# Computer Algebra for Concrete Mathematics

## Summer semester 2023

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In this lecture basic skills and techniques will be elaborated which are relevant to simplify formulas related to enumeration. Special emphasis is put on tools that support the student for the analysis of algorithms (best case, worst case and average case). In particular, the participant gets acquainted to apply these computer algebra tools to non-trivial examples.

The content of the lecture can be summarized by the following key words:

- algorithmic treatment of formal power series;

- c-finite and holonomic functions/sequences;

- recurrence solving;

- basic aspects of asymptotics;

- symbolic summation.

A major emphasis of the lecture is to present the basic notions, to develop the basic ideas of the underlying algorithms and to put computer algebra into action for concrete examples.

In addition, many of the topics discussed in the lecture can be found in the books

- *Concrete Mathematics - A Foundation for Computer Science* by R.L.Graham, D.E.Knuth and O.Patashnik (Addison-Wesley, 1994),

- *The Concrete Tetrahedron* by Manuel Kauers and Peter Paule (Springer Wien, 2011).

# Contents

# Lecture from March 07, 2023

## 1  Introduction

In this lecture we will deal in strong interaction with generating functions, recurrences, asymptotics and summation. This interplay can be visualized also as follows:
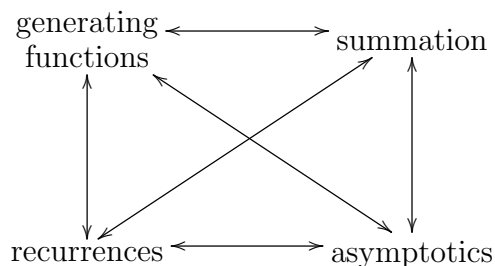


In the introduction we will illustrate the interaction of recurrences, summation and asymptotics. The machinery of generating functions (formal power series) will be introduced in Section 3 below.

### 1.1  A case study for SelectionSort and summation

We start with an example of the well-known sorting algorithm SelectionSort.

**Example 1.1.** We apply SelectionSort to $(5, 3, 8, 2)$. This means that we find a smallest element from the list/array and move it to the beginning. Afterwards we proceed with the list/array where the first entry is ignored. In this way we get the following operation steps:

$$5 \quad 8 \quad 3 \quad \boxed{2}$$
$$\downarrow$$
$$2 \mid 8 \quad \boxed{3} \quad 5$$
$$\downarrow$$
$$2 \quad 3 \mid 8 \quad \boxed{5}$$
$$\downarrow$$
$$2 \quad 3 \quad 5 \mid 8$$
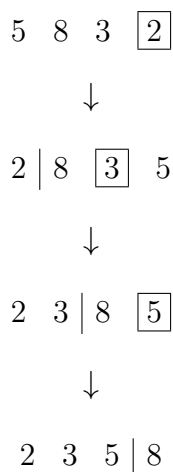
Remark: if one deals with an array, one can move the smallest element to the right place by swapping elements accordingly.

In the following (for SelectionSort and later for QuickSort) we will estimate the time complexity by counting the number of comparisons. In this regard, the following consideration is immediate: In order to find the smallest element 2 in $(5, 8, 3, 2)$ one needs 3 comparisons; to find the

smallest element 3 in $(8, 3, 5)$, one needs 2 comparisons; and to find the smallest element 5 in $(8, 5)$, one needs 1 comparison. In total

$$3 + 2 + 1 = 6$$

comparisons are needed. In general, if

$$F(n) = \text{number of pairwise comparisons to selection-sort } n \text{ elements } (n \geq 0)$$

we get

$$
\begin{aligned}
F(n) \quad &= \quad (n-1) + (n-2) + \cdots + 2 + 1 \\
&= \quad \sum_{k=1}^{n} (n - k) \\
&\overset{k \to n-k}{=} \quad \sum_{k=1}^{n-1} k \\
&\overset{\text{why?}}{=} \quad \frac{n(n-1)}{2}.
\end{aligned}
$$

**Notation 1.2.** *We will use the following conventions:*

$$\mathbb{N} := \{0, 1, 2, 3, \dots\}$$
$$\mathbb{N}^* := \{1, 2, 3, \dots\}$$
$$\sum_{k=a}^{b} f(k) := 0 \ \ if \ a > b$$
$$\prod_{k=a}^{b} f(k) := 1 \ \ if \ a > b.$$

One way to simplify the Gauss sum $\sum_{k=1}^{n} k$ (or to show that it equals to $\frac{n(n-1)}{2}$) is to sum it twice (in the usual and in the reversed order):

$$
\begin{array}{ccccccccccc}
 & 1 & + & 2 & + & 3 & + & \dots & + & n & \\
+ & n & + & (n-1) & + & (n-2) & + & \dots & + & 1 & \\
\hline
= & (n+1) & + & (n+1) & + & (n+1) & + & \dots & + & (n+1) & = n(n+1).
\end{array}
$$

This gives

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Alternatively, one can consider the picture (the black balls represent the sum)

$$1 \quad +\, 2 \quad +\, 3 \quad +\, 4 \quad +\, 5 \quad \ldots \qquad +\, n \qquad\qquad = n(n+1)$$

$$\left.\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \ldots & \bullet \\
\circledast & \bullet & \bullet & \bullet & \bullet & \ldots & \bullet \\
\circledast & \circledast & \bullet & \bullet & \bullet & \ldots & \bullet \\
\circledast & \circledast & \circledast & \bullet & \bullet & \ldots & \vdots \\
\circledast & \circledast & \circledast & \circledast & \bullet & \ldots & \bullet \\
\circledast & \circledast & \circledast & \circledast & \circledast & \ldots & \bullet \\
\circledast & \circledast & \circledast & \circledast & \circledast & \ldots & \circledast
\end{array}\right\} n+1$$

$$\underbrace{\phantom{xxxxxxxxxxxxxxxxxxxx}}_{n}$$

in order to extract the identity.

**HW 1.** Try to apply the Gauß-method to sum

(a) $\sum_{k=0}^{n}(2k+1)$

(b) $\sum_{k=1}^{n} k^2$

(c) $\sum_{k=1}^{n} k^3$

Find and prove a formula for (a), (b) and (c).

Note that the above proof can be also reflected with the following sum manipulations:

$$\sum_{k=1}^{n-1} k \stackrel{k \to n-k}{=} \sum_{k=1}^{n-1}(n-k) \;=\; \sum_{k=1}^{n-1} n - \sum_{k=1}^{n-1} k$$

$$= \; n \underbrace{\sum_{k=1}^{n-1} 1}_{n-1} - \sum_{k=1}^{n-1} k$$

$$\Downarrow$$

$$2\sum_{k=1}^{n-1} = n(n-1).$$

Throughout this lecture the harmonic numbers will play a central role.

**Definition 1.3.** *For $n \in \mathbb{N}$, we define*

$$H_n = \sum_{k=1}^{n} \frac{1}{k};$$

*note that with our convention from above we have $H_0 = 0$.*

**HW 2.** Prove for all $n \in \mathbb{N}$ that

$$\sum_{k=0}^{n-1} \frac{k}{(k+1)(k+2)} = H_n - \frac{2n}{n+1}.$$

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**HW 3.** Let $f : \mathbb{Z} \to \mathbb{C}$ and $a, b \in \mathbb{Z}$ with $a \leq b$.

1. For

$$S(a, b) := \sum_{k=a}^{b} (f(k+1) - f(k))$$

   show that

$$S(a, b) = f(b+1) - f(a).$$

2. Suppose in addition that $f(k) \neq 0$ for all $k$ with $a \leq k \leq b$. For

$$P(a, b) := \prod_{k=a}^{b} \frac{f(k+1)}{f(k)}$$

   show that

$$P(a, b) = \frac{f(b+1)}{f(a)}.$$

**HW 4.** Use the previous homework to find a closed form for

$$a_n := \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right).$$

**BP 1.** Consider the function $\exp : \mathbb{R} \to \mathbb{R}$ defined by

$$x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Prove: there is no rational function $r(x) \in \mathbb{R}(x)$ (i.e., $r(x) = \frac{p(x)}{q(x)}$ for polynomials $p, q \in \mathbb{R}[x]$) such that

$$\exp(x) = r(x) \quad \forall x \in U$$

where $U \subseteq \mathbb{R}$ is some non-empty open interval.

**HW 5.** Given a tower of $n$ discs, initially stacked in decreasing size on one of three pegs. Transfer the entire tower to one of the other pegs, moving only one disc at each step and never moving a larger one onto a smaller one. Find $a_n$, the minimal number of moves ($n \geq 0$).
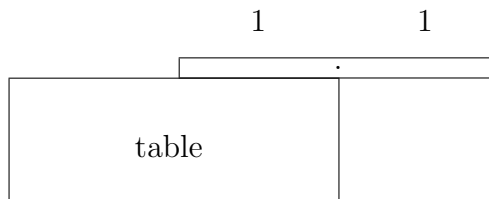
**HW 6.** How many slices of pizza can a person maximally obtain by making $n$ straight cuts with a pizza knife. Let $P_n$ ($n \geq 0$) be that number.

**BP 2.** Prove that there is no rational function $r(x) \in \mathbb{C}(x)$ such that
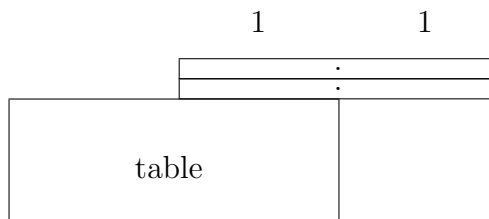
$$H_n = r(n)$$

for all $n \in \mathbb{N}$ with $n \geq \lambda$ for some $\lambda \in \mathbb{N}$.

**Example 1.4.** What is the maximal possible overhang of $n$ cards (beer coaster)?

Let us suppose that a card has length 2. Then with 1 card we get the overhang 1:



If we are given two cards, we start with



and observe that we cannot move the top card further. Thus we move the card below further and further away, and it is not difficult to see that the balance point is reached with the overhang $1 + \frac{1}{2} = \frac{3}{2}$. Thus we get the following picture:
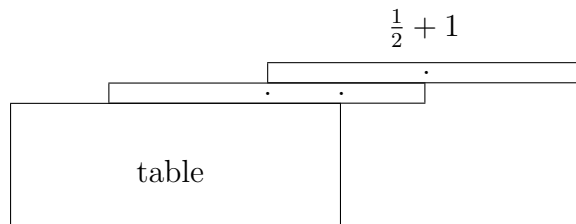


If we are given three cards, the balance point will be at position $1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

With 4 cards the overhang will be $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$, which is more than one card length. With 52 cards, we get the overhang $H_{52} = 2.27 \cdot 2$, i.e., it is twice the card length. In general, the maximal overhang of $n$ cards is $H_n$.

*What happens if we stack more and more cards. Can we make the overhang arbitrarily large?*

Questions like this lead immediately to asymptotic considerations.

## 1.2 Some basic notions for asymptotics

We start with the big-O notation.

**Definition 1.5.** *For[1] $g : \mathbb{N} \to \mathbb{R}$ we define*

$$O(g) = \{f : \mathbb{N} \to \mathbb{R} \mid \exists c_f \in \mathbb{R},\ n_0 \in \mathbb{N}\ \forall n \geq n_0 :\ |f(n)| \leq c_f\,|g(n)|\}.$$

**Example 1.6.** We have

$$2n \in O(n), \qquad 2n - 5 \in O(n), \qquad 2^{2^{2^{2^{2^{2^2}}}}} \cdot n \in O(n), \qquad \frac{1}{2^{2^{2^{2^{2^{2^2}}}}}} \cdot n \in O(n).$$

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[1]Note that the function $g$ can be also interpreted as a the sequence $(g(n))_{n \geq 0}$ with entries from $\mathbb{R}$.

The following theorem states that the harmonic numbers and the log-function grow similarly fast.

**Theorem 1.7.** *We have*
$$H_n - \log(n) - \gamma \in O(\tfrac{1}{n});$$
*here* $\log(n) := \ln(n)$ *denotes the natural logarithm (to the basis e) and* $\gamma = 0,5772156\cdots \in \mathbb{R}$ *is Euler's constant[2]*

*Proof.* For a proof see the book Concrete Mathematics by Graham/Knuth/Patashnik. □

**Example 1.8.** Going back to Example 1.4 we conclude with

$$\lim_{n\to\infty} \log(n) = \infty$$

that the overhang can be (theoretically) arbitrary long if sufficiently many cards are available. However, the overhang grows dramatically low. E.g., suppose that the card length is 10cm (i.e., one unit corresponds to 5cm). Then we can produce an overhang of more than 100m with $n$ cards, if
$$2000 \le H_n \le \log(n) + 1$$
holds. Thus $n \ge e^{1999}$, i.e., we must take around $1.429 \cdot 10^{868}$ cards; however, one estimates that there are only around $10^{80}$ atoms in the universe...

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[2]So far it is not known, if $\gamma \in \mathbb{Q}$ or if $\gamma$ is irrational.

# Lecture from March 14, 2023

In particular, we can conclude with this theorem that there is a constant $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ (actually one can choose $n_0 = 1$) such that

$$|\underbrace{H_n - \log(n) - \gamma}_{:=a_n \in \mathbb{R}}| \leq \frac{c}{n}$$

holds for all $n \geq n_0 = 1$. This implies that $(a_n)_{n \geq 0}$ converges (in the analysis sense) to 0, i.e.,

$$\lim_{n \to \infty} a_n = 0$$

$$\Updownarrow$$

$$\lim_{n \to \infty} (H_n - \log(n) - \gamma) = 0$$

$$\Updownarrow$$

$$\lim_{n \to \infty} (H_n - \log(n)) = \gamma.$$

**Definition 1.9.** *For $a, b : \mathbb{N} \to \mathbb{R}$ we define*

$$a(n) \sim b(n) \quad \Leftrightarrow \quad \lim_{n \to \infty} \frac{a(n)}{b(n)} = 1.$$

*In this case, we also say that $a(n)$ and $b(n)$ are asymptotically equal.*

We remark that $\sim$ is an equivalence relations.

**HW 7.** Show that $H_n \sim \log(n)$.

Another important fact is Stirling's formula that can be stated as follows; the proof can be found again in Concrete Mathematics.

**Theorem 1.10.**
$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Note: Since $n^n = (e^{\log(n)})^n = e^{n \log(n)}$, we get

$$n! \sim e^{n(\log(n)-1)} \sqrt{2\pi n}.$$

## 1.3   A case study for QuickSort and recurrences

We turn to QuickSort and sort the array

$$44 \quad 13 \quad 9 \quad \boxed{29} \quad 71 \quad 67 \quad 5 \quad 8 \quad 15$$

with $n = 9$ elements. In each step we split the array by choosing a pivot element at random. Here we take, e.g., 29. We swap it to the right side and focus on the remaining array:

$$44 \quad 13 \quad 9 \quad 15 \quad 71 \quad 67 \quad 5 \quad 8 \,\big|\, 29.$$

Next, we split the array: all elements which are smaller than 29 are moved to the left-hand side, and all elements which are larger than 29 are moved to the right-hand side. Here the trick is to start with the corner entries.

$$\boxed{44} \quad 13 \quad 9 \quad 15 \quad 71 \quad 67 \quad 5 \quad \boxed{8} \,\big|\, 29$$

In this case, both entries are on the wrong side. Thus we swap them and get

$$8 \,\big|\, 13 \quad 9 \quad 15 \quad 71 \quad 67 \quad 5 \,\big|\, 44 \quad 29$$

Now we repeat this procedure for the remaining entries: we zoom in until we find again two bad entries. In this case,

$$8 \quad 13 \quad 9 \quad 15 \,\big|\,\boxed{71} \quad 67 \quad \boxed{5} \,\big|\, 44 \quad 29$$

and swap it:

$$8 \quad 13 \quad 9 \quad 15 \quad 5 \,\big|\, 67 \,\big|\, 71 \quad 44 \quad 29.$$

Eventually, the two lines meet and we are done: we only have to move the pivot element, which is on the right most place, to the correct position, namely at the position where the two lines meet:

$$8 \quad 13 \quad 9 \quad 15 \quad 5 \,\big|\, 29 \,\big|\, 71 \quad 44 \quad 67$$

Summarizing, we obtained our split: all elements which are smaller than 29 are left of it and all elements which are large are right to it. In total, we needed $n - 1 = 8$ comparisons to obtain this split. Note that the element 29 is already at the correct position of the to be sorted array. Thus we have to repeat this tactic for the remaining sub-arrays. E.g., we obtain the following splits (applying the method recursively and choosing particular pivot elements):

$$44 \quad 13 \quad 9 \quad \boxed{29} \quad 71 \quad 67 \quad 5 \quad 8 \quad 15$$

$$\downarrow 8 \text{ comparisons}$$

$$8 \quad 13 \quad 9 \quad 15 \quad \boxed{5} \,\big|\, 29 \,\big|\, 71 \quad 44 \quad \boxed{67}$$

$$\downarrow 4+2=6 \text{ comparisons}$$

$$5 \,\big|\, 13 \quad \boxed{9} \quad 15 \quad 8 \,\big|\, 29 \,\big|\, 44 \,\big|\, 67 \,\big|\, 71$$

$$\downarrow 3 \text{ comparisons}$$

$$5 \,\big|\, 8 \,\big|\, 9 \,\big|\, \boxed{15} \quad 13 \,\big|\, 29 \,\big|\, 44 \,\big|\, 67 \,\big|\, 71$$

$$\downarrow 1 \text{ comparison}$$

$$5 \,\big|\, 8 \,\big|\, 9 \,\big|\, 13 \,\big|\, 15 \,\big|\, 29 \,\big|\, 44 \,\big|\, 67 \,\big|\, 71$$

Summarizing, we needed in total

$$8 + 6 + 3 + 1 = 18$$

comparisons to QuickSort the above array (with the particularly chosen pivot elements marked with a box).

We recall that SelectionSort needs $\frac{n(n-1)}{2}\big|_{n=9} = 9 \cdot 4 = 36$ comparisons. So there is an improvement (as the name QuickSort suggests).

Note further that QuickSort strongly depends on the choice of the pivot elements. In particular, one can construct, e.g., a worst case scenario as follows: Take an already sorted array and apply QuickSort by choosing always the right most element. Then we obtain the worst split: the pivot element is right most and all other elements are on the left-hand side. Thus QuickSort is applied to $n - 1$ elements in the next step. Applying this argument iteratively, we need in this worst case situation

$$(n - 1) + (n - 2) + (n - 3) + \cdots + 1 = \frac{n(n - 1)}{2}$$

comparisons. In other words: QuickSort behaves in the worst case like the slow algorithm SelectionSort.

In order to see (and prove) the improvement of QuickSort in contrast to, e.g., SelectionSort, we have to explore the average case. Here we assume that the pivot elements are chosen arbitrarily and that after the split it is equally likely that the pivot element is at position $1, 2, \ldots, n - 1$ or $n$. Under this assumption, we are interested in the following counting:

$F(n) =$ the avarage number of pairwise comparisons to quicksort $n$ elements $(n \geq 0)$.

The base case of QuickSort is the empty array or an array of length 1. Here the array is already sorted and no comparisons are necessary. Thus we have

$$F(0) = F(1) = 0.$$

For an array of length $n = 2$ QuickSort will perform exactly one comparison, i.e., we can set

$$F(2) = 1.$$

More generally, if we are given an array of length $n$, we choose a pivot element (at random) and calculate in the divide step (of our Divide and Conquer strategy) a split. For this task we need $n - 1$ comparisons (independently of the choice of the pivot element). By our assumption it is equally likely that the pivot element will be at position 1, at position 2,..., or at position $n$. Thus we get

$$F(n) = \underbrace{n - 1}_{\substack{\text{comparisons} \\ \text{for the split}}} + \frac{1}{n}\sum_{k=1}^{n} F_k(n);$$

here $F_k(n)$ denotes the average number of comparisons that are needed to sort the derived array where $k - 1$ elements are left to the pivot element and $n - k$ are right to the pivot element. More precisely, $F_k(n)$ is determined by the property that QuickSort is applied to each of the two subarrays: namely, the average number of comparisons are $F(k - 1)$ for the left array and the average number of comparisons are $F(n - k)$ for the right array. Thus we have

$$F_k(n) = F(k - 1) + F(n - k)$$

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and in total we get

$$F(n) = n - 1 + \frac{1}{n}\sum_{k=1}^{n}(F(k-1) + F(n-k)), \quad n \geq 1$$

Note that this formula (together with $F(0) = 0$) allows us to compute all values of $F(n)$. E.g., we compute

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $\dots$ |
|------|---|---|---|----------------|------------------|-----------------|-------------------|-------------------|--------------------|--------------------|---------------------|---------------------|----------------------|----------------------|----------------------|---------|
| $F(n)$ | 0 | 0 | 1 | $\frac{8}{3}$ | $\frac{29}{6}$ | $\frac{37}{5}$ | $\frac{103}{10}$ | $\frac{472}{35}$ | $\frac{2369}{140}$ | $\frac{2593}{126}$ | $\frac{30791}{1260}$ | $\frac{32891}{1155}$ | $\frac{452993}{13860}$ | $\frac{476753}{12870}$ | $\frac{499061}{12012}$ | $\dots$ |

$$(1)$$

Unfortunately, this formula (or the produced numbers) do not provide any information how good QuickSort really is.

