
```
Out[40]=

$$(1 - z)(1 + 2 z)$$

In[41]:=
```
roots = Union[ z /. Solve[R == 0]]
```
Out[41]=

$$\left\{-\frac{1}{2}, 1\right\}$$

In[42]:=
```
factors = Table[PolynomialGCD[q, (p - z D[q, x]) /. {z → roots[[ii]]}], {ii, Length[roots]}]
```
Out[42]=

$$\left\{\frac{1}{2} + \frac{x}{2}, x\right\}$$

In[43]:=
```
Sum[roots[[ii]] Log[factors[[ii]]], {ii, Length[roots]}]
```
Out[43]=

$$\text{Log}[x]-\frac{1}{2}\ \text{Log}\left[\frac{1}{2}+\frac{x}{2}\right]$$

In[44]:=
```
Integrate[p / q, x]
```
Out[44]=

$$\text{Log}[x]-\frac{1}{2}\ \text{Log}\left[1+x\ \right]$$

Example : $h=\frac{1}{x^2-2}$

In[45]:=
```
p = 1;
q = x^2 - 2;
```

In[47]:=
```
R = Resultant[q, p - z D[q, x], x]
```
Out[47]=

$$1-8\,z$$

In[48]:=
```
roots = Union[ z /. Solve[R == 0]]
```
Out[48]=

$$\left\{-\frac{1}{2\ \sqrt{2}},\ \frac{1}{2\ \sqrt{2}}\right\}$$

In[49]:=
```
Table[(p - z D[q, x]) /. {z → roots[[ii]]}, {ii, Length[roots]}]
```
Out[49]=

$$\left\{1+\frac{x}{\sqrt{2}},\ 1-\frac{x}{\sqrt{2}}\right\}$$

In[50]:=
```
factors = Table[PolynomialGCD[q,
    (p - z D[q, x]) /. {z → roots[[ii]]}, Extension → Automatic], {ii, Length[roots]}]
```
Out[50]=

$$\left\{\sqrt{2}+x,\ \sqrt{2}-x\right\}$$

In[51]:=
```
Sum[roots[[ii]] Log[factors[[ii]]], {ii, Length[roots]}]
```
Out[51]=

$$\frac{\text{Log}\left[\sqrt{2}-x\right]}{2\ \sqrt{2}}-\frac{\text{Log}\left[\sqrt{2}+x\right]}{2\ \sqrt{2}}$$

In[52]:=
```
Integrate[p / q, x]
```
Out[52]=

$$\frac{\text{Log}\left[\sqrt{2}-x\right]-\text{Log}\left[\sqrt{2}+x\right]}{2\ \sqrt{2}}$$

Example 5.6.2 (Bronstein)

Dy = 1/x, i.e., y = Log(x)

In[53]:=
```
p = (2 y^2 - y - x^2);
q = (y^3 - x^2 y);
Dq = 3 y^2 / x - 2 x y - x;
```

In[56]:=
```
FactorSquareFree[q]
```

Out[56]=

$-y\,(x\;\;-y\;\;)$

In[57]:=
```
R = Resultant[q, p - z Dq, x] // Factor
```

Out[57]=

$-(-1+y)\;\;y\;\;(-1+2z)(1+2z)$

In[58]:=
```
roots = Union[z /. Solve[R == 0, z]]
```

Out[58]=

$\left\{-\frac{1}{2}, \frac{1}{2}\right\}$

In[59]:=
```
factors = Table[
  PolynomialGCD[q, (p - z Dq) /. {z → roots[[ii]]}, Extension → Automatic], {ii, Length[roots]}]
```

Out[59]=

$\left\{\frac{1}{2} - \frac{y}{2x}, \frac{1}{2} + \frac{y}{2x}\right\}$

In[60]:=
```
Dfactors = {-(1 - y) / (2 x^2), (1 - y) / (2 x^2)}
```

Out[60]=

$\left\{\frac{-1+y}{2x}, \frac{1-y}{2x}\right\}$

In[62]:=
```
roots[[1]] × Dfactors[[1]] / factors[[1]] // Factor
```

Out[62]=

$-\frac{-1+y}{2x(x-y)}$

In[63]:=
```
u = p / q - Sum[roots[[ii]] × Dfactors[[ii]] / factors[[ii]], {ii, Length[roots]}] // Factor
```

Out[63]=

$\frac{1}{y}$

In[64]:=
```
Sum[roots[[ii]] Log[factors[[ii]]], {ii, Length[roots]}]
```

Out[64]=

$$-\frac{1}{2}\ \mathrm{Log}\left[\frac{1}{2}-\frac{y}{2\ x}\right]+\frac{1}{2}\ \mathrm{Log}\left[\frac{1}{2}+\frac{y}{2\ x}\right]$$

In[65]:=
```
Integrate[(2 Log[x]^2 - Log[x] - x^2) / (Log[x]^3 - x^2 Log[x]), x]
```

Out[65]=

$$-\frac{1}{2}\ \mathrm{Log}[x-\mathrm{Log}[x]]+\frac{1}{2}\ \mathrm{Log}[x+\mathrm{Log}[x]]+\mathrm{LogIntegral}[x]$$