In a preprocessing step we simplify the found recurrence further. A first observation is that the summand consists of two parts which are equal:

$$\sum_{k=1}^{n} F(k-1) \overset{k \to n-k}{=} \sum_{k=1}^{n} F(n-k).$$

Thus we get

$$\begin{aligned}
F(n) &= n - 1 + \frac{1}{n}\sum_{k=1}^{n}(F(k-1) + F(n-k)) \\
&= n - 1 + \frac{1}{n}\sum_{k=1}^{n}(F(k-1) + \sum_{k=1}^{n}F(n-k)) \\
&= n - 1 + \frac{2}{n}\sum_{k=1}^{n}F(k-1) \\
&= n - 1 + \frac{2}{n}\sum_{k=0}^{n-1}F(k).
\end{aligned}$$

Second, we eliminate the indefinite sum with upper bound $n$ (where the summand itself is free of $n$). This can be accomplished easily if the factor in front of the sum is also free of $n$. To accomplish this latter requirement, we first multiply our recurrence with $n$ and get

$$n\,F(n) = n(n-1) + \boxed{2}\sum_{k=0}^{n-1}F(k) \quad n \geq 1.$$

Given this special form, we shift the recurrence in $n$, i.e., replace $n$ by $n+1$ yielding (where the constant in front of the sum does not change!)

$$(n+1)\,F(n+1) = (n+1)n + \boxed{2}\sum_{k=0}^{n}F(k) \quad n \geq 0$$

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Finally, we subtract both and obtain

$$(n+1)\,F(n+1) - nF(n) = 2n + \boxed{2}(\underbrace{\sum_{k=0}^{n} F(k) - \sum_{k=0}^{n-1} F(k)}_{=F(n)}) \quad n \geq 1.$$

Summarizing, we obtained

$$(n+1)F(n+1) - (n+2)F(n) = 2n, \quad n \geq 1$$

with the initial value $F(1) = 0$. One can easily check that the found recurrence is also valid for $n = 0$ with $F(0) = F(1) = 0$. Thus we obtain

$$\begin{aligned} (n+1)F(n+1) - (n+2)F(n) &= 2n, \quad n \geq 0 \\ F(0) &= 0. \end{aligned} \tag{2}$$

By construction this simplified recurrence produces again the sequence $(F(n))_{n \geq 0}$ where the first values are printed in (1). More precisely, we can use the formula

$$F(n) \leftarrow \begin{cases} \frac{n+1}{n}F(n-1) + 2\frac{n-1}{n} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0. \end{cases}$$

In the following we will explore its behavior further by solving the found recurrence. For first-order linear recurrences (here $F(n)$ and $F(n+1)$ arise linearly) we can exploit the following method, also called *"variation of constants"*.

Step 1: Find a closed form for the homogeneous equation:

$$(n+1)H(n+1) - (n+2)H(n) = 0.$$

Unrolling it and proper cancellations give

$$\begin{aligned} H(n+1) &= \frac{n+2}{n+1}H(n) = \frac{n+2}{n+1} \cdot \frac{n+1}{n}H(n-1) = \dots \\ &= \frac{n+2}{\cancel{n+1}} \cdot \frac{\cancel{n+1}}{\cancel{n}} \cdot \frac{\cancel{n}}{\cancel{n-1}} \dots \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{1}H(0) = (n+2)H(0), \end{aligned}$$

and thus

$$H(n) = (n+1)H(0).$$

Using the homogeneous solution $(n+1)$ (we ignore the constant $H(0)$), we make the following ansatz in

Step 2: Find a sequence $G(n)$ such that

$$F(n) = (n+1)G(n)$$

holds for all $n \geq 0$.

As a consequence it follows that

$$0 = F(0) = G(0)$$

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and

$$\underbrace{(n+1)}_{}\overbrace{(n+2)G(n+1)}^{=F(n+1)}-(n+2)\overbrace{(n+1)G(n)}^{=F(n)} = 2n, \quad n \geq 0$$

$$\Updownarrow$$

$$G(n+1) - G(n) = \frac{2n}{(n+1)(n+2)}, \quad n \geq 0.$$

Using telescoping (see HW 3) it follows that

$$G(n) - \underbrace{G(0)}_{=0} = \sum_{k=0}^{n-1} \frac{2k}{(k+1)(k+2)}, \quad n \geq 0$$

and with summation (HW 2) we get

$$G(n) = 2H_n - \frac{4n}{n+1}, \quad n \geq 0. \tag{3}$$

Thus

$$F(n) = (n+1)G(n) = 2(n+1)H_n - 4n, \quad n \geq 0. \tag{4}$$

Finally, we show that the average number $F(n)$ to quicksort $n$ elements is asymptotically equal to $2n\log(n)$; as a consequence it also follows that $F(n) \in O(n\log(n))$.

**Theorem 1.11.**

$$F(n) \sim 2n\log(n).$$

*Proof.* We have

$$\begin{aligned} F(n) &= 2(n+1)H_n - 4n \\ &\sim 2(n+1)H_n \\ &\sim 2nH_n \\ &\sim 2n\log(n). \end{aligned}$$

The last equivalence follows by $H_n \sim \log(n)$; see HW 7. $\qquad\square$

**Example 1.12.** For an array of length $n = 100$ SelectionSort needs $\frac{n(n-1)}{2} = 4950$ comparisons and QuickSort (in average) $2n\log(n) = 921.03$ comparisons; this looks not like a big deal. However, if we take, e.g., $n = 1000000$, then SelectionSort needs $\sim 5 * 10^{11}$ comparisons, and QuickSort $\sim 2.7631 * 10^7$ comparison; so there is the speedup factor 18095.
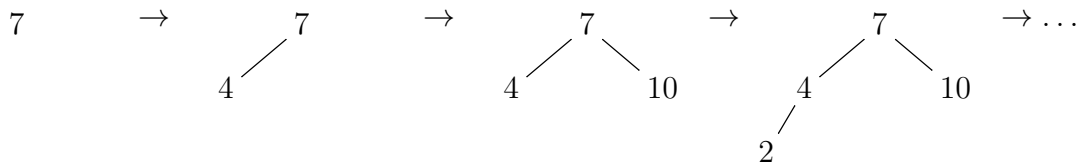
# Lecture from March 21, 2023

## 1.4  A case study for Binary Search Trees (BST)

**Definition 1.13.** *A binary search tree is a binary tree where in each subtree with root $x$ all the elements in its left subtree are smaller and all elements in its right subtree are larger than $x$.*

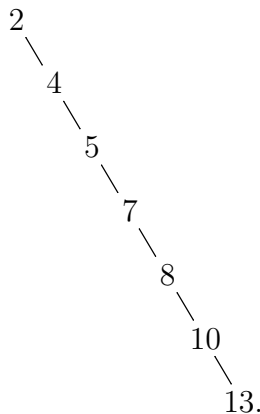**Example 1.14.** The binary search tree

$$\tag{5}$$

```
          7
        /   \
       4     10
      / \    /  \
     2   5  8   13
```

represents the set $\{2, 4, 5, 7, 8, 10, 13\}$. It can be generated by inserting step-wise the elements, e.g., in the order $7 \to 4 \to 10 \to 2 \to 5 \to 8 \to 13$:

```
                 7                7                 7
  7     →       /        →       / \       →       / \       → ...
              4               4    10           4    10
                                                /
                                               2
```

In the tree (5) the paths $P_1 = (7)$, $P_2 = (7, 10)$, or $P_3 = (7, 4, 5)$ go from the root 7 to the root 7, to the inner node 10 or to the leaf 5 (via the inner node 4), respectively. $P_1$ has path length 1, $P_2$ has path length 2 and $P_3$ has path length 3.

An alternative tree can be generated by inserting step-wise the elements in the order $2 \to 4 \to 5 \to 7 \to 8 \to 10 \to 13$:

```
2
 \
  4
   \
    5
     \
      7
       \
        8
         \
         10
           \
           13.
```

In worst case a binary search tree turns to a linked list (see the second tree in the example). Thus finding an element in such a worst case scenario requires $n$ operations.

However, the average time complexity is much better in a randomly chosen tree. More precisely, take the set $S = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ and suppose that we are given a binary search trees in which the elements $S$ are stored at random. This means that they are generated by a certain order $b_1 \to b_2 \to \cdots \to b_n$ (with $\{a_1, a_2, \ldots, a_n\} = \{b_1, b_2, \ldots, b_n\}$) where each insertion order $b_1 \to b_2 \to \cdots \to b_n$ is equally likely. E.g., the element $b_1$ equals the element $a_1$ with probability $1/n$, it equals $a_2$ with probability $1/n$, etc.

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Now define

$$P(n) = \text{the avarage path length of such a binary search tree with } n \text{ nodes.}$$

As illustrated in the above example a path starts at the root and walks down a tree and stops at an inner node or a leaf; in particular, the path length is the number of nodes that are visited during the walk (including the root). Then the average cost to find an element[3] $c \in S$ (randomly chosen) in a tree (randomly generated) equals precisely $P(n)$: it is simply the average path length (the path ends at $c$).

Similarly to the average number of operations $F(n)$ in QuickSort, we will explore the average number $P(n)$ to find an element in a binary search tree. Obviously we have

$$P(1) = 1.$$

Next, consider the following special case: the first element that we insert in our random construction is $a_{i+1}$ where $0 \le i < n$. After inserting randomly the remaining $n-1$ elements we obtain a tree with the following shape:

$$b_1 = a_{i+1} \qquad\qquad (6)$$



where the elements $S_1 = \{a_1, \dots, a_i\}$ are stored in $T_1$ and $S_2 = \{a_{i+1}, \dots, a_n\}$ are stored in $T_2$. Now consider the average path length $\delta_{a_{i+1}}$ from the root $a_{i+1}$ to any node $c$ randomly chosen from $S$. Note that $c = b_1 = a_{i+1}$ has probability $\frac{1}{n}$, $c \in S_1$ has probability $\frac{i}{n}$, and $c \in S_2$ has probability $\frac{n-i-1}{n}$. Thus we obtain three cases:

1. With probability $\frac{1}{n}$ the path length is 1 (we go from $c = a_{i+1}$ to $c$).

2. With probability $\frac{i}{n}$ we obtain a path of at least length two. It start at $a_{i+1}$, goes to another element in $S_1$ and from this node the average path length is $P(i)$ to end up at $c$. Thus the average path length equals $1 + P(i)$.

3. Analogously to case 2, we obtain with probability $\frac{n-i-1}{n}$ the average path length $1 + P(n - i - 1)$.

In total it follows that

$$\begin{aligned}
\delta_{a_{i+1}} &= \tfrac{1}{n} + \tfrac{i}{n}(P(i) + 1) + \tfrac{n-i-1}{n}(P(n-i-1) + 1) \\
&= 1 + \tfrac{i}{n}P(i) + \tfrac{n-i-1}{n}P(n-i-1).
\end{aligned}$$

In order to get a complete formula (recursion for $P(n)$) we note that each special case (6) with $b_1 = a_{i+1}$ for $0 \le i < n$ is equally likely, i.e., arises with probability $\frac{1}{n}$. Thus we get

$$P(n) = \sum_{i=0}^{n-1} \frac{1}{n} \delta_{a_{i+1}}.$$

---

[3]It seems nonsense to search for an element in the tree if one assumes that it is in. However, $c$ might be a data base key and one is interested in extracting the data that is attached to $c$. Thus one has to find the element $c$ in the tree which is stored together with a pointer that refers to the desired data.

**HW 8.** Show that

$$P(n) = 1 + \frac{2}{n^2} \sum_{i=0}^{n-1} i\, P(i).$$

**HW 9.** Show that

$$n^2\, P(n) - (n-1)(n+1)P(n-1) = 2n - 1, \quad n \geq 2.$$

**HW 10.** Solve the recurrence in closed form (i.e., in terms of the harmonic numbers). More precisely, perform the following steps:

1. Compute a solution $H(n) \in \mathbb{Q}(n)$ of the homogeneous version

$$n^2\, H(n) - (n-1)(n+1)H(n-1) = 0.$$

2. Make the ansatz $P(n) = H(n)G(n)$ which leads to

$$G(n+1) - G(n) = r(n) \qquad r \geq l$$

for some $r(n) \in \mathbb{Q}(n)$ and $l \in \mathbb{N}$. By the telescoping trick this gives

$$G(n) - G(l) = \sum_{k=l}^{n-1} r(k)$$

with some explicitly given $G(l) \in \mathbb{Q}$ and thus

$$F(n) = H(n) \left( \sum_{k=l}^{n-1} r(k) + G(l) \right).$$

3. Simplify the sum further in term of the harmonic numbers.

**Theorem 1.15.**

$$P(n) \in O(\log(n)),$$
$$P(n) \sim 2\log(n).$$

*Proof.* **HW 11**. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Summarizing, given an arbitrary binary search tree with $n$ elements, one finds an element (that occurs in the tree) in $O(\log(n))$ operations. A similar result can be derived for the case that the element is not stored and one wants to verify this fact.

# 2* Recall: basic notions from algebra

In the following we repeat some basic notions and constructions from algebra that will arise in the lecture. In particular, they are useful to tackle the bonus problems stated below.

**Recall 2.1.** Let $\mathbb{A}$ be a set with two operations $+ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$. $(\mathbb{A}, +, \cdot)$ is called a *field* if the following properties hold:

1. $\forall a, b, c \in \mathbb{A} : (a + b) + c = a + (b + c)$;

2. $\forall a, b \in \mathbb{A} : a + b = b + a$;

3. $\exists 0 \in \mathbb{A} \, \forall a \in \mathbb{A} : 0 + a = a$;

4. $\forall a \in \mathbb{A} \, \exists b \in \mathbb{A} : a + b = 0$;

5. $\forall a, b, c \in \mathbb{A} : (a \cdot b) \cdot c = a \cdot (b \cdot c)$;

6. $\forall a, b \in \mathbb{A} : a \cdot b = b \cdot a$;

7. $\exists 1 \in \mathbb{A} \setminus \{0\} \, \forall a \in \mathbb{A} : 1 \, a = a$;

8. $\forall a \in \mathbb{A} \setminus \{0\} \, \exists b \in \mathbb{A} \setminus \{0\} : a \, b = 1$;

9. $\forall a, b, c \in \mathbb{A} : a \cdot (b + c) = a \cdot b + a \cdot c$.

If the operations are clear from the context, one simply writes $\mathbb{A}$ for the field. Often one neglects $\cdot$ and simply writes $a \, b$ instead of $a \cdot b$.

Note: Properties (1)–(4) imply that $(\mathbb{A}, +)$ is a commutative (abelian) group, and properties (5)–(8) imply that also $(\mathbb{A} \setminus \{0\}, \cdot)$ is a commutative (abelian) group. Finally, property (9) states that the operations $+$ and $\cdot$ interact distributively.

**Example 2.2.** The rational numbers $(\mathbb{Q}, +, \cdot)$, the real numbers $(\mathbb{R}, +, \cdot)$ or the complex numbers the real numbers $(\mathbb{C}, +, \cdot)$ are fields with the usual operations. Furthermore, the set of rational functions $(\mathbb{Q}(n), +, \cdot)$, $(\mathbb{R}(n), +, \cdot)$ or $(\mathbb{C}(n), +, \cdot)$ (see BPs 1 and 2) form fields with the usual operations.

Often one is given algebraic structures which are equipped with operations that do not satisfy all the properties required for a field. In this lecture we deal mostly with commutative rings with 1.

**Recall 2.3.** Let $\mathbb{A}$ be a set with two operations $+ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$. $(\mathbb{A}, +, \cdot)$ is called a *commutative ring with 1* if the properties (1)–(7) and (9) from Recall 2.1 hold. If the operations are clear from the context, one simply writes $\mathbb{A}$ for the ring.

Note: a commutative ring with 1 is a field if all non-zero elements are invertible (i.e., property (8) in Recall 2.1 holds). In general, one can distinguish two types of rings.

**Recall 2.4.** A commutative ring $\mathbb{A}$ with 1 is called *integral domain* if the following holds:

$$\forall a, b \in \mathbb{A} : a \cdot b = 0 \Rightarrow a = 0 \vee b = 0.$$

If the ring is not an integral domain, one finds two such elements $a, b \in \mathbb{A} \setminus \{0\}$ with $a \cdot b = 0$. Such elements are also called zero-divisors. In this case, $\mathbb{A}$ is also called a *a commutative ring with 1 and zero-divisors.*

Note that a commutative ring with 1 which contains zero-divisors cannot be a field. Contrary, an integral domain (i.e., a commutative ring with 1 and without zero-divisors) might be a field. Even better, if it is an integral domain but not a field yet, one can build the set of quotients. Then this extended set forms a field and contains $\mathbb{A}$ as subring. More precisely, one can carry out the following construction.

**Recall 2.5.** Let $(\mathbb{A}, +, \cdot)$ be an integral domain and define the set of quotients

$$Q(\mathbb{A}) = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{A}, b \in \mathbb{A} \setminus \{0\} \right\}.$$

Since $\mathbb{A}$ is an integral domain, it follows that for any $b_1, b_2 \in \mathbb{A} \setminus \{0\}$ we have $b_1 \cdot b_2 \neq 0$. Thus we can define the operation $\oplus : Q(\mathbb{A}) \times Q(\mathbb{A}) \to Q(\mathbb{A})$ defined by

$$\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_1 \cdot b_2 + a_2 \cdot b_1}{b_1 \cdot b_2} \tag{7}$$

and the operation $\odot : Q(\mathbb{A}) \times Q(\mathbb{A}) \to Q(\mathbb{A})$ defined by

$$\frac{a_1}{b_1} \odot \frac{a_2}{b_2} = \frac{a_1 \cdot a_2}{b_2 \cdot b_2}. \tag{8}$$

One can verify that $(Q(\mathbb{A}), \oplus, \odot)$ is a field (i.e., all 9 properties in Recall 2.1 hold). $Q(\mathbb{A})$ is also called the *quotient field of $\mathbb{A}$. Note that $\mathbb{A}$ is contained in $Q(\mathbb{A})$ by identifying $a \in \mathbb{A}$ with $\frac{a}{1} \in Q(\mathbb{A})$. In particular, for any $a, b \in \mathbb{A}$ we have $a \oplus b = a + b$ and $a \odot b = a \cdot b$. This means that the ring $\mathbb{A}$ is contained in the field $Q(\mathbb{A})$. Usually, one reuses for $Q(\mathbb{A})$ again the operations $+$ and $\cdot$ from the integral domain $\mathbb{A}$.*

**Example 2.6.** The set of integers $\mathbb{Z}$ forms a ring with the usual addition and multiplication. The ring is an integral domain, i.e., for any $a, b \in \mathbb{Z} \setminus \{0\}$ we have $a \cdot b \neq 0$. The set of quotients $Q(\mathbb{Z})$ is nothing else than $\mathbb{Q}$. Together with the operations (7) and (8) one obtains a field, also called the field of rational numbers. Obviously, $\mathbb{Z}$ is contained in $\mathbb{Q}$ by identifying $\frac{a}{1} \in \mathbb{Q}$ with $a \in \mathbb{Z}$.

A ring with zero divisors cannot be turned to a field: one can find at least two denominators whose multiplication turns to zero. Such "exotic" rings will arise in the next section.

Sometimes one is faced with a commutative (abelian) group $(\mathbb{A}, +)$, i.e., with an operation $+ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ with the properties (1)-(4) from Recall 2.1, but a multiplication $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ with the properties of a ring or a field does not exist. However, in many cases one can determine a field $\mathbb{K}$ together with a scalar operation $* : \mathbb{K} \times \mathbb{A} \to \mathbb{A}$ yielding a vector space.

**Recall 2.7.** Let $(\mathbb{K}, +, \cdot)$ be field and let $\mathbb{A}$ be a set with two operations $+ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and $* : \mathbb{K} \times \mathbb{A} \to \mathbb{A}$.

$(\mathbb{A}, +, *)$ is called a vector space over $\mathbb{K}$ if properties (1)–(4) from Recall 2.1 hold and in addition the following properties hold:

1. $\forall a \in \mathbb{A} \; \forall \lambda, \mu \in \mathbb{K} : (\lambda \cdot \mu) * a = \lambda * (\mu * a)$;

2. $\forall a \in \mathbb{A} : 1 * a = a$ (here 1 is the neutral element in $\mathbb{K}$);

3. $\forall a, b \in \mathbb{A} \; \forall \lambda \in \mathbb{K} : \lambda * (a + b) = \lambda * a + \lambda * b$;

4. $\forall a \in \mathbb{A} \; \forall \lambda, \mu \in \mathbb{K} : (\lambda + \mu) * a = \lambda * a + \mu * a$.

$*$ is also called a *scalar multiplication.*

**Example 2.8.** $\mathbb{R}^3$ is a vector space over $\mathbb{R}$. E.g.,

$$(1, 0, 0) + (-1) * (0, 1, 0) = (1, 0, 0) + (0, -1, 0) = (1, -1, 0).$$

More generally take any field $\mathbb{K}$ and $n \in \mathbb{N}^*$. Then $\mathbb{K}^n$ (the set of vectors of length $n$ with entries from $\mathbb{K}$) forms a vector space over $\mathbb{K}$.

# 3   Generating functions and formal power series

In this lecture we will often deal with sequences $(a_n)_{\geq 0}$ (see, e.g., (1) with $a_n = F(n)$). As we will see later, it is often more convenient to work with a generating function

$$(a_n)_{n \geq 0} \mapsto \sum_{n=0}^{\infty} a_n \, x^n.$$

For instance, the generating function of $a_n = 1$ with $n \geq 0$ yields

$$(a_n)_{n \geq 0} \mapsto \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.} \qquad (GS)$$

In analysis the identity (GS) can be interpreted as follows by considering the functions

- $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{1-x}$;

- $g : (-1, 1) \to \mathbb{R}$ defined by $g(x) = \lim_{N \to \infty} \sum_{n=0}^{N} x^n$; by the ratio test the power series $g(x)$ has the convergence radius $r = 1$.

Then (GS) in analysis means

$$g(x) = f(x) \qquad \forall x \in (-1, 1).$$

In algebra the interpretation will be elaborated in Example 3.12 below.

**Definition 3.1.** *(including notations)*
$\mathbb{K}$ *denotes a field containing the rational numbers $\mathbb{Q}$ as subfield; typical examples are $\mathbb{K} = \mathbb{Q}$, the rational function field $\mathbb{K} = \mathbb{Q}(n)$ with rational coefficients (like $\frac{n^3+1}{2n^2+3}$), $\mathbb{K} = \mathbb{R}$ or the rational function field $\mathbb{K} = \mathbb{R}(n)$ with real coefficients (like $\frac{n^3+\sqrt{2}}{2n^2+\pi}$).*
*The set of sequences with entries from $\mathbb{K}$ is denoted by*

$$\mathbb{K}^{\mathbb{N}} := \{(a_n)_{n \geq 0} \mid a_n \in \mathbb{K}\}.$$

In the following we will explore step-wise more and more operations that can be applied to the set $\mathbb{K}^{\mathbb{N}}$. We start with the following simple versions.

**Definition 3.2.** *For $(a_n)_{n \geq 0}, (b_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}}$ and $\lambda \in \mathbb{K}$ we define*

$$(a_n)_{n \geq 0} + (b_n)_{n \geq 0} := (a_n + b_n)_{n \geq 0},$$
$$\lambda * (a_n)_{n \geq 0} := (\lambda a_n)_{n \geq 0}.$$

**Example 3.3.** We have

$$(1, 0, 0, 0, \dots) + (-1) * (0, 1, 0, 0, \dots) = (1, 0, 0, 0, \dots) + (0, -1, 0, 0, \dots) = (1, -1, 0, 0, \dots).$$

**Lemma 3.4.** $(\mathbb{K}^{\mathbb{N}}, +, *)$ *is a vector space over* $\mathbb{K}$.

*Proof.* **BP 3**. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Remark 3.5.** For $m \in \mathbb{N}^*$ set

$$\tilde{K}_m = \{(a_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}} \mid a_l = 0 \ \forall l \geq m\}.$$

Then $\tilde{K}_m$ and $\mathbb{K}^m$ are isomorphic as vector spaces, i.e., up to renaming of the objects (taking only the first $m$ entries from each element of $\tilde{K}_m$)

$$(a_0, a_1, \ldots, a_{m-1}, 0, 0, 0, \ldots) \mapsto (a_0, a_1, \ldots, a_{m-1})$$

they are the same. Summarizing, the vector space $\mathbb{K}^{\mathbb{N}}$ contains $\tilde{K}_m$ and thus $\mathbb{K}^m$ as a special case ($\mathbb{K}^m$ is the well known vector space known from linear algebra; it is a subspace of $\mathbb{K}^{\mathbb{N}}$).

**Example 3.6.** For $\mathbb{K} = \mathbb{R}$ and $m = 3$ we conclude that

$$\tilde{\mathbb{R}}_3 = \{(a_n)_{n \geq 0} \in \mathbb{R}^{\mathbb{N}} \mid a_l = 0 \ \forall l \geq 3\}$$

and $\mathbb{R}^3$ are isomorphic. For instance, we identify $(1, -1, 0, 0, \ldots) \in \tilde{\mathbb{R}}_3$ with $(1, -1, 0) \in \mathbb{R}^3$.

In contrast to the scalar multiplication $*$ for $\mathbb{K}^{\mathbb{N}}$, one can define more flexible operations for $\mathbb{K}^{\mathbb{N}}$. As will be seen later, the Cauchy product can be considered as a generalization of the scalar multiplication.

**Definition 3.7.** *For* $(a_n)_{n \geq 0}, (b_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}}$ *we define the* Hadamard product $\odot : \mathbb{K}^{\mathbb{N}} \times \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ *by*

$$(a_n)_{n \geq 0} \odot (b_n)_{n \geq 0} := (a_n \, b_n)_{n \geq 0}$$

*and the* Cauchy product $\cdot : \mathbb{K}^{\mathbb{N}} \times \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ *by*

$$(a_n)_{n \geq 0} \cdot (b_n)_{n \geq 0} := (c_n)_{n \geq 0}$$

*where*

$$c_n = \sum_{k=0}^{n} a_k \, b_{n-k}.$$

**Example 3.8.** Consider $(a_n)_{n \geq 0}$ with $a_n = 1$ for $n \geq 0$ and $(b_n)_{n \geq 0}$ with $b_0 = 1$, $b_1 = -1$ and $b_n = 0$ for $n \geq 2$. Then

$$(a_n)_{n \geq 0} \odot (b_n)_{n \geq 0} = (1, 1, 1, 1, \ldots) \odot (1, -1, 0, 0, \ldots) = (1, -1, 0, 0, \ldots)$$

and

$$(a_n)_{n \geq 0} \cdot (b_n)_{n \geq 0} = (1, 1, 1, 1, \ldots) \cdot (1, -1, 0, 0, \ldots) = (c_0, c_1, c_2, \ldots)$$

with

$$c_n = \sum_{k=0}^{n} a_k \, b_{n-k}.$$

More precisely,

$$c_0 = 1 \cdot 1 = 1$$
$$c_1 = a_0\,b_1 + a_1\,b_0 = -1 + 1 = 0$$
$$c_2 = a_0\,b_2 + a_1\,b_1 + a_2\,b_0 = 0 + -1 + 1 = 0$$
$$c_3 = a_1\,b_3 + a_1\,b_2 + a_2\,b_1 + a_3\,b_0 = 0 + 0 - 1 + 1 = 0$$
$$\vdots$$
$$c_l = 0 \quad \forall l \geq 1.$$

Thus

$$(a_n)_{n \geq 0} \cdot (b_n)_{n \geq 0} = (1,1,1,1,\dots) \cdot (1,-1,0,0,\dots) = (1,0,0,0,\dots). \tag{9}$$

**Theorem 3.9.** $(\mathbb{K}^{\mathbb{N}}, +, \odot)$ *is a commutative ring with[4] 1, but not an integral domain.*

*Proof.* **BP 4**. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Recall: A ring $R$ with 1 is an integral domain if

$$\forall a, b \in R : a \cdot b = 0 \Rightarrow a = 0 \vee b = 0$$

If there are two elements $a, b \in R$ with $a \neq 0 \neq b$ and $a\,b = 0$, then and $a$ and $b$ are called zero divisors. Obviously, $R$ is an integral domain if and only if it has no zero-divisors.

**Example 3.10.** $(\mathbb{K}^{\mathbb{N}}, +, \odot)$ contains zero-divisors: $(a_n)_{n \geq 0} = (1,0,1,0,1,0,\dots)$ and $(b_n)_{n \geq 0} = (0,1,0,1,0,1,\dots)$ are not the zero-sequence but their Hadamard product produces the zero-sequence:

$$(a_n)_{n \geq 0} \odot (b_n)_{n \geq 0} = (1,0,1,0,1,0,\dots) \odot (0,1,0,1,0,1,\dots) = (0,0,0,0,0,\dots).$$

**Theorem 3.11.** $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ *is a commutative ring with[5] 1, it is even an integral domain.*

*Proof.* **BP 5**: Show that $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ is a commutative ring with 1.
**HW 12**: Show that it is even an integral domain[6]. $\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

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**Notation.** For $(a_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}}$ we also write

$$(a_n)_{n \geq 0} =: \sum_{n=0}^{\infty} a_n\,x^n = a(x)$$

with $x$ an indeterminate (variable).
For $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ we shall write $(\mathbb{K}[[x]], +, \cdot)$ or $\mathbb{K}[[x]]$ for short and will call it also the *ring of formal power series over* $\mathbb{K}$.
Note: $(1,0,0,0,\dots) = 1\,x^0 + 0\,x^1 + 0\,x^2 + 0\,x^3 + \cdots = 1$ is the 1-element in $\mathbb{K}[[x]]$. If it is clear from the context, we simply write $a\,b$ instead of $a \cdot b$.

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[4]Note: $(0,0,0,\dots)$ is the zero element and $(1,1,1,1,\dots)$ is the 1 element.
[5]Note: $(0,0,0,\dots)$ is the zero element and $(1,0,0,0,\dots)$ is the 1 element.
[6]Hint: To prove that the ring is integral, we show that it has no zero divisors. Namely, suppose that $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are both not the zero-sequence (as warm up example suppose that $a_0 \neq 0 \neq b_0$). Then show that at least one entry in $(a_n)_{n \geq 0} \cdot (b_n)_{n \geq 0}$ is not zero.

**Example 3.12.** Rewriting (9) in its formal power series notation gives

$$\left(\sum_{n=0}^{\infty} x^n\right)(1-x) = 1.$$

In summary, the interpretation of (GS) in the algebra sense means that $1-x$ and $\sum_{n=0}^{\infty} x^n$ are the multiplicative inverses to each other, i.e., $\sum_{n=0}^{\infty} x^n$ multiplied by $1-x$ equals the 1-element. In short,

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

**Remark 3.13.** We define

$$\mathbb{K}[x] = \left\{ \sum_{n=0}^{\infty} a_n\, x^n \in \mathbb{K}[[x]] \mid a_k = 0 \text{ for all } k \geq \delta \text{ for some } \delta \in \mathbb{N} \right\} \subseteq \mathbb{K}[[x]]$$

and call it the set of polynomials. Take two such polynomials $a(x) = \sum_{n=0}^{m_1} a_n\, x^n$ and $b(x) = \sum_{n=0}^{m_2} n_n\, x^n$ from $\mathbb{K}[x]$ (we define $a_k = 0$ for $k > m_1$ and $b_k = 0$ for $k > m_2$). Then the Cauchy product simplifies to

$$a(x) \cdot b(x) = \sum_{n=0}^{m_1+m_2} c_n x^n$$

with $c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$. In Mathematica the multiplication $a(x), b(x) \in \mathbb{Q}[x]$ can be carried out, e.g., by `Expand[a*b]`. Note that $(\mathbb{K}[x], +, \cdot)$ itself is a ring with $1 \in \mathbb{K}[x]$ called the *polynomial ring over $\mathbb{K}$. This follows from the fact that for all $a, b \in \mathbb{K}[x]$ we have that $a + b \in \mathbb{K}[x]$ and $a \cdot b \in \mathbb{K}[x]$ i.e., $(\mathbb{K}[x], +, \cdot)$ is a subring of $(\mathbb{K}[[x]], +, \cdot)$. Furthermore, since $\mathbb{K}[[x]]$ has no zero divisors, also its subset $\mathbb{K}[x]$ has no zero-divisors. In other words, $(\mathbb{K}[x], +, \cdot)$ itself is an integral domain.*

**Remark 3.14.** Take $\lambda \in \mathbb{K} \subseteq \mathbb{K}[[x]]$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{K}[[x]]$ and consider the Cauchy product

$$\lambda \cdot b(x) := (\lambda, 0, 0, \dots) \cdot (b_0, b_1, b_2, \dots) = (c_n)_{n \geq 0}$$

with

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

with $a_0 = \lambda$ and $a_l = 0$ for $l \geq 1$. Then $c_n = \lambda\, b_n$ and hence

$$\lambda \cdot b(x) = (\lambda b_0, \lambda b_1, \lambda b_2, \dots) = \sum_{n=0}^{\infty} (\lambda\, b_n) x^n.$$

In other words, restricting $\cdot : \mathbb{K}[[x]] \times \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ to $\cdot : \mathbb{K} \times \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ yields precisely our scalar multiplication $* : \mathbb{K} \times \mathbb{K}[[x]] \to \mathbb{K}$ introduced in Definition 3.2 and by Lemma 3.4 it follows that $(\mathbb{K}[[x]], +, \cdot)$ with $\cdot : \mathbb{K} \times \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ is a vector space over $\mathbb{K}$.

**Lemma 3.15.** *Take $\lambda \in \mathbb{K}$, $m \in \mathbb{N}$ and $a(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{K}[[x]]$. Then*

$$(\lambda\, x^m) \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} \lambda a_n x^{n+m} = \sum_{n=m}^{\infty} \lambda a_{n-m} x^n.$$

*Proof.* **HW 13**. □

**Definition 3.16.** *(coefficient functional)*
*For $k \in \mathbb{N}$ we define*

$$[x^k] \sum_{n=0}^{\infty} a_n x^n = a_k.$$

*As shortcut we write*

$$a(0) := [x^0]a(x) = a_0.$$

**Lemma 3.17.** *For $k \in \mathbb{N}$ and $a(x), b(x) \in \mathbb{K}[[x]]$,*

$$[x^k](a(x) + b(x)) = [x^k]a(x) + [x^k]b(x),$$
$$[x^k](\lambda \, a(x)) = \lambda [x^k]a(x).$$

*Proof.* **HW 14**. □

**Remark 3.18.** By the properties from Lemma 3.17 it follows that for a fixed $k \in \mathbb{N}$ the map

$$[x^k] : \mathbb{K}[[x]] \to \mathbb{K}$$

is a linear map.

**HW 15.** In $(\mathbb{K}[x], +, \cdot)$ prove

1. $\left( \sum_{n=0}^{\infty} c^n x^n \right) (1 - cx) = 1$          $(c \in \mathbb{K})$

2. $\left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right) \left( \sum_{k=0}^{\infty} \frac{(-1)^n}{n!} x^n \right) = 1.$

# Lecture from April 18, 2023

**Definition 3.19.** *Let $R$ be a commutative ring with $1$ containing $\mathbb{Q}$. For $r \in R$ and $k \in \mathbb{N}$ we define the* falling factorial

$$r^{\underline{k}} = r(r-1)(r-2) \cdots (r-k+1), \quad k \geq 1$$
$$r^{\underline{0}} = 1$$

and the *raising factorial*

$$r^{\bar{k}} = r(r+1)(r+2) \cdots (r+k-1), \quad k \geq 1$$
$$r^{\bar{0}} = 1.$$

In addition, for $r \in R$ and $k \in \mathbb{Z}$ we define the *binomial coefficient with*

$$\binom{r}{k} := \begin{cases} \frac{r^{\underline{k}}}{k!} & k \geq 0 \\ 0 & k < 0. \end{cases}$$

**HW 16.** Show for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}$ that

$$\binom{z+1}{k} = \binom{z}{k} + \binom{z}{k-1}.$$

**Definition 3.20.** *The following formal power series deserve a short-cut notation:*

$$\exp(x) := \sum_{n=0}^{\infty} \frac{1}{n!} x^n \in \mathbb{K}[[x]],$$

$$\exp(-x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \in \mathbb{K}[[x]],$$

$$\log(1+x) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \in \mathbb{K}[[x]],$$

$$\log(1-x) := -\sum_{n=1}^{\infty} \frac{1}{n} x^n \in \mathbb{K}[[x]].$$

**Remark 3.21.** From HW 15 we know that

$$\exp(x) \cdot \exp(-x) = 1,$$

i.e., $\exp(-x)$ is the multiplicative inverse of $\exp(x)$:

$$\exp(x)^{-1} := \exp(-x).$$

**Remark 3.22.** In analysis we know that

$$\frac{d}{dx} \log(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} n x^{n-1} = -\sum_{n=1}^{\infty} x^{n-1} = -\frac{1}{1-x}$$

with $|x| < 1$.

The last two remarks motivate us to consider the operations differentiation and division also in our formal ring $\mathbb{K}[[x]]$ in more details.

## 3.1 Differentiation and division

**Definition 3.23.** *Let $(R, +, \cdot)$ be a commutative ring with $1$ (containing $\mathbb{Q}$ as a subring). Let $D : R \to R$ be a function such that*

$$D(a + b) = D(a) + D(b)$$
$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

*for all $a, b \in R$. Then $D$ is called a (formal) derivative on $R$, and the pair $(R, D)$ is called a differential ring.*

**Lemma 3.24.** *Consider $D_x : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ with*

$$D_x\Big(\sum_{n=0}^{\infty} a_n x^k\Big) = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n.$$

*Then $(\mathbb{K}[[x]], D_x)$ is a differential ring.*

*Proof.* **HW 17**. □

**Example 3.25.**

$$D_x \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(n+1)x^n = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This motivates the notation from above: for $\exp(x) := \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ we have $D_x \exp(x) = \exp(x)$.

**Definition 3.26.** *We define the formal integration $\int_x : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ by*

$$\int_x \sum_{n=0}^{\infty} a_n\, x^n \mapsto \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}x^n.$$

**Remark 3.27.** In analysis this formal integration is equivalent to the integration of a power series from $0$ to $x$.

The action of $D_x$ and $\int_x$ on the sequence representation of a formal power series is nothing else than the shift of the sequence (up to some normalizing factor) to the left or right:

$$D_x(a_0, a_1, a_2, a_3, \dots) = (1\, a_1, 2\, a_2, 3\, a_3, \dots)$$

$$\int_x (a_0, a_1, a_2, \dots) = (0, \tfrac{a_0}{1}, \tfrac{a_1}{2}, \tfrac{a_2}{3}, \dots).$$

This yields

**Theorem 3.28.** *For all $a(x) \in \mathbb{K}[[x]]$ we have*
1. $D_x \int_x a(x) = a(x)$             *"Fundamental Theorem of Calculus I"*
2. $\int_x D_x a(x) = a(x) - a(0)$      *"Fundamental Theorem of Calculus II"*
3. $[x^n]a(x) = \frac{1}{n!}\left((D_x)^n a(x)\right)|_{x=0}$    *"Taylor's formula"*

*Proof.* With

$$D_x \int_x (a_0, a_1, a_2, \dots) = D_x(0, \tfrac{a_0}{1}, \tfrac{a_1}{2}, \tfrac{a_2}{3}, \dots) = (a_0, a_1, a_3, \dots)$$

property 1 follows. With

$$\int_x D_x(a_0, a_1, a_2, a_3, \dots) = \int_x (1\, a_1, 2\, a_2, 3\, a_3, \dots) = (0, a_1, a_2, a_3, \dots)$$

and

$$
\begin{aligned}
a(x) = (a_0, a_1, a_2, a_3, \dots) &= (a_0, 0, 0, 0, \dots) + (0, a_1, a_2, a_3, \dots) \\
&= (a_0, 0, 0, 0, \dots) + \int_x D_x(a_0, a_1, a_2, a_3, \dots) \\
&= a(0) + \int_x D_x a(x)
\end{aligned}
$$

property 2 is established. Finally, observe that

$$
\begin{aligned}
(D_x)^n(a_0, a_1, a_2, a_3, a_4, a_5, \dots) &= (D_x)^{n-1} D_x(a_0, a_1, a_2, a_3, a_4, a_5, \dots) \\
&= (D_x)^{n-1}(1\,a_1, 2\,a_2, 3\,a_3, 4\,a_4, 5\,a_5, \dots) \\
&= (D_x)^{n-2} D_x(1\,a_1, 2\,a_2, 3\,a_3, 4\,a_4, 5\,a_5, \dots) \\
&= (D_x)^{n-2}(1 \cdot 2\,a_2, 2 \cdot 3\,a_3, 3 \cdot 4\,a_4, 4 \cdot 5\,a_5 \dots) \\
&= (D_x)^{n-3} D_x(1 \cdot 2\,a_2, 2 \cdot 3\,a_3, 3 \cdot 4\,a_4, 4 \cdot 5\,a_5 \dots) \\
&= (D_x)^{n-3}(1 \cdot 2 \cdot 3\,a_3, 2 \cdot 3 \cdot 4\,a_4, 3 \cdot 4 \cdot 5\,a_5, \dots) \\
&= \dots = (n! a_n, \tfrac{n!}{1!} a_{n+1}, \tfrac{n!}{2!} a_{n+2}, \tfrac{n!}{3!} a_{n+3}, \dots)
\end{aligned}
$$

which gives $(D_x)^n(a_0, a_1, a_2, a_3, \dots)|_{x=0} = n! a_n = n! [x^n] a(x)$, and thus proves property 3. $\qquad\square$

The following property is particular strong: one can invert a formal power series if and only if the constant term does not vanish.

**Theorem 3.29.** *[multiplicative inverse] Let $a(x) \in \mathbb{K}[[x]]$. Then:*

$$
\boxed{\textit{There exists a } b(x) \in \mathbb{K}[[x]] \textit{ with } a(x) \cdot b(x) = 1} \quad \Leftrightarrow \quad \boxed{a(0) \neq 0}.
$$

*Proof.* Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$.
$\Rightarrow$: Suppose that there is a $b(x) = \sum_{n=0}^{\infty} b_n x^n$ with

$$
a(x) \cdot b(x) = 1.
$$

Then by the definition of the Cauchy product,

$$
a_0 \, b_0 = 1
$$

and thus, since $\mathbb{K}$ is a field, $a(0) = a_0 \neq 0$.
$\Leftarrow$ Suppose that $a_0 \neq 0$. We construct $b(x) = b_n x^n$ such that

$$
1 x^0 + 0 x^1 + 0 x^2 + \dots = 1 = a(x) \cdot b(x) = \left( \sum_{n=0}^{\infty} a_n x^n \right) \cdot \left( \sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k \, b_{n-k}. \quad (10)
$$

By coefficient comparison in equation (10) at $[x^0]$ we get $a_0 \, b_0 = 1$ and thus with $a_0 \neq 0$ we get (in the field $\mathbb{K}$)

$$
\boxed{b_0 = \frac{1}{a_0},} \tag{11}
$$

i.e., $b_0$ is determined. Furthermore, by coefficient comparison in equation (10) at $[x^n]$ with $n \geq 1$ we get

$$
0 = \sum_{k=0}^{n} a_k \, b_{n-k} = a_0 \, b_n + \sum_{k=1}^{n} a_k \, b_{n-k}
$$

and hence

$$
\boxed{b_n = -\frac{1}{a_0} \sum_{k=1}^{n} a_k \, b_{n-k}.} \tag{12}
$$

Summarizing,

- given $a_0$, we can determine $b_0$ with (11),

- given $a_0, a_1$ and $b_0$ we can use the formula in (12) to determine $b_1$,

- given $a_0, a_1, a_2$ and $b_0, b_1$, we can use the formula in (12) to determine $b_2$,

- ...

and thus all $b_n$ can be determined iteratively for $a(x) = \sum_{n=0}^{\infty} a_n x^n$. $\qquad\square$

Note 1: If the standard operations in $\mathbb{K}$ are computable and if the coefficients $a_0, a_1, \ldots, a_l$ ($a_0 \neq 0$) are given explicitly (can be computed by an algorithm), also the first coefficients $b_0, b_1, \ldots, b_l$ ($b_0 \neq 0$) can be computed. In particular, we get

$$\left(\sum_{n=0}^{l} a_l\right)\left(\sum_{n=0}^{l} b_l\right) = \boxed{1} + \underbrace{0\,x^1 + 0\,x^2 + \cdots + 0\,x^l}_{=0} + c_{l+1}x^{l+1} + c_{l+2}x^{l+2}$$

where $c_{l+1}, c_{l+2}, \ldots$ are error terms (which arise since we only considered the first $l+1$ coefficients of $a(x)$ and $b(x)$).

Note 2: Let $b(x), \tilde{b}(x) \in \mathbb{K}[[x]]$ such that

$$a(x) \cdot b(x) = 1 = a(x) \cdot \tilde{b}(x).$$

Then $b(x) \cdot (a(x) \cdot b(x)) = b(x) \cdot (a(x) \cdot \tilde{b}(x))$ and thus

$$b(x) = (a(x) \cdot b(x)) \cdot b(x) = (b(x) \cdot a(x)) \cdot b(x)$$
$$= \boxed{b(x) \cdot (a(x) \cdot b(x)) = b(x) \cdot (a(x) \cdot \tilde{b}(x))}$$
$$= (b(x) \cdot a(x)) \cdot \tilde{b}(x) = (a(x) \cdot b(x)) \cdot \tilde{b}(x) = \tilde{b}(x),$$

i.e., $b(x) = \tilde{b}(x)$. Consequently, if $a(x)$ has a multiplicative inverse $b(x)$ (i.e., if $a(x) \neq 0$), then it is unique and motivates the following notation.

**Notation.** Let $a(x) \in \mathbb{K}[[x]]$ with $a(0) \neq 0$. Then for the multiplicative inverse $b(x) \in \mathbb{K}[[x]]$ of $a(x)$ (determined, e.g., by the above theorem) we also write

$$a^{-1}(x) := b(x) \text{ or } \frac{1}{a(x)} := b(x).$$

**Example.** We have $(1-x) \cdot \sum_{n=0}^{\infty} x^n = 1$, i.e., the multiplicative inverse of $1-x$ is $\sum_{n=0}^{\infty} x^n$ and we write

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Furthermore, the multiplicative inverse of $\sum_{n=0}^{\infty} x^n$ is $(1-x)$ and we write

$$\left(\sum_{n=0}^{\infty} x^n\right)^{-1} = \frac{1}{\sum_{n=0}^{\infty} x^n} = 1 - x.$$

**Example.** By HW. 15 we have $\exp(x) \cdot \exp(-x) = 1$. Thus $\exp(-x)$ is the multiplicative inverse of $\exp(x)$, i.e.,

$$\exp(x)^{-1} = \frac{1}{\exp(x)} = \exp(-x);$$

furthermore, $\exp(x)$ is the multiplicative inverse of $\exp(-x)$, i.e.,

$$\exp(-x)^{-1} = \frac{1}{\exp(-x)} = \exp(x).$$

**HW 18.** Let $\exp(c\,x) := \sum_{n=0}^{\infty} \frac{c^n}{n!} x^n$. For $a, b \in \mathbb{K}$ show:

$$\exp(a\,x)\exp(b\,x) = \exp((a+b)x).$$

Hint for possible solutions of the HWs below: Compute the first coefficients $b_n$ and use Sloan's database `https://oeis.org` to find a closed form.

**HW 19.** Find a closed form for the coefficients in the multiplicative inverse of $(1-2x)^2 \in \mathbb{K}[[x]]$.

**HW 20.** Find a closed form for the coefficients in the multiplicative inverse of $(1-x)^3 \in \mathbb{K}[[x]]$.

**HW 21.** Find a closed form for the coefficients in the multiplicative inverse of $\exp(2x) \in \mathbb{K}[[x]]$.

## Lecture from April 25, 2023

### 3.2 Finding closed forms for generating functions

**Example** Find a closed form for

$$H(x) = \sum_{n=0}^{\infty} H_n x^n \in \mathbb{K}[[x]]$$

where $H_n$ denote the harmonic numbers ($H_0 = 0$).

$\boxed{\text{TACTIC:}}$ Find a (functional) equation for $H(x)$.

**A) by using the recurrence for $H_n$:**

$$H_{n+1} = H_n + \frac{1}{n+1}$$

with $H_0 = 0$.
Note: we stay in $\mathbb{K}[[x]]$ (no analysis!)

Using the recurrence (in the second line) we get

$$H(x) = \sum_{n=0}^{\infty} H_n \, x^n = \sum_{n=1}^{\infty} H_n \, x^n = \sum_{n=0}^{\infty} H_{n+1} \, x^{n+1}$$

$$= \sum_{n=0}^{\infty} \left( H_n + \frac{1}{n+1} \right) x^{n+1}$$

$$= x \sum_{n=0}^{\infty} H_n \, x^n + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$= x H(x) - \log(1 - x)$$

and thus

$$\boxed{H(x) = -\frac{1}{1 - x} \log(1 - x).} \tag{13}$$

Note 1: Here we use that $(1 - x)\frac{1}{1-x} = 1$.

Note 2: We consider $-\frac{1}{1-x} \log(1 - x)$ as a closed form for $H(x)$ since it is the product of well known formal power series which we gave already special names:

- the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$;

- the logarithmic power series $\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$; see also Definition 3.20.

**B) by using an extra lemma (which we will use later again)**

**Lemma 3.30.** *We have[7]*

$$\sum_{n=0}^{\infty} x^n \left( \sum_{k=0}^{n} a_k \right) = \frac{1}{1 - x} \sum_{n=0}^{\infty} a_n \, x^n$$

*or equivalently (by multiplying with $1 - x$)*

$$(1 - x) \sum_{n=0}^{\infty} x^n \left( \sum_{k=0}^{n} a_k \right) = \sum_{n=0}^{\infty} a_n x^n.$$

---

[7]The Cauchy product of $\frac{1}{1-x}$ with a formal power series $a(x) = \sum_{n=0}^{\infty} a_n \, x^n$ yields a formal power series whose coefficients are $\sum_{k=0}^{n} a_k$.

*Proof.*

$$\text{LHS} = (1 - x) \sum_{n=0}^{\infty} x^n \left( \sum_{k=0}^{n} a_k \right)$$

$$= \underbrace{\sum_{n=0}^{\infty} x^n \left( \sum_{k=0}^{n} a_k \right)}_{x^0 a_0 + \sum_{n=1}^{\infty} x^n \left( \sum_{k=0}^{n} a_k \right)} - \underbrace{\sum_{n=0}^{\infty} x^{n+1} \left( \sum_{k=0}^{n} a_k \right)}_{\sum_{n=1}^{\infty} x^n \left( \sum_{k=0}^{n-1} a_k \right)}$$

$$= x_0 \, a_0 + \sum_{n=1}^{\infty} x^n \left( \underbrace{\sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k}_{a_n} \right)$$

$$= \sum_{n=0}^{\infty} a_n x^n = \text{RHS}.$$

$\square$

Remark: The special case of the above lemma with

$$a_k = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ if } k \geq 1 \end{cases}$$

gives

$$(1 - x) \sum_{n=0}^{\infty} x^n = 1.$$

Another special case can be used for $H(x)$:

$$H(x) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k \quad \text{where } a_k = \begin{cases} 0 \text{ if } k = 0 \\ \frac{1}{k} \text{ if } k \geq 1 \end{cases}$$

$$= \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n$$

$$= \frac{1}{1-x} \left( 0 + \sum_{n=1}^{\infty} \frac{x^n}{n} \right)$$

$$= -\frac{1}{1-x} \log(1 - x).$$

**HW 22.** Consider the formal power series $f(x) = \frac{1}{(1-x)^2} \log(1 - x) \in \mathbb{Q}[[x]]$. Express the coefficients $f_n \in \mathbb{Q}$ of $f(x) = \sum_{k=0}^{\infty} f_n x^n$ in terms of the harmonic numbers $H_n$.

# 4    Application: binary trees

We will apply the tools from the previous section in order to explore binary trees.

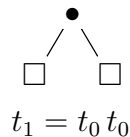**Definition 4.1.** *A binary tree is*

- *a single external node (denoted by □)*

- *or an internal node (denoted by •) that is connected to two binary trees: a left and a right subtree.*

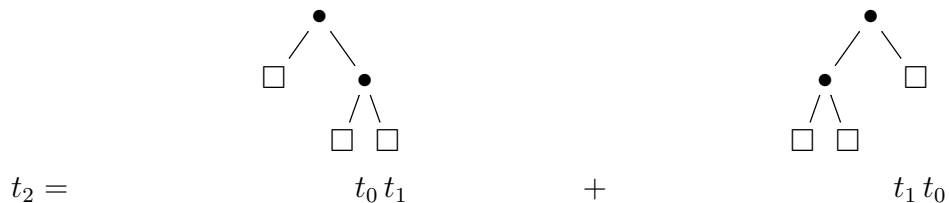To warm up, we consider all trees with $n$ external nodes for $n = 1, 2, 3, 4$.
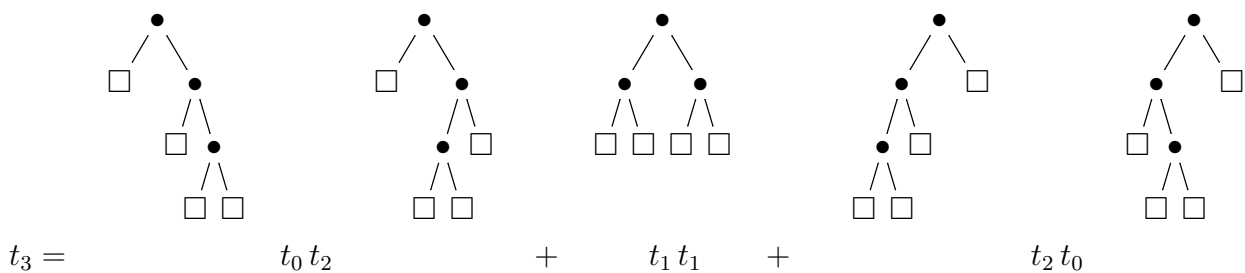For $n = 1$ we get:

$$□$$
$$t_0$$

For $n = 2$ we get:

$$t_1 = t_0\, t_0$$

For $n = 3$ we get:

$$t_2 = \qquad\qquad t_0\, t_1 \qquad\qquad + \qquad\qquad t_1\, t_0$$

And for $n = 4$ we get:

$$t_3 = \qquad\qquad t_0\, t_2 \qquad + \qquad t_1\, t_1 \qquad + \qquad\qquad t_2\, t_0$$

In the following we want to explore

$$\boxed{t_n := \text{the number of binary trees with } n + 1 \text{ external nodes.}}$$

Looking at the above enumeration we get

| $n$ | 0 | 1 | 2 | 3 | ... | $n$ | ... |
|-----|---|---|---|---|-----|-----|-----|
| $t_n$ | 1 | 1 | 2 | 5 | ... | ? | ... |

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The main goal is to produce a general formula for $t_n$. Note that we can produce all trees with $n + 1$ external nodes by considering all trees of the form

$$\underset{\substack{\text{a binary}\\\text{tree with }k+1\\\text{external nodes}}}{\diagdown} \bullet \underset{\substack{\text{a binary}\\\text{tree with }n-k\\\text{external nodes}}}{\diagup}$$
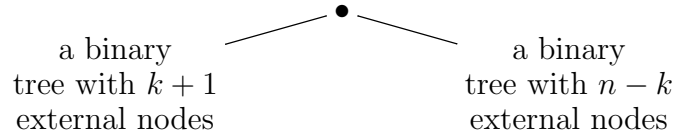
with $k = 0, 1, 2, \ldots, n - 1$. Summarizing, we get

$$t_0 = 1$$
$$t_n = \sum_{k=0}^{n-1} t_k \, t_{n-k-1}. \tag{14}$$

For instance,

$$t_1 = t_0 \, t_0 = 1$$
$$t_2 = t_0 \, t_1 + t_1 \, t_0 = 1 + 1 = 2$$
$$t_3 = t_0 \, t_2 + t_1 \, t_1 + t_2 \, t_0 = 2 + 1 + 2 = 5;$$

note that this formula is also reflected in the graphical enumeration from above.

In general, we can now calculate any value $t_n$ with $n \in \mathbb{N}$ using this recursion formula. But we can do much better!

After this preparation step we can activate our formal power series engine by defining the generating function

$$t(x) = \sum_{n=0}^{\infty} t_n \, x^n \in \mathbb{Q}[[x]].$$

Hence

$$t(x) = t_0 \, x^0 + \sum_{n=1}^{\infty} x^n \sum_{k=0}^{n-1} t_k \, t_{n-1-k}$$
$$= 1 + \sum_{n=0}^{\infty} x^{n+1} \sum_{k=0}^{n} t_k \, t_{n-k}$$
$$= 1 + x \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} t_k \, t_{n-k}$$

and by the Cauchy product we get

$$= 1 + x \, t(x)^2.$$

Summarizing, we obtain the following functional equation:

$$\boxed{x \, t(x)^2 - t(x) + 1 = 0.} \tag{15}$$

As it turns out, the following trick makes our life easier: multiply the equation with $x$. This gives

$$x^2 \, t(x)^2 - x \, t(x) + x = 0.$$

Thus if we define

$$T(x) := x \, t(x) \in \mathbb{Q}[[x]]$$

we get

$$T(x)^2 - T(x) + x = 0$$

$$\|$$

$$\underbrace{T(x)^2 - 2T(x) \cdot \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2}_{=\left(T(x) - \frac{1}{2}\right)^2} - \tfrac{1}{4} + x$$

which is equivalent to

$$(T(x) - \tfrac{1}{2})^2 = \tfrac{1}{4}(1 - 4x).$$

$$\updownarrow$$

$$(2T(x) - 1)^2 = 1 - 4x.$$

**Lemma 4.2.** *Let $g(x) \in \mathbb{K}[[x]]$ with $g(0) = 1$. Then there is a unique $f(x) \in \mathbb{K}[[x]]$ with $f(x)^2 = g(x)$ and $f(0) = 1$. In addition, there is exactly one other solution which is $-f(x)$.*

*Proof.* **HW 23** (Hint: adapt the proof of Theorem 3.29). □

With this lemma it follows that we can take $f(x) \in \mathbb{K}[[x]]$ with $f(0) = 1$ such that

$$f(x)^2 = 1 - 4x. \tag{16}$$

In particular, we get

$$2T(x) - 1 = \pm f(x)$$

$$\updownarrow$$

$$T(x) = \frac{1}{2} \pm \frac{1}{2} f(x).$$

In the following we will write for the unique $f(x) \in \mathbb{Q}[[x]]$ with $f(0) = 1$ and $f(x)^2 = 1 - 4x$ also

$$\sqrt{1 - 4x} := f(x).$$

Thus we get

$$T(x) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4x};$$

Looking at the constant term on both sides shows that

$$T(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4x};$$

the plus version would have given $[x^0](\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - 4x}) = 1$, but we have $[x^0]T(x) = 0$.

Note that the formal power series $\sqrt{1-4x}$ (which can be determined by the above lemma) has a nice closed form. In the moment we present it without a proof and keep it as

**Conjecture A** (for a proof see later)

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n}(-1)^n 4^n x^n \in \mathbb{Q}[[x]].$$

Suppose that our conjecture is correct. Then we can continue as follows:

$$\sum_{n=0}^{\infty} t_n x^n = t(x) = \frac{1}{x}(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4x}) \tag{17}$$

$$= \frac{1}{x}\frac{-1}{2}\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n}(-1)^n 4^n x^n$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n}(-1)^n 4^n x^{n-1}$$

Warning: In (17) we divided by $x$ which is not allowed in $\mathbb{Q}[[x]]$. Formal details will come later. Here we can justify it (somehow) since the constant term vanishes and thus the division is possible by shifting the formal power series (considered as a sequence in $\mathbb{Q}^{\mathbb{N}}$) to the left.

Consequently,

$$t_n = -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-1)^{n+1}4^{n+1}$$

$$\overset{\textbf{HW 24}}{=} \frac{1}{n+1}\binom{2n}{n}$$

holds for all $n \in \mathbb{N}$. Summarizing we obtain

**Conjecture B**

$$\boxed{t_n = \frac{1}{n+1}\binom{2n}{n}}$$

Remark: Conjecture B would turn to a theorem if Conjecture A is verified. Conversely, if Conjecture B can be verified, also Conjecture A is true since we can trace back the consequences. In other words, verifying Conjecture A or B would give the full result.

In the following we verify Conjecture B (which also shows that Conjecture A holds).

**Proof of Conjecture B with symbolic summation.** Define

$$s_n = \frac{1}{n+1}\binom{2n}{n}.$$

Then $s_n$ counts the number of trees with $n + 1$ external nodes if and only if it satisfies the old recurrence (14), i.e., if and only if we have[8]

$$s_0 = 1$$

$$s_n = \sum_{k=0}^{n-1} s_k \, s_{n-k-1}.$$

Obviously $s_0 = 1$ holds. What remains to verify is the second equation which reads as follows:

$$\frac{1}{n+1}\binom{2n}{n} = \sum_{k=0}^{n-1} \frac{1}{k+1}\binom{2k}{k}\frac{1}{n-k}\binom{2n-2k-2}{n-k-1}. \tag{18}$$

We verify this identity with the help of the summation package[9] **Sigma**. First, we load it into Mathematica:

In[1]:= $<<$ **Sigma.m**

> Sigma - A summation package by Carsten Schneider © RISC-JKU

Next we insert the sum on the right-hand side of (18) into Mathematica. Note that we consider even a more general sum by choosing the upper summation bound to be $A$ (and not the special value $n-1$):

In[2]:= **mySum = SigmaSum[1/(k+1)SigmaBinomial[2k, k]/(n−k)SigmaBinomial[2n−2k−2, n−k−1], {k, 0, A}]**

Out[2]= $\displaystyle\sum_{k=0}^{A} \frac{\binom{2k}{k}\binom{-2-2k+2n}{-1-k+n}}{(1+k)(-k+n)}$

Finally, we apply the function call **SigmaReduce** to the sum and obtain the following simplification:

In[3]:= **sol = SigmaReduce[mySum]**

Out[3]= $\displaystyle\frac{(-1+2n)\binom{-2+2n}{-1+n}}{n(1+n)} + \frac{(1+2A)(2+2A-n)\binom{2A}{A}\binom{-2-2A+2n}{-1-A+n}}{(1+A)n(1+n)}$

Specializing $A$ to $n-1$ gives

In[4]:= **sol = sol/.{A → n − 1}//Together**

Out[4]= $\text{sol} = \displaystyle\frac{2(-1+2n)\binom{-2+2n}{-1+n}}{n(1+n)}$

With sharp eyes one can see already that this result can be transformed to the left-hand side of (18). However, this can be done automatically with **Sigma** by telling it to use $\binom{2n}{n}$:

In[5]:= **SigmaReduce[sol, n, Tower → {SigmaBinomial[2n, n]}]**

Out[5]= $\text{sol} = \displaystyle\frac{1}{n+1}\binom{2n}{n}$

Note that we not only proved the identity (18), but given the right-hand side, we discovered the simplified left-hand side. This completes the proof of Conjecture B and yields

**Theorem 4.3.** *For $n \in \mathbb{N}$,*

$$t_n = \frac{1}{n+1}\binom{2n}{n}.$$

---

[8]Note that the old recurrence defines uniquely the sequence $t_n$. If $s_n$ is a solution of this recurrence, then the sequence $s_n$ must agree with the sequence $t_n$.

[9]You can download the package from `https://risc.jku.at/sw/sigma/`

# Lecture from May 2, 2023

**HW 25.** Simplify

1. $\sum_{k=0}^{n} k\,k!$;

2. $\sum_{k=0}^{n}(-1)^k \binom{m}{k}$;

3. $\sum_{k=0}^{n}(-1)^k \binom{m}{k} H_k$.

**HW 26.** Simplify

1. $\sum_{k=0}^{n} H_k^2$;

2. $\sum_{k=0}^{n}(H_{m+k})^2$;

3. $\sum_{k=0}^{n} H_k^3$.

Last week we used the symbolic summation package `Sigma` to simplify a sum (and posed various homeworks to simplify similar sums). In this regard, the following remark is relevant.

**Remark 4.4.** Simplifications of indefinite sums like

$$\sum_{k=0}^{n} H_k = (n+1)H_n - n$$

for $n \in \mathbb{N}$ often find simple verifications.

For instance, let $L(n)$ be the left-hand side and $R(n)$ be the right-hand side of the above identity. Then the strategy is to show that both sides satisfy the same recurrence (telescoping equation).

By definition we get for the left-hand side:

$$L(n+1) - L(n) = H_{n+1} \quad \text{for } n \in \mathbb{N},$$
$$L(0) = 0.$$

We verify that also the right-hand side satisfies the same recurrence. Namely we get

$$\begin{aligned}
R(n+1) - R(n) &= (n+2)H_{n+1} - (n+1)H_n + n \\
&= (n+2)\big(H_n + \tfrac{1}{n+1}\big) - (n+1)H_n - 1 \\
&= H_n + 1 + \frac{1}{n+1} - 1 \\
&= H_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$ and $R(0) = (0+1)H_0 - 0 = 0$. Thus both sides satisfy the same recurrence with the same initial value and thus both sides must agree:

$$L(n) = R(n) \quad \forall n \in \mathbb{N}.$$

We remark that the observation in the above remark is another important feature that can be applied often in computer algebra calculations. Software packages like `Sigma` might be based on rather complicated algorithms and non-trivial software, and thus one might doubt that the result is correct. However, the verification can be carried out independently of the original code either by hand (as above) or by much simpler code that we can trust.

**HW 27.** Verify the correctness of the first identity that you have derived in HW 26.

In the second part of this section we will emphasis that with further computer algebra tools many of the hand-calculations of the last lecture can be skipped. Let us recall, what we accomplished last week. We started to produce the first values of $t_n$ and obtained the table:

| $n$ | 0 | 1 | 2 | 3 | ... | $n$ | ... |
|---|---|---|---|---|---|---|---|
| $t_n$ | 1 | 1 | 2 | 5 | ... | ? | ... |

Inspired by this enumeration

- we produced the general formula (14) which allows one to calculate any value for $t_n$;

- we looked at its generating function $T(x) = \sum_{n=0}^{\infty} t_n x^n$ and obtained the closed form

$$T(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4x} \tag{19}$$

  with

$$\sqrt{1 - 4x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n}(-1)^n 4^n x^n \in \mathbb{Q}[[x]];$$

- finally, we derived

$$t_n = \frac{1}{n+1}\binom{2n}{n}.$$

To establish these formulas we did hand calculations based on classical generating function technologies. However, we used also modern computer algebra tools and verified correctness with the package `Sigma`: namely, we proved Conjecture B (and thus Conjecture A).

In the second part of this section we will be even more extreme. Starting with the discovered formula (14) we produce some more values, say

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $t_n$ | 1 | 1 | 2 | 5 | 14 | 42 | 132. |

As it turns out, this data is completely sufficient to discover all the remaining ingredients! More precisely, we load another package[10] from the Algorithmic Combinatorics Group at RISC:

In[6]:= **<< RISC'GeneratingFunctions'**

> Package GeneratingFunctions version 0.8 written by Christian Mallinger Copyright Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

Then given the first values of $t_n$ (i.e., for $n = 0, 1, 2, 3, 4, 5$) one can guess the linear recurrence

$$-2(1 + 2n)a(n) + (2 + n)a(1 + n) = 0 \tag{20}$$

for $t_n$ as follows:

---

[10]You can download the package `GeneratingFunctions` from `https://risc.jku.at/sw/generatingfunctions/`.

In[7]:= **treeRE = GuessRE[{1, 1, 2, 5, 14, 42}, a[n]]**

Out[7]= $\{\{-2(1 + 2n)a[n] + (2 + n)a[1 + n] == 0, a[0] == 1\}, "ogf"\}$

One might be scared that the guess of the recurrence is wrong. But one can use, e.g. the data of $t_5 = 132$ (which has not been used for the guessing) and verifies that the recurrence holds not only for $n = 0, 1, 2, 3, 4$ but also for $n = 5$:

$$-2(1 + 2n)a(n) + (2 + n)a(1 + n) = -22a(5) + 7a(6) = 0.$$

This gives more confidence. Of course, one can repeat this game and test further values for $n = 7, 8, 9, 10$ and get even more confidence. Still this is not a proof...

However, we can easily solve the discovered recurrence (20) with $a(0) = 1$ (see HW 28) and get

$$a_n = \frac{1}{n+1}\binom{2n}{n}.$$

**HW 28.** Given the sequence $a(n)$ defined by

$$-2(2n + 1)a(n) + (n + 2)a(n + 1) = 0$$

and $a(0) = 1$. Show that $a(n) = \frac{1}{n+1}\binom{2n}{n}$ holds.

Thus we get for free (modulo some clever computer algebra packages) the desired result, i.e., Conjecture 6. Finally, we can activate our summation tools and verify correctness (as we did last week already).

> SUMMARY: by simply pressing some buttons one can guess and afterwards verify the result by means of computer algebra.

We conclude this section by observing that we can use again computer algebra to get also the closed form (19). Namely, by using again the package `GeneratingFunctions` with the command

In[8]:= **treeDE = RE2DE[treeRE[[1]], a[n], t[x]]**

Out[8]= $\{-1 - (-1 + 2x)t[x] - (-x + 4x^2)t'[x] == 0, t[0] == 1\}$

we calculate a differential equation that is satisfied by the formal power series $t(x) = \sum_{n=0}^{\infty} t_n x^n$. Formally this means that

$$0 = -1 - (-1 + 2x)t(x) - (-x + 4x^2)D_x t(x)$$

$$= -1 - (-1 + 2x)\sum_{n=0}^{\infty} t_n x^n + (-x + 4x^2)\sum_{n=1}^{\infty} t_n n x^{n-1}$$

holds.

Finally, we solve this differential equation with the Mathematica function `DSolve` and get the desired closed form:

In[9]:= **DSolve[treeDE, t[x], x]**

Out[9]= $\{\{t[x] \to \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4x}\}\}$

**Remark 4.5.** Sometimes it is convenient to work not with inhomogeneous recurrences like

$$H_{n+1} - H_n = \frac{1}{n+1}$$

or equivalently

$$(n+1)H_{n+1} - (n+1)H_n = 1$$

but with homogeneous versions (with possibly higher order). Namely, taking its shifted version

$$(n+2)H_{n+2} - (n+2)H_{n+1} = 1$$

and subtracting both versions gives

$$(n+2)H_{n+2} - (2n+3)H_{n+1} + (n+1)H_n = 0. \tag{21}$$

**HW 29.** Consider the QuickSort recurrence

$$(n+1)F_{n+1} - (n+2)F_n = 2n, \quad n \geq 0$$

and transform it to a homogeneous recurrence (of higher order).
Hint use the trick from above (shift and subtract) twice.

**HW 30.** Compute a differential equation for the generating function $Q(x) = \sum_{n=0}^{\infty} F_n x^n$ where $F_n$ are the average comparisons to quicksort an array with $n$ elements.
Hint: use, e.g., the homogeneous recurrence from the previous homework.

**HW 31.** Compute a differential equation for the generating function $H(x) = \sum_{n=0}^{\infty} H_n x^n$ (e.g., with RE2DE) and solve it (e.g., with DSolve). Compare your result with $H(x) = -\frac{1}{1-x}\log(1-x)$ derived in (13).

# 5 Formal power series: Part 2

## 5.1 Connection to analysis

In general, I promised that you will be dispensed from analysis. However, today the situation is slightly different. The bad news is that we will connect our formal world to analysis, i.e., you will see analysis arguments in this lecture :-( The good news is that this yields new constructions to use cleverly computer algebra. The second good news is that this lecture will be an exception, i.e., no analysis will appear from next week on. Summary: stay relaxed.

Let us start with the following simple observation: The formal power series world and the analysis world (with power series) are different.

**Example 5.1.** Consider the formal power series $\sum_{n=0}^{\infty} n! x^n \in \mathbb{K}[[x]]$. If we restrict to $\mathbb{K} = \mathbb{R}$, we come close to the analysis world. However, by the ratio test

$$\lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n\to\infty} |(n+1)x| = \begin{cases} \infty & x \neq 0 \\ 0 & x = 0, \end{cases}$$

the series converges only for $x = 0$. Thus as function the power series can be used only trivially.

**Example 5.2.** The power series $f(x) = \sum_{n=0}^{\infty} \frac{(1+x)^n}{n!}$ in analysis converges for all $x \in \mathbb{R}$ by the ratio test:

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1} n!}{(n+1)! x^n} \right| = \lim_{n\to\infty} \left| \frac{x+1}{n} \right| = 0.$$

Thus we can define the non-trivial function

$$f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$$

which is the exponential function:

$$f(x) = \exp(1+x) = e^{x+1} = e\, e^x.$$

However the series is not an element in the formal power series ring $\mathbb{R}[[x]]$.

BUT: numerous series live in BOTH worlds!

**Example 5.3.** In analysis we have

$$y'(x) = \frac{1}{1+x}$$

for all $x \in ]-1, \infty[$ for the function $y : ]-1, \infty[ \to \mathbb{R}$ with $y(x) = \log(1+x)$.

Does this relation carry over to $\mathbb{K}[[x]]$? The answer is YES. Namely, for the formal power series

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \in \mathbb{K}[[x]],$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \in \mathbb{K}[[x]]$$

and the derivative $D_x$ given in Definition 3.26 we have

$$D_x \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} D_x x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} n x^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1} = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}.$$

The following observation is crucial: For $x \in \mathbb{C}$ with $|x| < 1$ (in particularly for $x \in \mathbb{R}$) the functions $\log(1+x)$ and $\frac{1}{x+1}$ (in the analysis world) have power series (Taylor series) representations:

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Thus the formal power series world and the power series world from analysis match here. Furthermore, our formal definition of the differentiation $D_x$ in $\mathbb{K}[[x]]$ is the same as in the analysis world.

More precisely, the following property holds.

**Theorem 5.4.** *Let* $f(x) = \sum_{n=0}^{\infty} f_n x^n$ *be a power series* $(a_n \in \mathbb{R})$ *with convergency radius* $r > 0$, *i.e.,* $f(x) \in \mathbb{R}$ *for all* $|x| < r$. *Then* $f(x)$ *is arbitrarily differentiable In particular, for* $k \in \mathbb{N}^*$ *we have*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} f_n n(n-1)(n-2)\ldots(n-k+1) x^{n-k}$$

*for all* $x \in \mathbb{R}$ *with* $|x| < r$.

*Remark:* the above statement holds also for complex power series $(a_n \in \mathbb{C})$ where $|x| < r$ means that $x \in \mathbb{C}$ is inside the circle with radius $r$ centered around 0.

An important consequence of Theorem 5.4 is the following property: for all $n \in \mathbb{N}$ and $x \in \mathbb{C}$ with $|x| < r$ we have

$$[x^n] f^{(k)}(x) = [x^0] D^k f(x) = f^{(k)}(0)$$
$$= f_k k(k-1)(k-2)\ldots(k-k+1) x^0 = f_k k!.$$

Hence

$$f_k = \frac{f^{(k)}(0)}{k!} = \frac{[x^0] D^k f(x)}{k!}$$

for $k \in \mathbb{N}$ which is nothing else than Taylor's formula in the analysis world. Note that we have obtained this formula already earlier in Theorem 3.28 in the formal power series world.

```
(************************************************************************)
(*          WARING: hardcore for complex analysis fans          *)
(*...
```

Let $D = \{x \in \mathbb{C} \mid |x| < r\}$ for some $0 < r \in \mathbb{R}$ and suppose that $f : D \to \mathbb{C}$ is complex differentiable at all points $x \in D$. Then $f$ is infinitely many times (complex) differentiable in $D$ and

$$f(x) = \sum_{n=0}^{\infty} f_n x^n \quad \forall x \in D$$

with

$$f_n = \frac{f^{(n)}(0)}{n!}. \tag{22}$$

**\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*)**

All what one should remember is that many functions are complex differentiable and thus have a power series representation. In particular, there is a formula (see (22) which is again Taylor's formula) that can be used (in particular in computer algebra systems).

**Example 5.5.** The function $f(x) = e^{-x}$ is complex differentiable for $D = \mathbb{C}$. Thus one can compute the coefficients of its power series by (22):

$$f_n = \frac{f^{(n)}(0)}{n!} = \frac{(e^{-x})^{(n)}|_{x=0}}{n!} = \frac{(-1)^n e^0}{n!} = \frac{(-1)^n}{n!}$$

and gets

$$e^{-x} = \sum_{n=0}^{\infty} f_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \tag{23}$$

for each $x \in \mathbb{C}$.

We can use this observation to activate again computer algebra (and ignore all the analysis calculations). Namely, instead of using analysis arguments (compute a formula for $\frac{f^{(n)}(0)}{n!}$ using the known differentiation rules), we can rely again on our Mathematica tools from above. Executing

In[10]:= **fSeries = Series[e$^{-x}$, {x, 0, 10}]**

Out[10]= $1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \dfrac{x^4}{24} - \dfrac{x^5}{120} + \dfrac{x^6}{720} - \dfrac{x^7}{5040} + \dfrac{x^8}{40320} - \dfrac{x^9}{362880} + \dfrac{x^{10}}{3628800} + 0\left(x^{11}\right)$

we compute the coefficients of the power series expansion of $e^{-x}$ up to the coefficient of $x^{10}$. Let us rewrite this representation in list form

In[11]:= **fL = CoefficientList[fSeries, x]**

Out[11]= $\{1, -1, \dfrac{1}{2}, -\dfrac{1}{6}, \dfrac{1}{24}, -\dfrac{1}{120}, \dfrac{1}{720}, -\dfrac{1}{5040}, \dfrac{1}{40320}, -\dfrac{1}{362880}, \dfrac{1}{3628800}\}$

In other words, we used Mathematica to compute the list $(f_0, f_1, \ldots, f_{10})$. Then using the package

In[12]:= **<< RISC'GeneratingFunctions'**

Package GeneratingFunctions version 0.8 written by Christian Mallinger Copyright Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

we can guess a recurrence of the coefficients $f_n$ by

In[13]:= **rec = GuessRE[fL, f[n]]**

Finally, we solve the recurrence with

In[14]:= **RSolve[rec[[1]], f[n], n]**

Out[14]=  $\{\{f[n] \to \dfrac{(-1)^n}{\text{Pochhammer}[2, n-1]}\}\}$

**Remark 5.6.** Let $R$ be a commutative ring with 1, $x \in R$ and $n \in \mathbb{N}$. The Pochhammer symbol is defined by

$$(x)_n := x^{\bar{n}}.$$

In Mathematica it is denoted by `Pochhammer[x,n]`.

With

$$f_n = \frac{(-1)^n}{(2)_{n-1}} = \frac{(-1)^n}{n!}$$

we have rediscovered (23) by utilizing our computer algebra toolbox.

**HW 32.** For the function $f(x) = \frac{1+2x}{1-2x}$ there exists a complex series expansion. Find it.

**HW 33.** For the function $f(x) = \left(\frac{1+x}{1-x}\right)^2$ there exists a complex series expansion. Find it.

**HW 34.** For the function $f(x) = \sqrt{\frac{1+x}{1-x}}$ there exists a complex series expansion. Find it.

**HW 35.** For the function $f(x) = \log(\frac{1+x}{1-x})$ there exists a complex series expansion. Find it.

**BP 6.** For the above functions $f(x)$ and complex series expansions find (the maximal) $r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} f_n\, x^n \quad |x| < r.$$

# Lecture from May 9, 2023

We are now ready to state the following crucial connection between the formal and analysis world.

**Theorem 5.7.** *[Transfer principle] Let $a(x)$ and $b(x)$ be complex functions such that*

$$a(x) = b(x)$$

*for all $|x| < r$ for some $r > 0$. Suppose that $a(x)$ and $b(x)$ have power series representations such that for all $|x| < r$ we have*

$$
\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{C}. \\
b(x) &= \sum_{n=0}^{\infty} b_n x^n, \quad b_n \in \mathbb{C}.
\end{aligned}
\tag{24}
$$

*Then*

$$a_n = b_n \quad \forall n \in \mathbb{N}.$$

In other words, if we are given two formal power series $a(x), b(x) \in \mathbb{C}[[x]]$ with (24) and we are suspicious that they agree in the ring of formal power series $\mathbb{C}[[x]]$, i.e.,

$$a(x) = (a_0, a_1, a_2, \dots) = (b_0, b_1, b_2, \dots) = b(x), \tag{25}$$

then we can either try to prove this identity formally in $\mathbb{C}[[x]]$ or we can use analysis! Namely, if $a(x)$ and $b(x)$ can be evaluated for each $|x| < r$ for some $r > 0$ and the corresponding functions equal for all $|x| < r$ (e.g., by looking up some formulas in an analysis text book), then it follows automatically that also (25) holds.

In the following example we utilize the following well known result from analysis.

**Lemma 5.8.** *If $f(x) = \sum_{n=0}^{\infty} f_n x^n$ and $g(x) = \sum_{n=0}^{\infty} g_n x^n$ are power series with radius of convergence $r_f > 0$ and $r_g > 0$, respectively. Then*

$$
\begin{aligned}
f(x) + g(x) &= \sum_{n=0}^{\infty} (f_n + g_n) x^n, \\
f(x) g(x) &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} f_k \, g_{n-k}
\end{aligned}
$$

*holds for $|x| < \min(r_f, r_g)$.*

**HW 36.** Verify that the real function $A : \,]-1, 1[ \, \to \mathbb{R}$ with $x \mapsto \frac{e^{-x}}{1-x}$ satisfies

$$A'(x) = \frac{x}{1-x} A(x), \quad A(0) = 1.$$

Note: By the same rules it follows that $A$ (as complex function with inputs inside of the unit circle) satisfies this differential equation.

**Example 5.9.** Let $D = \{x \in \mathbb{C} \mid |x| < 1\}$ be the points inside of the unit circle and consider the complex function $A : D \to \mathbb{C}$ with

$$A(x) = \frac{e^{-x}}{1 - x}.$$

By HW 36 it follows that $A(x)$ satisfies the differential equation

$$A'(x) = \frac{x}{1 - x} A(x)$$

with $A(0) = 1$.

As a consequence, we can carry over this result to the formal power series world as follows.

- $A(x)$ has a power series representation. Namely,

  - $e^{-x}$ has a power series representation for $x \in \mathbb{C}$, see (23).
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ has a power series representation for all $x \in D$.
  - Thus by Lemma 5.8 $A(x)$ has a power series representation. Namely,

  $$A(x) = \Big( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \Big) \Big( \sum_{n=0}^{\infty} x^n \Big) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

  holds for all $x \in D$.

- $a(x) = A'(x)$ has a power series representation by Theorem 5.4.
  More precisely, we have

  $$A'(x) = \sum_{n=0}^{\infty} a_n x^n$$

  with

  $$a_n = (n+1) \sum_{k=0}^{n+1} \frac{(-1)^k}{k!}.$$

- $b(x) = \frac{x}{1-x} A(x)$ has a power series representation by Lemma 5.8. More precisely, with $f(x) = A(x)$ and $g(x) = \frac{1}{1-x}$ we have

  $$\frac{x}{1-x} A(x) = x \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \underbrace{\sum_{i=0}^{k} \frac{(-1)^i}{i!}}_{f_k} \cdot \underbrace{1}_{g_{n-k}} = \sum_{n=0}^{\infty} b_n x^n$$

  with

  $$b_n = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \frac{(-1)^i}{i!}$$

  for all $x \in D$.

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Consequently by the Transfer Principle (Theorem 5.7) it follows that

$$a_n = b_n \quad \forall n \in \mathbb{N}.$$

In particular it follows that

$$D_x A(x) = \frac{x}{1-x} A(x) \tag{26}$$

holds in the formal power series ring $\mathbb{C}[[x]]$ with the formal power series $A(x), \frac{x}{1-x} \in \mathbb{C}[[x]]$. As a byproduct we get from $a_n = b_n$ the non-trivial identity

$$(n+1) \sum_{k=0}^{n+1} \frac{(-1)^k}{k!} = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \frac{(-1)^i}{i!}. \tag{27}$$

**BP 7.** Prove the identity (27) for all $n \in \mathbb{N}$ without analysis arguments (e.g., with symbolic summation). In this way you will verify (26) purely formally (i.e., without using the transfer principle from analysis).

## 5.2 Quotients of formal power series

As we have learned earlier (see Theorem 3.11) the ring of formal power series $(\mathbb{K}[[x]], +, \cdot)$ with the Cauchy product $\cdot$ forms an integral domain. Thus we can follow the construction of Recall 2.5 and obtain the field of quotients of $\mathbb{K}[[x]]$.

In various instances such a construction is non-trivial. E.g., the field of quotients of the polynomial ring $\mathbb{K}[x]$ yields the rational function field $\mathbb{K}(x)$ and typical elements are $\frac{x+1}{x^2+x+1}$. However, in our particular situation of formal power series the situation is surprisingly simple: Due to Theorem 3.29 most of the elements in $\mathbb{K}[[x]]$ are already invertible. In particular, those elements which are not invertible can be easily turned to invertible elements.

More precisely, consider two formal power series

$$a(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{K}[[x]] \setminus \{0\},$$

$$b(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{K}[[x]] \setminus \{0\}.$$

Then there are integers $\alpha, \beta \in \mathbb{N}$ and $A(x), B(x) \in \mathbb{K}[[x]]$ with

$$\begin{aligned} a(x) &= x^\alpha A(x) \text{ where } A(0) \neq 0, \\ b(x) &= x^\beta B(x) \text{ where } B(0) \neq 0. \end{aligned} \tag{28}$$

**Example 5.10.** For $b(x) = x^3 - x^4 = x^3(1-x) \in \mathbb{K}[[x]]$ we can choose $\beta = 3$ and $B(x) = 1 - x$ and get $b(x) = x^\beta B(x)$ with with $B(0) = 1 \neq 0$

Given such (28) we can form the quotient construction (compare Recall 2.5) and get

$$\frac{a(x)}{b(x)} = \frac{x^\alpha}{x^\beta} \frac{A(x)}{B(x)}. \tag{29}$$

Note that $B(0) \neq 0$ and thus $B(x)$ has the invertible element $B(x)^{-1} \in \mathbb{K}[[x]]$ by Theorem 3.11. Consequently,

$$\frac{a(x)}{b(x)} = \frac{x^{\alpha}}{x^{\beta}} \frac{A(x)}{B(x)} = x^{\alpha - \beta} \underbrace{A(x) \cdot B(x)^{-1}}_{\in \mathbb{K}[[x]]}.$$

If $\alpha \geq \beta$ we get $x^{\alpha - \beta} \in \mathbb{K}[x] \subseteq \mathbb{K}[[x]]$ and thus the quotient in (29) is an element in $\mathbb{K}[[x]]$. The only "critical" situation is $\alpha < \beta$.

**Example 5.11.** Take $a(x) = 1$ and $b(x) = x^3 - x^4$. Then $b(x) = x^3 B(x)$ with $B(x) = 1 - x$. Since $B(0) \neq 0$ we can invert $B(x)$ in $\mathbb{K}[[x]]$ and get

$$B(x)^{-1} = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots.$$

Thus we have

$$\frac{a(x)}{b(x)} = \frac{1}{x^3 B(x)} = x^{-3} B(x)^{-1}$$
$$= x^{-3}(1 + x + x^2 + x^3 + \dots) = x^{-3} + x^{-2} + x^{-1} + x^0 + \dots$$

This motivates us to define the following set.

**Definition 5.12** (formal Laurent series)**.**

$$\mathbb{K}((x)) := \{(a_n)_{n \in \mathbb{Z}} \mid a_n \in \mathbb{K} \text{ and } a_n \neq 0 \text{ for only finitely many } \textbf{negative} \text{ indices}\};$$

*here we denote the two-sided zero sequence as usual by* $0 = (\dots, 0, 0, 0, \dots) \in \mathbb{K}((x))$. *Furthermore, we define the order of* $a = (a_n)_{n \in \mathbb{Z}} \in \mathbb{K}((x)) \setminus \{0\}$ *by*

$$ord(a) = \min\{n \in \mathbb{Z} \mid a_n \neq 0\} \in \mathbb{Z};$$

*for the special case* $a = 0$ *we define* $ord(0) = \infty$. *In particular, for* $a \neq 0$ *we use the notation*

$$a = a(x) = \sum_{n = ord(a)}^{\infty} a_n x^n.$$

**Example 5.13.** We have

$$\frac{1}{x^3 + x^4} = \frac{1}{x^3} \frac{1}{1 - x} = x^{-3}(1 + x + x^2 + \dots)$$
$$= x^{-3} + x^{-2} + x^{-1} + x^0 + \dots = \sum_{n=-3}^{\infty} x^n \in \mathbb{K}((x)).$$

**Remark 5.14.** Consider

$$R = \{(a_n)_{n \in \mathbb{Z}} \in \mathbb{K}((x)) \mid a_n = 0 \text{ for all } n < 0\} \subseteq \mathbb{K}((x)).$$

Then $R$ is isomorphic to $\mathbb{K}^{\mathbb{N}}$. More precisely, we can identify $a = (\dots, 0, 0, a_0, a_1, a_2, \dots)$ with $(a_0, a_1, a_2, \dots)$. In this way we can say that $\mathbb{K}^{\mathbb{N}}$ (which is equal to $R$) is contained in $\mathbb{K}((x))$.

The operations $+$ and $\cdot$ (Cauchy product) of $\mathbb{K}[[x]]$ naturally extend to $\mathbb{K}((x))$ and yield a commutative ring $(\mathbb{K}((x)), +, \cdot)$ where each element (except the 0 element) is multiplicative invertible. Thus we get the following theorem.

**Theorem 5.15.** $(\mathbb{K}((x)), +, \cdot)$ *is a field which contains* $(\mathbb{K}[[x]], +, \cdot)$ *as subring.*

*Proof.* **HW 37** (Basically everything is said above, just repeat the construction once more carefully...) $\qquad\square$

Remark: The quotient field $(\mathbb{K}((x)), +, \cdot)$ of $(\mathbb{K}[[x]], +, \cdot)$ is also called the field or Laurent series.

**Remark 5.16.** If one wants to implement the set $\mathbb{K}((x))$, one has to encode the elements $a \in \mathbb{K}((x)) \setminus \{0\}$. Note that for $d = \mathrm{ord}(a) \in \mathbb{Z}$ we have

$$a = \sum_{n=d}^{\infty} a_n\, x^n = a_d x^d + a_{d+1} x^{d+1} + a_{d+2} x^{d+2} + \cdots = x^d(a_d + a_{d+1}x + a_{d+2}x^2 + \dots)$$

with $a_d \neq 0$. This yields the following natural data structure (implementation of $\mathbb{K}((x))$): we take the tuple $(d, f)$ with $d \in \mathbb{Z}$ and $f \in \mathbb{K}^{\mathbb{N}}$ where $f = (a_d, a_{d+1}, a_{d+2}, \dots) \in \mathbb{K}^{\mathbb{N}}$ with $a_d \neq 0$. Of course, the infinite list $f$ cannot be stored physically. But, depending on the application, sufficiently many entries of $f$ can be stored, e.g., in a linked list.
This convention is also useful if one simply wants to store an element $a(x)$ from $\mathbb{K}[[x]]$ (and not from $\mathbb{K}((x))$): as above one can write $a(x) = x^d A(x)$ with $A(0) \neq 0$ where $d = \mathrm{ord}(a) \in \mathbb{N}$. I.e., one drops the first $d-1$ zero elements of $a$ at the beginning and starts to memorize the list $A$ where the first entry is non-zero . In addition to $A$ one needs $d \in \mathbb{N}$ as offset to recover the full representation of $a(x) = x^d A(x) \in \mathbb{K}[[x]]$.

## 5.3   Convergence in $\mathbb{K}[[x]]$ and $\mathbb{K}((x))$

I promised that we will avoid analysis aspects in this lecture (except in Subsection 5.1). Even so we will use the word "convergence" below, it has nothing to do with the analysis interpretation: it is totally formally defined and can be considered as a new formal operation (like $+$, the Cauchy product $\cdot$, $D_x$, inversion, etc.) in the setting of formal power series.
In the following we restrict to $\mathbb{K}[[x]]$, the extension to $\mathbb{K}((x))$ is immediate.

**Definition 5.17.** *Consider a sequence in* $\mathbb{K}[[x]]$ *(i.e., a sequence of formal power series[11]):* $(a_k(x))_{k \geq 0}$ *with* $a_k(x) \in \mathbb{K}[[x]]$.
*Then we say that* $(a_k(x))_{k \geq 0}$ *converges to a formal power series* $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{K}[[x]]$ *if the following holds:*
*For all* $n \in \mathbb{N}$ *there exists a starting point* $l(= l(n)) \in \mathbb{N}$ *such that*

$$\begin{aligned}
f_n &= [x^n]a_l(x) \\
&= [x^n]a_{l+1}(x) \\
&= [x^n]a_{l+2}(x) \\
&= a.s.o.
\end{aligned}$$

---

[11]More precisely, a sequence of sequences.

*In other words: from a certain point on, namely $l$, the $n$th coefficient in the sequence $(a_k(x))_{k \geq 0}$ does not change and equals the corresponding coefficient $f_n$ of $f(x)$.*

*We say that $(a_k(x))_{k \geq 0}$ is convergent if there exists a formal power series $f(x) \in \mathbb{K}[[x]]$ such that $(a_k(x))_{k \geq 0}$ converges to $f(x)$.*

Note: if $(a_k(x))_{k \geq 0}$ converges to $f_1(x) \in \mathbb{K}[[x]]$ and $f_2(x) \in \mathbb{K}[[x]]$ then $f_1(x) = f_2(x)$. This uniqueness property enables us to introduce the following notation/definition/operation.

**Definition 5.18.** *Let $(a_k(x))_{k \geq 0}$ be convergent. Then the uniquely determined $f(x) \in \mathbb{K}[[x]]$ to which $(a_k(x))_{k \geq 0}$ converges is denoted by*

$$\lim_{k \to \infty} a_k(x) := f(x).$$

As mentioned already above, we have introduced a new "fancy" operation $\lim_{k \to \infty} a_k(x)$ for a sequence $(a_k(x))_{k \geq 0}$ in $\mathbb{K}[[x]]$ which is convergent (i.e., the $n$ coefficient stabilizes within the series/list of formal power series $(a_0(x), a_1(x), a_2(x), \dots)$).

**Example 5.19.** Consider the formal power series $a_k(x) := \frac{1 - x^k}{1 - x} \in \mathbb{K}[[x]]$ with $k \in \mathbb{N}$. Note that we get

$$
\begin{aligned}
a_k &= (1 - x^k)(1 + x + x^2 + x^3 + \dots) \\
&= (1 + x + x^2 + x^3 + \dots) - (x^k + x^{k+1} + x^{k+2} + x^{k+3} + \dots) \\
&= 1 + x + x^2 + \cdots + x^{k-1} \in \mathbb{K}[x].
\end{aligned}
$$

In other words (in sequence notation $\mathbb{K}^{\mathbb{N}}$) we have

$$
\begin{aligned}
(a_k(x))_{k \geq 0} = \Big( &(0, 0, 0, 0, 0, \dots), \\
&(1, 0, 0, 0, 0, \dots) \\
&(1, 1, 0, 0, 0, \dots) \\
&(1, 1, 1, 0, 0, \dots) \\
&(1, 1, 1, 1, 0, \dots), \dots \Big)
\end{aligned}
$$

Based on this pattern we conjecture that $(a_k(x))_{k \geq 0}$ is convergent with

$$\lim_{k \to \infty} a_k(x) = (1, 1, 1, 1, \dots) = \sum_{n=0}^{\infty} f_n x^n = \frac{1}{1 - x} \quad \text{with } f_n = 1.$$

Let us prove this observation. Let $n \in \mathbb{N}$ be arbitrary but fixed. Then for $l = n + 1$ we get

$$
\begin{aligned}
f_n = 1 = [x^n] a_l(x) = [x^n] \underbrace{a_{n+1}(x)}_{1 + x + \cdots + x^n} \\
= [x^n] \underbrace{a_{n+2}(x)}_{1 + x + \cdots + x^{n+1}} \\
= [x^n] \underbrace{a_{n+3}(x)}_{1 + x + \cdots + x^{n+2}} \\
= a.s.o.
\end{aligned}
$$

$\square$

In other words, we can define the formal power series $\frac{1}{1-x}$ by the limit of $(a_k(x))_{k \geq 0}$.

This example generalizes to

**Theorem 5.20.** *Let* $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{K}[[x]]$. *Then*

$$f(x) = \lim_{n \to \infty} \underbrace{(f_0 + f_1\, x + \cdots + f_n\, x^n)}_{\in \mathbb{K}[[x]]}.$$

*Proof.* **HW 38**. $\square$

In particular, we get limit theorems (analogously to analysis) and thus can nicely calculate with our new formal limit operation.

**Theorem 5.21.** *If* $(a_k(x))_{k \geq 0}$ *and* $(b_k(x))_{k \geq 0}$ *are convergent, then* $(a_k(x))_{k \geq 0} + (b_k(x))_{k \geq 0}$ *and* $(a_k(x))_{k \geq 0} \cdot (b_k(x))_{k \geq 0}$ *are convergent. In particular, we have*

$$\lim_{k \to \infty} (a_k(x) + b_k(x)) = \big( \lim_{k \to \infty} a_k(x) \big) + \big( \lim_{k \to \infty} b_k(x) \big),$$
$$\lim_{k \to \infty} (a_k(x) \cdot b_k(x)) = \big( \lim_{k \to \infty} a_k(x) \big) \cdot \big( \lim_{k \to \infty} b_k(x) \big).$$

**BP 8.** Show the above theorem for the $+$ operation.

## Lecture from May 16, 2023

The final operation in $\mathbb{K}[[x]]$, that we will consider in this lecture, is as follows: given two formal power series, say

$$a(x) = \sum_{n=0}^{\infty} a_n\, x^n \in \mathbb{K}[[x]]$$

and

$$b(x) = \sum_{n=0}^{\infty} b_n\, x^n \in \mathbb{K}[[x]],$$

we want to carry out the composition

$$(a \circ b)(x) = a(b(x)) = \sum_{n=0}^{\infty} a_n b(x)^n.$$

The following questions arise: Under which assumptions can one define such an operation in $\mathbb{K}[[x]]$ by using only finitely many field operations in $\mathbb{K}$? Is $a(b(x))$ again a formal power series? In order to justify and clarify these aspects, we will rely on our lim-construction from above.

**Definition 5.22.** *Let* $(b_n(x))_{n \geq 0}$ *be a sequence of formal power series in* $\mathbb{K}[[x]]$ *and define for* $N \in \mathbb{N}$,

$$a_N(x) := \sum_{n=0}^{N} b_n(x) \in \mathbb{K}[[x]].$$

*If the newly defined sequence $((a_N(x))_{N \geq 0}$ of formal power series in $\mathbb{K}[[x]]$ is convergent, i.e., there is a formal power series $b(x) \in \mathbb{K}[[x]]$ with*

$$\lim_{N \to \infty} a_N(x) = b(x),$$

*we define*

$$\sum_{n=0}^{\infty} b_n(x) := b(x) \left( = \lim_{N \to \infty} \sum_{n=0}^{N} a_n(x) \right).$$

In this case, show that

**HW 39.**

$$[x^k]b(x) = \sum_{n=0}^{\infty} [x^k]b_n(x) (= \lim_{N \to \infty} \sum_{n=0}^{N} [x^k]b_n(x));$$

and

**HW 40.**

$$D_x b(x) = \sum_{n=0}^{\infty} D_x b_n(x) \left( = \lim_{N \to \infty} \sum_{n=0}^{N} D_x b_n(x) \right).$$

**HW 41.** Show that for the sequence $(b_n(x))_{n \geq 0}$ with $b_n(x) = \frac{(1+x)^n}{n!} \in \mathbb{K}[[x]]$ the limit $\sum_{n=0}^{\infty} b_n(x) \left( = \lim_{N \to \infty} \sum_{n=0}^{N} b_n(x) \right)$ does not exist.

A slight generalization of the last homework yields the following criterion when such a limit construction exists.

**Theorem 5.23.** *Let $(b_n(x))_{n \geq 0}$ be a sequence of formal power series in $\mathbb{K}[[x]]$. Then the following statements are equivalent:*

1. $\left( \sum_{n=0}^{N} b_n(x) \right)_{N \geq 0}$ *is convergent, i.e., there exists a $b(x) \in \mathbb{K}[[x]]$ with*

$$\sum_{n=0}^{\infty} b_n(x) = b(x);$$

2. $\lim_{n \to \infty} b_n(x) = 0;$

3. $\lim_{n \to \infty} ord(b_n(x)) = \infty.$

In other words, we can build an infinite sum of power series, if the order of the summands get larger and larger. This property will be crucial to define the composition in the next section. However, we will give a direct argument below – so do not worry if you are lost with Theorem 5.23...

## 5.4 Composition in $\mathbb{K}[[x]]$ (resp. $\mathbb{K}((x))$)

We are now ready to define the composition of

$$a(x) = \sum_{n=0}^{\infty} a_n\, x^n \in \mathbb{K}[[x]]$$

and

$$b(x) = \sum_{n=0}^{\infty} b_n\, x^n \in \mathbb{K}[[x]]$$

by using the construction from above.

The only extra requirement is that

$$b(0) = 0,$$

or equivalently, $b_0 = 0$ holds. With this property we can pullout $x$ and can write

$$b(x) = x\,\tilde{b}(x)$$

for some $\tilde{b}(x) \in \mathbb{K}[[x]]$. Now define

$$b_n(x) := a_n\, b(x)^n = a_n\, x^n \tilde{b}(x)^n \in \mathbb{K}[[x]].$$

Then one can easily see that $\mathrm{ord}(b_n(x)) \geq n$ and thus

$$\lim_{n\to\infty} \mathrm{ord}(b_n(x)) = \infty.$$

Consequently, we can apply Theorem 5.23 and it follows that there is a formal power series $b(x) \in \mathbb{K}[[x]]$ with

$$b(x) = \sum_{n=0}^{\infty} b_n(x) = \sum_{n=0}^{\infty} a_n\, b(x)^n = a(b(x)) = (a \circ b)(x).$$

Note that this construction enables one to compute explicitly the coefficients of $b(x)$. More precisely, with the following arguments we can neglect Theorem 5.23 and can justify our above construction directly:

$$
\begin{aligned}
[x^n] \sum_{n=0}^{\infty} a_n\, b(x)^n ={} & [x^n] \sum_{n=0}^{\infty} a_n\, x^n\, \tilde{b}(x)^n \\
={} & [x^m] a_0 x^0 \tilde{b}(x)^0 \\
& + [x^m] a_1 x^1 \tilde{b}(x)^1 \\
& \;\;\vdots \\
& + [x^m] a_m x^m \tilde{b}(x)^m \\
& + \underbrace{[x^m] a_{m+1} x^{m+1} \tilde{b}(x)^{m+1}}_{0} + \underbrace{[x^m] a_{m+2} x^{m+2} \tilde{b}(x)^{m+2}}_{0} + \underbrace{\cdots}_{0}
\end{aligned}
\tag{30}
$$

In other words, only the first $m + 1$ summands have to be considered in order to get the $m$th coefficient. More precisely, for each of the $m + 1$ summands $a_i x^i \tilde{b}(x)^i$ with $i = 0, \ldots, m$ the $m$th coefficient can be calculated by using only finitely many operations of the addition and the Cauchy product in $\mathbb{K}[[x]]$. In particular, these latter operations, can be accomplished by using only the first coefficients of $a(x)$ and $b(x)$ and a finite number of field operations in $\mathbb{K}$.

The construction of the composition can be summarized in

**Definition 5.24.** *Let $a(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{K}[[x]]$ and $b(x) \in \mathbb{K}[[x]]$ with $b(0) = 0$. Then the composition is defined by*

$$(a \circ b)(x) = a(b(x)) = \sum_{n=0}^{\infty} a_n b(x)^n \in \mathbb{K}[[x]].$$

**Example 5.25.** Let $f(x) = \frac{1}{1-x}$ and $g(x) = x^2$. Since $g(0) = 0$, we can built the composition

$$f(g(x)) = \sum_{k=0}^{\infty} (x^2)^k = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1 - x^2}.$$

**HW 42.** Let $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{1-x} - 1$. Calculate the first 20 coefficients of $f(g(x))$.

Summarizing, we have introduced a new operation, namely composition $a(b(x))$ of $a(x)$ and $b(x)$ that is always possible if $b(0) = 0$. Conversely, if $b(0) \neq 0$ and $a(x)$ is not a polynomial, the above observation (30) does not work, i.e., cannot be defined: the calculation of the $n$th coefficient cannot be carried out by using only a finite number of field operations in $\mathbb{K}$.

**Remark 5.26.** In analysis more general operations are possible: if a series converges, one can define an infinite sum whose summands are from $\mathbb{R}$. In short, if certain series converge, more general compositions might be possible with $b(0) \neq 0$ and $a(x) \notin \mathbb{R}[[x]] \setminus \mathbb{R}[x]$. This case has been presented, e.g., in Example 5.2.

We conclude with another nice property which is connected to the inversion w.r.t. composition.

**Example 5.27.** Consider

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \in \mathbb{K}[[x]],$$

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \in \mathbb{K}[[x]].$$

We observe the following:

- Since $\mathrm{ord}(\log(1 + x)) = 1$, we can carry out the composition: there is a formal power series $g(x) \in \mathbb{K}[[x]]$ with
$$g(x) = \exp(\log(1 + x)).$$

- Using analysis we get
$$\exp(\log(1 + x)) = 1 + x$$

  as functions.

Thus by the transfer principle (see Theorem 5.7) we conclude that
$$\exp(\log(1 + x)) = g(x) = 1 + x$$

holds also in $\mathbb{K}[[x]]$. Now define
$$b(x) := \exp(x) - 1 \in \mathbb{K}[[x]],$$
$$B(x) = \log(1 + x) \in \mathbb{K}[[x]]$$

with $\mathrm{ord}(b(x)) = 1 = \mathrm{ord}(B(x))$. Then by $\exp(\log(1 + x)) = 1 + x$ we conclude that
$$b(B(x)) = x.$$

Analogously we can conclude that
$$x = B(b(x)) = \log(\exp(x))$$

holds.

More generally, we have the following theorem.

**Theorem 5.28.** *Let $b(x) \in \mathbb{K}[[x]]$ with[12] $\mathrm{ord}(b(x)) = 1$. Then there is a unique $B(x) \in \mathbb{K}[[x]]$ with $\mathrm{ord}(B(x)) = 1$ such that*
$$\boxed{B(b(x)) = x = b(B(x)).}$$

**Remark 5.29.** The above theorem is constructive: the corresponding $B$ can be computed with the Lagrange inversion formula; further details are skipped.

**Example 5.30.** Consider the formal power series
$$T(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4x} = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n}(-1)^n 4^n x^n \in \mathbb{K}[[]x]]$$

with $\mathrm{ord}(T(x)) = 1$. By Theorem 5.28 we conclude that there is a unique $S(x) \in \mathbb{K}[[x]]$ with $\mathrm{ord}(S(x)) = 1$ and
$$S(T(x)) = T(S(x)) = x.$$

**HW 43.** Find $S(x) \in \mathbb{K}[[x]]$. (Hint use the property $T(x)^2 - T(x) + x = 0$.)

Let us summarize the operations in $\mathbb{K}[[x]]$ that have been introduced in this lecture:

1. component-wise addition $+$ (see Definition 3.2);

2. scalar multiplication $*$ (see Definition 3.2);

---

[12]Note that $b(0) = 0$ holds.

3. Hadamard products ∘ (see Definition 3.7);

4. Cauchy products · (see Definition 3.7);

5. coefficient functional $[x^n]$ (see Definition 3.16);

6. differentiation/integration $D_x, \int_x$ (see Lemma 3.24 and Definition 3.26);

7. multiplicative inverse (see Theorem 3.29);

8. limit construction (see Definition 5.18);

9. composition (see Definition 5.24).

10. inversion of composition (see Theorem 5.28).

# 6  c-finite sequences

Later we will consider recurrences where the coefficients may depend on $n$ (see, e.g., (2) and (21)). But in this section we will restrict to recurrences which are homogeneous and where the coefficients do not depend on the shift-variable.

**Definition 6.1.** *We call a sequence $(a_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}}$ c-finite of order $r$ if there exists a linear recurrence of order $r$ with constant coefficients that has $(a_n)_{n \geq 0}$ as solution. This means that there are $c_0, \ldots, c_r \in \mathbb{K}$ with $c_r \neq 0$ such that*

$$c_r \, a_{n+r} + c_{r-1} \, a_{n+r-1} + \cdots + c_0 \, a_n = 0 \quad (n \in \mathbb{N}).$$

*We call such a recurrence with constant coefficients also c-finite recurrence.*

The Fibonacci numbers $F_n$ with $n \geq 0$ are one of the most prominent examples of c-finite sequences. They appear in Leonardo di Pisa's work ($\sim$ 1200) and can be defined by the following c-finite recurrence relation:

$$\boxed{\begin{aligned} &F_{n+2} = F_{n+1} + F_n \qquad (n \geq 0) \\ &F_0 = 0 \\ &F_1 = 1. \end{aligned}} \tag{31}$$

Together with the initial values we get the following first values

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|-----|---|---|---|---|---|---|---|----|----|-----|
| $F_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | ... |

$F_n$ counts, e.g., the number of rabbit couples under the following assumptions: a fresh born couple needs 2 month to generate its first couple. From that month on it regularly produces a new couple each month. In this model we suppose in addition that all the born couples behave the same and never die. Then the tree of growth looks as follows:
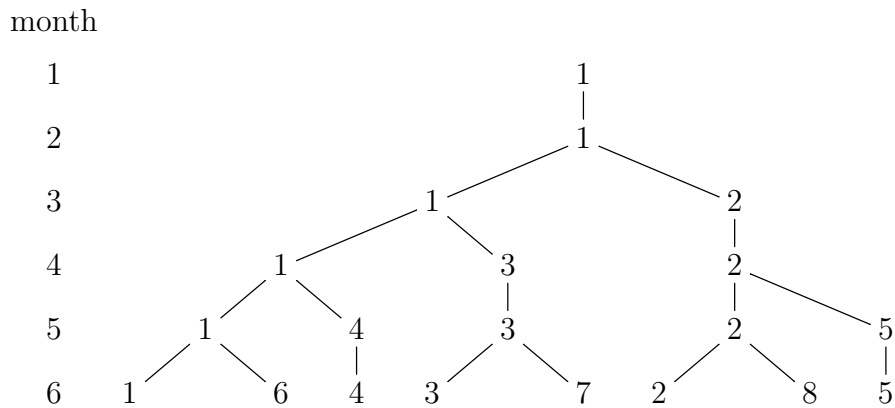
**Example 6.2.** Given $F_n$ by its defining recurrence relation and initial values (31), we can derive its generating function as follows:

$$
\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} F_n\, x^n \\
&= F_0 + F_1\, x + \sum_{n=0}^{\infty} F_{n+2}\, x^{n+2} \\
&= x + \sum_{n=2}^{\infty} (F_n + F_{n+1})\, x^{n+2} \\
&= x + \sum_{n=0}^{\infty} F_n\, x^{n+2} + \sum_{n=0}^{\infty} F_{n+1}\, x^{n+2} \\
&= x + x^2 \sum_{n=0}^{\infty} F_n\, x^n + x \sum_{n=1}^{\infty} F_{n+1}\, x^{n+1} \\
&= x + x^2 F(x) + x F(x)
\end{aligned}
$$

and thus

$$
(1 - x - x^2) F(x) = x
$$

or equivalently

$$
F(x) = \frac{x}{1 - x - x^2}. \tag{32}
$$

This means that $F(x)$ equals the inverse element of $1 - x - x^2$ times $x$. In short, the closed form of $F(x)$ is a rational function in $\mathbb{K}(x)$.

More generally, we obtain the following result.

**Theorem 6.3.** *Let* $(a_n)_{n \geq 0}$ *be a sequence in* $\mathbb{K}$.

$(a_n)_{n \geq 0}$ *satisfies a c-finite recurrence*

$$
a_{n+r} + c_{r-1} a_{n+r-1} + \cdots + \underbrace{c_0}_{\neq 0}\, a_n = 0, \quad n \in \mathbb{N}
$$

*with* $c_0, \ldots, c_{r-1} \in \mathbb{K}$

$$\Updownarrow$$

$$
\sum_{n=0}^{\infty} a_n\, x^n = \frac{p(x)}{1 + c_{r-1} x + \cdots + c_0\, x^r}
$$

*for some* $p(x) \in \mathbb{K}[x]$ *with* $\deg(p(x)) < r$.

*Proof.* **HW 44.** Hint: for $\Downarrow$ see the above example; for $\Uparrow$ do coefficient comparison in

$$
\left(1 + c_{r-1} x + \cdots + c_0\, x^r\right) \sum_{n=0}^{\infty} a_n\, x^n = p(x).
$$

$\square$

## Lecture from May 23, 2023

Note that the above theorem is constructive if the field operations in $\mathbb{K}$ can be computed. Namely, suppose that $(a_n)_{n\geq} \in \mathbb{K}^{\mathbb{N}}$ is given by a $c$-finite recurrence of order $r$

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + \underbrace{c_0}_{\neq 0} a_n = 0 \tag{33}$$

for some $c_0, \ldots, c_{r-1} \in \mathbb{K}$ and initial values $a_0, \ldots, a_{r-1} \in \mathbb{K}$. Then one can compute $p(x) \in \mathbb{K}[x]$ with $\deg(p(x)) < r$ such that

$$\sum_{n=0}^{\infty} a_n\, x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_0\, x^r}.$$

Conversely, suppose that we are a given formal power series $\sum_{n=0}^{\infty} a_n\, x^n$ whose closed form is a rational function $\frac{p(x)}{1+c_{r-1}x+\cdots+c_0\, x^r}$ for some $p(x) \in \mathbb{K}[x]$ with $\deg(p) < r$. Then the sequence $(a_n)_{\geq 0}$ satisfies the $c$-finite recurrence (33).

**Example 6.4** (cont. Ex, 6.2)**.** Consider the Fibonacci numbers with the defining equation (31). Then by Theorem 6.3 we know that

$$\sum_{n=0}^{\infty} F_n x^n = \frac{p(x)}{1 - x - x^2}$$

with $p(x) = \alpha\, x + \beta$ where $\alpha, \beta \in \mathbb{K}$. Equivalently we have

$$(1 - x - x^2)\sum_{n=0}^{\infty} F_n x^n = p(x). \tag{34}$$

By looking at the constant term on both sides we conclude

$$1 \cdot F_0 = p(0) = \beta$$

and thus $\beta = 0$. In order to get a constraint for $\alpha$ we apply $D_x$ to (34) (i.e., shift the sequence to the left) and take again the costant term. Then we get:

$$\text{LHS} = [x^0](-1 - 2x)\sum_{n=0}^{\infty} F_n x^n + (1 - x - x^2)\sum_{n=1}^{\infty} nF_n x^{n-1} = -F_0 + F_1 = 1,$$

$$\text{RHS} = [x^0]p'(x) = \alpha.$$

Thus $\alpha = 1$ and we get $p(x) = x$. In other words, we have rediscovered (rederived) the identity (35).

## 6.1  Solving $c$-finite recurrences

On page 13 you have seen how one can solve an arbitrary first-order recurrence using the "variation of constant" method. In the following we will restrict to $c$-finite recurrences but generalize to the situation that the recurrences may have any order.

More precisely, we will present two different (but related) tactics how one can compute a closed form solution of a given $c$-finite recurrence of order $r$. For simplicity, we consider again our Fibonacci sequence $(F_n)_{n\geq 0}$ defined by (31).

**Approach 1: usage of the rational function representation.**

Using Theorem 6.3 (or our explicit calculation in Example 6.2) we have

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}. \tag{35}$$

Exploiting this special closed form we can derive the solution of the recurrence

$$F_{n+2} - F_{n+1} - F_n = 0$$

together with its initial values $F_0 = 0$ and $F_1 = 1$ in closed form.

More precisely, let us factorize the denominator $1 - x - x^2$ in (35) in the form

$$1 - x - x^2 = (1 - r_+ x)(1 - r_- x);$$

here we can use any computer algebra system and get

$$r_+ = \frac{1 + \sqrt{5}}{2}, \qquad\qquad r_- = \frac{1 - \sqrt{5}}{2}.$$

Hence by partial fraction decomposition (PFD) we can write

$$F(x) = \frac{A}{1 - r_+ x} + \frac{B}{1 - r_- x}$$

for constants $A, B$, free $x$.

How can one determine $A, B$, i.e., how can one carry out the partial fraction decomposition? Here we indicate three possibilities:

1. Use your favorite computer algebra system. E.g., in Mathematica use the command Apart[F,x].

2. One can activate the Extended Euclidean algorithm. More precisely, let $p(x), q(x) \in \mathbb{K}[x]$ with $\gcd(p, q) = 1$. Then there are $\alpha(x), \beta(x) \in \mathbb{K}[x]$ such that

$$\alpha(x)p(x) + \beta(x)q(x) = 1(= \gcd(p(x), q(x)))$$

$$\Updownarrow$$

$$\frac{\alpha(x)}{q(x)} + \frac{\beta(x)}{p(x)} = \frac{1}{p(x)q(x)}.$$

Thus one can read off $A$ and $B$ (using some further polynomial divisions).

3. Often also the traditional method is fast. Looking at the ansatz

$$F(x) = \frac{A}{1 - r_+ x} + \frac{B}{1 - r_- x}$$

we get

$$
\begin{aligned}
A &= (1 - r_+ x)F(x)\big|_{x \mapsto \frac{1}{r_+}} \\
&= \frac{(1 - r_+ x)x}{(1 - r_+ x)(1 - r_- x)}\bigg|_{x \mapsto \frac{1}{r_+}} \\
&= \frac{x}{1 - r_- x}\bigg|_{x \mapsto \frac{1}{r_+}} \\
&= \frac{1/r_+}{1 - r_-/r_+} = \frac{1}{r_+ - r_-} = \frac{1}{\sqrt{5}}
\end{aligned}
$$

and

$$B = (1 - r_- x)F(x)\big|_{x \mapsto \frac{1}{r_-}} = \dots = \frac{1}{r_- - r_+} = -\frac{1}{\sqrt{5}}.$$

Thus

$$
\begin{aligned}
F(x) &= \frac{1}{\sqrt{5}}\frac{1}{1 - r_+ x} - \frac{1}{\sqrt{5}}\frac{1}{1 - r_- x} \\
&= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty} r_+^n x^n - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty} r_-^n x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}\left(r_+^n - r_-^n\right)x^n
\end{aligned}
$$

which gives the Euler–Binet formula (1843): for all $n \in \mathbb{N}$,

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(r_+^n - r_-^n\right).}$$

**Approach 2: finding all solutions via roots.**

We make the ansatz

$$F_n = \alpha^n$$

for an indeterminate $\alpha \in \mathbb{C}^*$. Thus

$$
\begin{aligned}
0 &= F_{n+2} - F_{n+1} - F_n \\
&= \alpha^{n+2} - \alpha^{n+1} - \alpha^n \\
&= \underbrace{\alpha^n}_{\neq 0}(\alpha^2 - \alpha - 1)
\end{aligned}
\tag{36}
$$

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and consequently $\alpha$ is a root of the polynomial

$$p(x) = x^2 - x - 1 = (x - r_+)(x - r_-). \tag{37}$$

Hence $r_+^n$ and $r_-^n$ are solutions of (36). In particular, by HW 45a,

$$G_n = A\, r_+^n + B r_-^n$$

with $A, B \in \mathbb{C}$ is a solution of (36).

Note:

$$V := \{A\, (r_+^n)_{n \geq 0} + B(r_-^n)_{n \geq 0} \mid A, B \in \mathbb{C}\}$$

is a vector space over $\mathbb{C}$. In particular, by HW 45b,

$$(r_+^n)_{n \geq 0}, \quad (r_-^n)_{n \geq 0}$$

is a basis of $V$. Moreover, by HW 46 it follows that $V$ contains all solutions of (36), i.e.,

$$V = \{(a_n) \in \mathbb{C}^n \mid a_{n+2} - a_{n+1} - a_n = 0 \text{ for all } n \geq 0\}.$$

**HW 45.** Let $r_+$ and $r_- \in \mathbb{C}$ be the roots of $q(x) = x^2 - x - 1 \in \mathbb{C}[x]$.

(a) Check that

$$A\, (r_+)^n + B\, (r_-)^n$$

with $A, B \in \mathbb{C}$ are solutions of the c-finite recurrence $a_{n+2} - a_{n+1} - a_n = 0$.

(b) In particular, show that $((r_+)^n)_{n \geq 0}$ and $((r_-)^n)_{n \geq 0}$ are linearly independent over $\mathbb{C}$.

**HW 46.** Define

$$V = \{(a_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}} \mid a_{n+2} - a_{n+1} - a_n = 0 \quad \forall n \in \mathbb{N}\}.$$

Show that $V = \{A\, ((r_+)^n)_{n \geq 0} + B\, ((r_-)^n)_{n \geq 0} \mid A, B \in \mathbb{C}\}$.

In the above example we were lucky: the polynomial in (37) has two different roots and thus yields two different solutions. What happens if the corresponding polynomial has roots with higher multiplicity. Let us consider the most degenerated case: a polynomial which has only one root (with non-trivial multiplicity).

**Example 6.5.** Consider the sequence $(a_n)_{n \geq 0}$ defined be the c-finite recurrence

$$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0 \tag{38}$$

and the initial values $a_0 = 0$, $a_1 = 1$ and $a_2 = 3$.

As above we make the ansatz

$$f_n(x) = x^n$$

with $x \in \mathbb{C}$ for a solution of (38). This gives

$$\begin{aligned} 0 =& f_{n+3}(x) - 3f_{n+2}(x) + 3f_{n+1}(x) - f_n(x) \\ =& x^n(x^3 - 3x^2 + 3x - 1) = x^n(x - \underbrace{1}_{=\alpha})^3 \end{aligned} \tag{39}$$

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for all $n \in \mathbb{N}$. Thus with $x = \alpha = 1$ we get a solution $(1^n)_{n \geq 0}$ of (38).

Unfortunately, there is only 1 root (but with multiplicity 3). However, a recurrence of order 3 has 3 linearly independent solutions. Can we determine the missing two solutions whose linear span provides all solutions?

The answer is **yes** by the following trick: We apply the differential operator $D_x$ to (39) and find

$$
\begin{aligned}
0 =& f'_{n+3}(x) - 3f'_{n+2}(x) + 3f'_{n+1}(x) - f'_n(x) \\
=& nx^{n-1}(x-1)^3 + x^n 3(x-1)^2 = (x-1)^2 q_1(x)
\end{aligned}
$$

with $q_1(x) = nx^{n-1}(x-1) + 3x^n = x^{n-1}(n(x-1) - x)$. Thus

$$
f'_n(x)|_{x \mapsto 1} = n \cdot 1^{n-1} = n
$$

is a solution of (38). Repeating the trick by applying $D_x^2$ to (39) we obtain

$$
\begin{aligned}
0 =& f''_{n+3}(x) - 3f''_{n+2}(x) + 3f''_{n+1}(x) - f''_n(x) \\
=& (x-1)^1 q_2(x)
\end{aligned}
$$

for some polynomial expression $q_2(x)$. Thus

$$
f''_n(x)|_{x \mapsto 1} = n(n-1) \cdot 1^{n-2} = n(n-1)
$$

is also a solution of (38). Note that applying this trick once more, i.e., applying $D_x^3$ to (39) does not yield any extra solution because on the right-hand side the factor $(x-1)$ disappears (which turned the right-hand side to zero when one sets $x$ to 1).

Summarizing, we find the set of solutions

$$
L = \{c_1(1)_{n \geq 0} + c_2(n)_{n \geq 0} + c_2(n(n-1))_{n \geq 0} \mid c_1, c_2, c_3 \in \mathbb{K}\}.
$$

As it turns out the three solutions are linearly independent, i.e.,

$$
c_1(1)_{n \geq 0} + c_2(n)_{n \geq 0} + c_2(n(n-1))_{n \geq 0} = 0
$$

if and only if $c_1 = c_2 = c_3 = 0$. Since there can be at most three linearly independent solutions, $L$ contains all solutions and thus $(a_n)_{n \geq 0} \in L$. Consequently, there exist $c_1, c_2, c_3 \in \mathbb{K}$ such that

$$
a_n = c_1 \, 1 + c_2 \, n + c_3 \, n(n-1) \quad (n \in \mathbb{N}).
$$

In order to determine the $c_i$ we look at the special cases $n = 0, 1, 2$:

$$
\begin{aligned}
0 &= c_1 \\
1 &= c_1 &&+ &&c_2 \\
3 &= c_1 &&+ &&2c_2 &&+ &&2c_3.
\end{aligned}
$$

Solving this linear system yields the unique choice $c_1 = 0$, $c_2 = 1$ and $c_3 = \frac{1}{2}$. Consequently

$$
a_n = n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1) \quad (n \in \mathbb{N})
$$

Note that we can verify this result also by the following argument: both sides are a solution of the same recurrence of order 3 and both sides agree with the first three values $n = 0, 1, 2$. Thus they must agree for all $n$.

Combining the ideas from the two examples above (take all roots $\alpha_1, \ldots, \alpha_k$ of the corresponding polynomial and extract further solutions by applying the differential operator $D_x^i$ for $i = 1, \ldots, m_i - 1$ where $m_i$ is the multiplicity of the root $\alpha_i$) produces all solutions of a given $c$-finite recurrence. More precisely, we obtain the following general result.

**Theorem 6.6.** *For $c_0, \ldots, c_{r-1} \in \mathbb{K}$ with $c_0 \neq 0$ suppose that*

$$p(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_1\,x + c_0 = (x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2} \ldots (x - \alpha_k)^{m_m}$$

*is a complete factorization (i.e., $m_i \in \mathbb{N}^*$ with $m_1 + m_2 + \cdots + m_k = r$ and the[13] $\alpha_i \in \mathbb{K}$ are pairwise different). Then the $r$ sequences*

$$
\begin{array}{ccccc}
(\alpha_1^n)_{n \geq 0} & (n\,\alpha_1^n)_{n \geq 0} & (n^2\,\alpha_1^n)_{n \geq 0} & \ldots & (n^{m_1-1}\,\alpha_1^n)_{n \geq 0} \\
(\alpha_2^n)_{n \geq 0} & (n\,\alpha_2^n)_{n \geq 0} & (n^2\,\alpha_2^n)_{n \geq 0} & \ldots & (n^{m_1-1}\,\alpha_2^n)_{n \geq 0} \\
\vdots & & & & \\
(\alpha_k^n)_{n \geq 0} & (n\,\alpha_k^n)_{n \geq 0} & (n^2\,\alpha_k^n)_{n \geq 0} & \ldots & (n^{m_k-1}\,\alpha_k^n)_{n \geq 0}
\end{array}
$$

*form a basis of the $\mathbb{K}$-vector space*

$$\{(a_n)_{n \geq 0} \mid a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1\,a_{n+1} + c_0\,a_n = 0 \text{ for all } n \geq 0\}.$$

**BP 9.** Prove Theorem 6.6 for the special case $r = 2$.


**BP 10.** Prove Theorem 6.6 for the special case $r = 3$.


# Lecture from June 6, 2023

## 6.2 Application: AVL-trees

Recall the definition of a binary search tree from Section 1.4: It is a binary tree where in each subtree with root $x$ all the elements in its left subtree are smaller than $x$ and all elements in its right subtree are larger than $x$.

We refine this definition as follows: an AVL-tree is a binary search tree in which for each node the heights of the two subtrees differ at most by 1.

**Example 6.7.** Consider the binary search tree



---
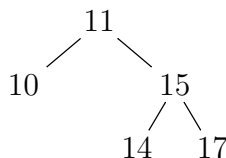
[13]Note that $p(x)$ must split into linear factors in $\mathbb{K}$; if this is not possible one has to extend the field $\mathbb{K}$.

which actually is an AVL-tree. However, after inserting 18 into this binary search tree we get

```
            11
          /    \
        10      15
               /  \
             14    17
                     \
                      18
```
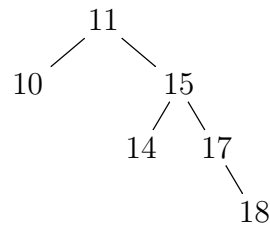
which is not an AVL-tree anymore: the right subtree of 11 has height 3 but the left subtree has height 1. As it turns out, one can rotate such an almost-AVL-tree (an AVL-tree which got spoiled by one extra insertion). More precisely, by choosing the first node to the right as new root (in our case 15) and changing some pointers, one gets another binary search tree

```
            15
          /    \
        11      17
       /  \       \
     10    14      18
```
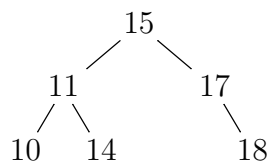
which is again an AVL-tree.

In general, the following holds for an AVL-tree $T$ with height $h$:

1. The insertion/deletion of an extra element in $T$ yields a new binary search tree (which is not necessarily an AVL tree). Here the worst case complexity is in $O(h)$ (in worst case one has to walk down the longest possible path which has length $h$).

2. In $O(1)$ steps (by changing some pointers) one can rebalance the derived binary search tree to an AVL-tree in case of an insertion; and in $O(h)$ steps one can rebalance the tree to an AVL-tree in case of a deletion; for further details we refer to the standard literature that deals with algorithms and data structures.

Combining the two steps one obtains the following result: in worst case one can insert/delete an element of an AVL-tree in $O(h)$ steps and obtains again an AVL-tree. In particular, one can search an element in an AVL-tree with at most $O(h)$ operations.

In this lecture we will not explore further this rebalancing process for insertion and deletion. But we will demonstrate the presented toolbox from above. Namely, we will show that the worst case height of an AVL tree with $n$ stored elements is $O(\log(n))$. In other words we can show that the search, the deletion and insertion of an element in an AVL-tree requires at most $O(\log(n))$ operations!

In order to obtain this result, we consider the worst situation: fix the height $h$ and construct the most stupid AVL-tree (in the sense of storing elements), i.e., a tree with as less nodes as possible. More precisely, we will compute

$$M(h) = \text{the minimal number of nodes in an AVL-tree with height } h.$$
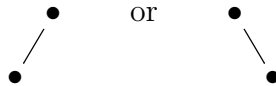
The only binary search tree of height 0 has exactly one element:
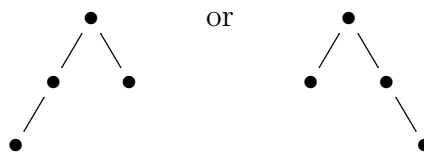
•

In particular, we have

$$M(0) = 1.$$

If one wants to construct an AVL-tree with height 1 with as less nodes as possible one ends up at



Thus

$$M(1) = 2.$$

If one wants to obtain an AVL-tree of height 2 with a minimal number of nodes, one finds the following two shapes:



Consequently

$$M(2) = 4.$$

More generally, if one wants to obtain an AVL-tree with height $h$ ($h \geq 2$) with as less nodes as possible, one needs one node for the root plus two subtrees where

- one must have height $h - 1$ (in order to get height $h$); here $M(h-1)$ are the minimal number nodes to manage this.

- the other subtree should have only height $h - 2$ – otherwise we spent too many nodes; here $M(h-2)$ is the minimal number of nodes.

In total, we get

$$
\begin{aligned}
M(h) &= 1 + M(h-1) + M(h-2) \qquad (h \geq 2) \\
M(0) &= 1, \\
M(1) &= 2.
\end{aligned}
\tag{40}
$$

This is all we need to run our $c$-finite machinery! First, we compute a homogeneous version (compare Remark 4.5) and get

$$
\begin{array}{ccccc}
M(h) & -M(h-1) & -M(h-2) & & = 1 \\
- & M(h-1) & -M(h-2) & -M(h-3) = 1 \\
\hline
M(h) & -2M(h-1) & & +M(h-3) = 0.
\end{array}
$$

Now we activate Theorem 6.6 that is built into Mathematica:

In[15]:= **RSolve[M[h] − 2M[h − 1] + M[h − 3] == 0, M[h], h]//FullSimplify**

Out[15]= $\left\{\left\{\texttt{M[h]} \rightarrow c_1 \left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^h + c_2 \left(\frac{1}{2}\left(1 + \sqrt{5}\right)\right)^h + c_3\right\}\right\}$

This means that

$$\left\{c_1 \left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^h + c_2 \left(\frac{1}{2}\left(1 + \sqrt{5}\right)\right)^h + c_3 \mid c_1, c_2, c_3 \in \mathbb{C}\right\}$$

provides the full solution space and thus there exist $c_1, c_2, c_3 \in \mathbb{C}$ such that

$$M(h) = c_1 \left( \frac{1}{2} \left( 1 - \sqrt{5} \right) \right)^h + c_2 \left( \frac{1}{2} \left( 1 + \sqrt{5} \right) \right)^h + c_3$$

holds for all $h \in \mathbb{N}$. Finally, looking at $h = 0, 1, 2$ we get the linear system

$$1 = c_1 + c_2 + c_3$$
$$2 = c_1 \frac{1}{2} \left( 1 - \sqrt{5} \right) + c_2 \frac{1}{2} \left( 1 + \sqrt{5} \right) + c_3$$
$$4 = c_1 \left( \frac{1}{2} \left( 1 - \sqrt{5} \right) \right)^2 + c_2 \left( \frac{1}{2} \left( 1 + \sqrt{5} \right) \right)^2 + c_3.$$

Solving it (using the Gauss algorithm) yields

$$c_1 = \frac{1}{5}(5 - 2\sqrt{5}),$$
$$c_2 = \frac{1}{5}(5 + 2\sqrt{5}),$$
$$c_3 = -1.$$

Remark: Providing in addition the initial values to `RSolve`, this calculation can be carried out in one stroke:

In[16]:= **RSolve[M[h] − 2M[h − 1] + M[h − 3] == 0**
&&**M[0] == 1&&M[1] == 2&&M[2] == 4, M[h], h]//FullSimplify**

Out[16]= $\frac{1}{5} \left( \left( 5 - 2\sqrt{5} \right) \left( \frac{1}{2} \left( 1 - \sqrt{5} \right) \right)^h + \left( 5 + 2\sqrt{5} \right) \left( \frac{1}{2} \left( 1 + \sqrt{5} \right) \right)^h - 5 \right)$

More precisely, we get

$$M(h) = -1 + \frac{5 - 2\sqrt{5}}{5} r_-^h + \frac{5 + 2\sqrt{5}}{5} r_+^h. \tag{41}$$

Since $r_- < 1$ one can easily see that $M(h)$ is asymptotically equal to $\frac{5 + 2\sqrt{5}}{5} r_+^h$, i.e.,

$$M(h) \sim \frac{5 + 2\sqrt{5}}{5} r_+^h.$$

Furthermore,

$$\frac{5 + 2\sqrt{5}}{5} r_+^h - 1 < M(h) < \frac{5 + 2\sqrt{5}}{5} r_+^h + 1$$

if $h$ is sufficiently large. Now take any AVL-tree with $n$ nodes and let $h$ be its maximal height. Then

$$M(h) \leq n.$$

**HW 47.** Show that that

$$h \leq 1.44 \log_2(n) + \text{const}.$$

This shows that the worst case to find/delete/insert and element is in $O(\log(n))$.

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## 6.3 Closure properties

The following powerful machinery allows one to prove beautiful identities in a jiffy.

**Theorem 6.8.** *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be c-finite sequences in $\mathbb{K}$ of order $r$ and $s$, respectively. Then the following holds.*

1. *$(a_n + b_n)_{n\geq 0}$ is c-finite of order $\leq r + s$.*

2. *$(\sum_{k=0}^{n} a_k\, b_{n-k})_{n\geq 0}$ is c-finite of order $\leq r + s$.*

   *In particular, we get the special case: $(\sum_{k=0}^{n} a_k)_{n\geq 0}$ is c-finite of order $\leq r + 1$.*

3. *$(a_n \cdot b_n)_{n\geq 0}$ is c-finite of order $\leq r \cdot s$.*

4. *For any $m \in \mathbb{N}^*$ and $p \in \{0, 1, \ldots, m-1\}$: $(a_{m\cdot n + p})_{n\geq 0}$ is c-finite of order $\leq r$.*

5. *For any $m \in \mathbb{N}^*$: $(a_{\lfloor \frac{n}{m} \rfloor})_{n\geq 0}$ is c-finite of order $\leq m \cdot r$.*

*In particular, if the c-finite recurrences of $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of order $r$ and $s$ are given explicitly and the field operations of $\mathbb{K}$ can be computed, the c-finite recurrences in statements (1)–(5) can be computed.*

The constructive versions of this theorem has been implemented, e.g., in Mallinger's package `GeneratingFunctions`.

**Example 6.9.** Prove
$$F_{2n} = 2\, F_n\, F_{n+1} - F_n^2, \quad n \in \mathbb{N}. \tag{42}$$
Proof strategy: derive a recurrence for

$$a_n := \underbrace{F_{2n}}_{\substack{\text{rec of} \\ \text{order} \leq 2}} - \underbrace{\underbrace{2\, F_n\, F_{n+1}}_{\substack{\text{rec of} \\ \text{order} \leq 4}} + \underbrace{F_n^2}_{\substack{\text{rec of} \\ \text{order} \leq 4}}}_{\text{rec of order} \leq 4 + 4 = 8} :$$

$$\underbrace{\phantom{a_n := F_{2n} - 2 F_n F_{n+1} + F_n^2}}_{\text{rec of order} \leq 2 + 8 = 10}$$

One option is to compute the recurrence of $a_n$ explicitly (see the homework below). However, for the proof of (42) or equivalently for the proof of

$$a_n = 0 \quad \forall n \geq 0 \tag{43}$$

it suffices to bound the order of recurrence of $a_n$. We start with the well known Fibonacci recurrence (36) which has order 2. Thus by part (3) of Theorem 6.8 it follows that $2\, F_n\, F_{n+1}$ and $-F_n^2$ have c-finite recurrences of order $\leq 4$. But then $2\, F_n\, F_{n+1} - F_n^2$ is c-finite of order $\leq 8$ by part (1) of Theorem 6.8. Finally, by part (4) of Theorem 6.8 it follows that $F_{2n}$ is c-finite of order $\leq 2$ and thus with part (1) of Theorem 6.8 the final sequence $a_n$ is c-finite of order $\leq 10$. Finally, we verify that the first 10 values of $a_n$ are zero

$$a_0 = a_1 = a_2 = \cdots = a_9 = 0$$

which proves (43) and thus (42).

*Remark.* Often students in the first year claim to have a proof of an identity by demonstrating that it is correct for the first instances. In general, this is of course nonsense and the teacher will show a tedious and lengthy proof. However in the *c*-finite world, this naive approach is completely valid! Check the first instances in order to obtain a rigorous proof. *Warning:* we are only allowed to do this by a strong theory that is hidden behind and which tells us how many values we have to check. So one first has to do the work (and prove Theorem 6.8) but afterwards one can prove identities as above in the simplest way (without cheating!).

**HW 48.** Use `GeneratingFunctions.m` (or another computer algebra package) to derive a *c*-finite recurrence for $(a_n)_{n \geq 0}$ with $a_n = F_{2n} - 2F_n F_{n+1} + F_n^2$.

**HW 49.** Prove $F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$ for $n \geq 1$.

**HW 50.** Prove $\sum_{n=0}^{\infty} F_n^3 x^n = \frac{x(1-2x-x^2)}{(1-4x-x^2)(1+x-x^2)}$.

**HW 51.** Use `GeneratingFunctions.m` (or another computer algebra package) to derive a *c*-finite recurrence for Kepler's identity[14]: $F_{n+1}F_{n-1} - F_n = (-1)^n$. Verify the correctness of the identity.

**HW 52.** Find/prove $\sum_{k=0}^{n} F_k = F_{n+2} - 1$ for $n \in \mathbb{N}$.

After exploiting this wonderful machinery, let us look behind the scene and prove it (at least in parts).

**Proof of Theorem 6.8.** Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be *c*-finite sequences in $\mathbb{K}$ of order $r$ and $s$, respectively. With Theorem 6.3 we get

$$\sum_{n=0}^{\infty} a_n x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + \underbrace{c_0}_{\neq 0} x^r}, \quad \deg(p) < r,$$

$$\sum_{n=0}^{\infty} b_n x^n = \frac{q(x)}{1 + d_{s-1}x + \cdots + \underbrace{d_0}_{\neq 0} x^s}, \quad \deg(q) < s$$

for some $p(x), q(x) \in \mathbb{K}[x]$.
(1) By Theorem 5.7 it follows that

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \frac{\tilde{p}(x)}{(1 + c_{r-1}x + \cdots + c_0 x^r)(1 + d_{s-1}x + \cdots + d_0 x^s)}$$

---

[14]This identity is often named as Cassini's identity. However, Peter Paule observed that this identity was discovered much earlier by Johannes Kepler. Since this lectures takes place at the Johannes Kepler University (JKU), it seems very appropriate to rename it accordingly and to give proper credits to Johannes Kepler.

with $\deg(\tilde{p}(x)) < r + s$. Thus by Theorem 6.3 there exists a recurrence of $(a_n + b_n)_{n \geq}$ of order[15] $\leq r + s$.

(2) By the Cauchy product and Theorem 5.7 it follows that

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k \, b_{n-k} = \Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \frac{p(x)\,q(x)}{(1 + c_{r-1}x + \cdots + c_0 x^r)(1 + d_{s-1}x + \cdots + d_0 x^s)}$$

with $\deg(p(x)q(x)) < r + s$. Thus by Theorem 6.3 there exists a recurrence of $(\sum_{k=0}^{n} a_k \, b_{n-k})_{n \geq}$ of order $\leq r + s$.

(3) For simplicity we restrict to the case $r = s = 2$. The general case can be carried over straightforwardly. The goal is to find a recurrence of the form

$$x_4 \, (a_{n+4} \, b_{n+4}) + x_3 \, (a_{n+3} \, b_{n+3}) + \cdots + x_0 \, (a_n \, b_n), \quad n \in \mathbb{N} \tag{44}$$

of order $\leq r + s = 4$ with $x_0, \ldots, x_4 \in \mathbb{K}$, not all zero.

Using the fact that $a_n$ and $b_n$ satisfy $c$-finite recurrences of order 2, say

$$a_{n+2} + c_1 \, a_{n+1} + c_0 \, a_n = 0$$
$$b_{n+2} + d_1 \, a_{n+1} + d_0 \, a_n = 0,$$

we can simplify the arising expressions in (44) as follows:

$$a_{n+2} b_{n+2} = (-c_1 \, a_{n+1} - c_0 \, a_n)(-d_1 \, b_{n+1} - d_0 \, b_n)$$
$$= \mathrm{LinComb}(\underbrace{a_{n+1} \, b_{n+1}}_{=h_{1,1}}, \underbrace{a_{n+1} \, b_n}_{=h_{1,0}}, \underbrace{a_n \, b_{n+1}}_{=h_{0,1}}, \underbrace{a_n \, b_n}_{=h_{0,0}}).$$

Similarly, we get

$$a_{n+3} b_{n+3} = \mathrm{LinComb}(a_{n+2} \, b_{n+2}, a_{n+2} \, b_{n+1}, a_{n+1} \, b_{n+2}, a_{n+1} \, b_{n+1})$$

and using the earlier reduction

$$= \mathrm{LinComb}(h_{1,1}, h_{1,0}, h_{0,1}, h_{0,0}).$$

Repeating these reductions yields

$$a_{n+4} b_{n+4} = \mathrm{LinComb}(h_{1,1}, h_{1,0}, h_{0,1}, h_{0,0}).$$

Hence (44) can be rewritten for some $\alpha_{i,j} \in \mathbb{K}$ as

$$0 = x_4 \big(\alpha_{1,1}^{(4)} h_{1,1} + \alpha_{1,0}^{(4)} h_{1,0} + \alpha_{0,1}^{(4)} h_{0,1} + \alpha_{0,0}^{(4)} h_{0,0}\big)$$
$$+ x_3 \big(\alpha_{1,1}^{(3)} h_{1,1} + \alpha_{1,0}^{(3)} h_{1,0} + \alpha_{0,1}^{(3)} h_{0,1} + \alpha_{0,0}^{(3)} h_{0,0}\big)$$
$$+ x_2 \big(\alpha_{1,1}^{(2)} h_{1,1} + \alpha_{1,0}^{(2)} h_{1,0} + \alpha_{0,1}^{(2)} h_{0,1} + \alpha_{0,0}^{(2)} h_{0,0}\big)$$
$$+ x_1 \big(\alpha_{1,1}^{(1)} h_{1,1} + \alpha_{1,0}^{(1)} h_{1,0} + \alpha_{0,1}^{(1)} h_{0,1} + \alpha_{0,0}^{(1)} h_{0,0}\big)$$
$$+ x_0 \big(\alpha_{1,1}^{(0)} h_{1,1} + \alpha_{1,0}^{(0)} h_{1,0} + \alpha_{0,1}^{(0)} h_{0,1} + \alpha_{0,0}^{(0)} h_{0,0}\big)$$

---

[15]On the right-hand side factors might cancel which could reduce the recurrence order.

$$h_{1,1}\big(\alpha_{1,1}^{(4)}x_4 + \alpha_{1,1}^{(3)}x_3 + \alpha_{1,1}^{(2)}x_2 + \alpha_{1,1}^{(1)}x_1 + \alpha_{1,1}^{(0)}x_0\big)$$

$$+h_{1,0}\big(\alpha_{1,0}^{(4)}x_4 + \alpha_{1,0}^{(3)}x_3 + \alpha_{1,0}^{(2)}x_2 + \alpha_{1,0}^{(1)}x_1 + \alpha_{1,0}^{(0)}x_0\big)$$

$$+h_{0,1}\big(\alpha_{0,1}^{(4)}x_4 + \alpha_{0,1}^{(3)}x_3 + \alpha_{0,1}^{(2)}x_2 + \alpha_{0,1}^{(1)}x_1 + \alpha_{0,1}^{(0)}x_0\big)$$

$$+h_{0,0}\big(\alpha_{0,0}^{(4)}x_4 + \alpha_{0,0}^{(3)}x_3 + \alpha_{0,0}^{(2)}x_2 + \alpha_{0,0}^{(1)}x_1 + \alpha_{0,0}^{(0)}x_0\big).$$

In particular, the above equation holds if the coefficients of $h_{i,j}$ are zero, i.e., if

$$\alpha_{1,1}^{(4)}x_4 + \alpha_{1,1}^{(3)}x_3 + \alpha_{1,1}^{(2)}x_2 + \alpha_{1,1}^{(1)}x_1 + \alpha_{1,1}^{(0)}x_0 = 0$$

$$\alpha_{1,0}^{(4)}x_4 + \alpha_{1,0}^{(3)}x_3 + \alpha_{1,0}^{(2)}x_2 + \alpha_{1,0}^{(1)}x_1 + \alpha_{1,0}^{(0)}x_0 = 0$$

$$\alpha_{0,1}^{(4)}x_4 + \alpha_{0,1}^{(3)}x_3 + \alpha_{0,1}^{(2)}x_2 + \alpha_{0,1}^{(1)}x_1 + \alpha_{0,1}^{(0)}x_0 = 0$$

$$\alpha_{0,0}^{(4)}x_4 + \alpha_{0,0}^{(3)}x_3 + \alpha_{0,0}^{(2)}x_2 + \alpha_{0,0}^{(1)}x_1 + \alpha_{0,0}^{(0)}x_0 = 0$$

$$\Updownarrow$$

$$\begin{pmatrix} \alpha_{1,1}^{(4)} & \alpha_{1,1}^{(3)} & \alpha_{1,1}^{(2)} & \alpha_{1,1}^{(1)} & \alpha_{1,1}^{(0)} \\ \alpha_{1,0}^{(4)} & \alpha_{1,0}^{(3)} & \alpha_{1,0}^{(2)} & \alpha_{1,0}^{(1)} & \alpha_{1,0}^{(0)} \\ \alpha_{0,1}^{(4)} & \alpha_{0,1}^{(3)} & \alpha_{0,1}^{(2)} & \alpha_{0,1}^{(1)} & \alpha_{0,1}^{(0)} \\ \alpha_{0,0}^{(4)} & \alpha_{0,0}^{(3)} & \alpha_{0,0}^{(2)} & \alpha_{0,0}^{(1)} & \alpha_{0,0}^{(0)} \end{pmatrix} \begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

holds. In other words we have to solve a linear system with given $\alpha_{i,j}$ and the unknowns $x_0, \ldots, x_4$. Let us lean back: we are given a linear system with 4 equations and 5 unknowns. Applying the Gauss method to such a system we will always compute a non-trivial solution $0 \neq (x_4, x_3, x_2, x_1, x_0) \in \mathbb{K}^5$! Going backwards the above arguments, it follows that we can compute a non-trivial recurrence (44) whose order is 4 or less (if $x_4 = 0$).

Note that the same argument can be applied to (1)–(2), that we proved differently, and to (4) and (5) that we leave as a fun-exercise.

# 7   Holonomic sequences and power series

Most sequences are *not c*-finite. One of the simplest examples are the harmonic numbers $H_n$. Its generating function is

$$\sum_{n=0}^{\infty} H_n x^n = -\frac{1}{1-x} \log(1-x) \notin \mathbb{K}(x)$$

(compare (13)), and thus it cannot be *c*-finite by Theorem 6.3. As already elaborated in Remark 4.5 the recurrence

$$H_{n+1} - H_n = \frac{1}{n+1}$$

can be turned to a homogeneous version:

$$(n+2)H_{n+2} - (2n+3)H_{n+1} + (n+1)H_n = 0.$$

In other words, $H_n$ satisfies a linear recurrence, not with constant coefficients (*c*-finite), but with polynomial coefficients in $\mathbb{K}[n]$. Such sequences are also called holonomic sequences.

**Definition 7.1.** $(a_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}}$ *is called* holonomic *(of order $r$ and degree $d$) if there exist polynomials $p_0(x), p_1(x), \ldots, p_r(x) \in \mathbb{K}[x]$ with $0 \neq p_r(x)$ and $\deg(p_i) < d$ for $0 \leq i \leq r$ such that*

$$p_r(n)\, a_{n+r} + p_{r-1}(n)\, a_{n+r-1} + \cdots + p_0(n)\, a_n = 0 \quad (n \in \mathbb{N}).$$

In Section 3.2 we used the inhomogeneous recurrence

$$H_{n+1} = H_n + \frac{1}{n+1}$$

with $H_0 = 0$ to determine the inhomogeneous 0-order linear differential equation

$$H(x) = -\frac{1}{1-x} \log(1-x)$$

where the right hand-side is non-trivial. Let us repeat the game with a slight variation by using

$$(n+1)H_{n+1} - (n+1)H_n - 1 = 0.$$

This gives

$$0 = \sum_{n=0}^{\infty} \left( (n+1)H_{n+1} - (n+1)H_n - 1 \right) x^n$$

$$= \underbrace{\sum_{n=0}^{\infty} (n+1)H_{n+1}\, x^n}_{H'(x)} - \underbrace{\underbrace{\sum_{n=0}^{\infty} nH_n\, x^n}_{\underbrace{\sum_{n=1}^{\infty} nH_n\, x^n}_{\underbrace{x\sum_{n=1}^{\infty} nH_n\, x^{n-1}}_{xH'(x)}}}} + \underbrace{\sum_{n=0}^{\infty} H_n x^n}_{H(x)} - \underbrace{\sum_{n=0}^{\infty} x^n}_{\frac{1}{1-x}}$$

$$= (1-x)H'(x) - H(x) - \frac{1}{1-x}$$

$$\Updownarrow$$

$$(1-x)^2 H'(x) - (1-x)H(x) = 1.$$

In other words, we obtain a first-order inhomogeneous linear differential equation. Similarly to the recurrence case (see Remark 4.5 using the shift operator) we can turn this equation to a homogeneous version by applying the differential operator $D_x$. In this way we get

$$-2(1-x)H'(x) + (1-x)^2 H''(x) + H(x) - (1-x)H'(x) = 0$$

or equivalently

$$(1-x)^2 H''(x) - 3(1-x)H'(x) + H(x) = 0. \tag{45}$$

This gives rise to the following definition.

**Definition 7.2.** $A(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{K}[[x]]$ *is holonomic (of order $r$ and degree $d$) if there exist polynomials $q_0(x), \dots, q_r(x) \in \mathbb{K}[x]$ with $q_r(x) \neq 0$ and $\deg(q_i) \leq r$ for $1 \leq i \leq r$ such that*

$$q_r(x)D_x^r A(x) + q_{r-1}(x)D_x^{r-1}A(x) + \cdots + a_0(x)D_x^0 A(x) = 0.$$

In other words, given the holonomic sequence $H_n$, also its generating function $H(x)$ is holonomic, i.e., satisfies a homogeneous linear differential equation with polynomial coefficients. More generally, we obtain the following result.

**Theorem 7.3.** *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{K}[[x]]$. Then the following holds:*

1. *If $(a_n)_{n \geq 0}$ is holonomic of order $r$ and degree $d$, then $A(x)$ is holonomic of order $\leq r + d$ and degree $\leq r + d$.*

2. *If $A(x) \in \mathbb{K}[[x]]$ is holonomic of order $r$ and degree $d$, then $(a_n)_{n \geq 0}$ is holonomic of order $\leq r + d$ and degree $\leq r$.*

*Proof.* Apply the classical method. The proof is algorithmic and the resulting algorithms are implemented in the package `GeneratingFunctions.m`. □

**Example 7.4.** We start with the recurrence of the harmonic numbers $a_n = a[n] = H_n$

In[17]:= **rec = a[n + 1] − a[n] == 1/(n + 1);**

and compute a linear differential equation for $A(x) = A[x] = \sum_{n=0}^{\infty} a_n x^n (= H(x))$ by using the package

In[18]:= **<< RISC'GeneratingFunctions'**

> Package GeneratingFunctions version 0.8 written by Christian Mallinger Copyright Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Linz, Austria

In[19]:= **de = RE2DE[rec, a[n], A[x]]**

Out[19]= $1 − (−1 + x)A[x] − (1 − 2x + x^2)A'[x] == 0$

As carried out above, we can compute its homogeneous version (see (45)):

In[20]:= **de = Collect[D[de[[1]], x], {A[x], A'[x], A''[x]}] == 0**

Out[20]= $−A[x] + (3 − 3x)A'[x] + (−1 + 2x − x^2)A''[x] == 0$

Note that we can go back to the coefficient (sequence) level. Given the differential equation `de` for $A(x)$ we can compute a recurrence for its coefficients $a_n$:

In[21]:= **DE2RE[de, A[x], a[n]]**

Out[21]= $−(1 + n)^2 a[n] + (1 + n)(3 + 2n)a[1 + n] − (1 + n)(2 + n)a[2 + n] == 0$

Removing the common factor $1 + n$ gives

$$(n + 2)a_{n+2} − (2n + 3)a_{n+1} + (n + 1)a_n = 0$$

which is nothing else than the already found recurrence (21) of $a_n = H_n$.

**Example 7.5.** We start with $a_n = n!$ and obtain with $\frac{a_{n+1}}{a_n} = n + 1$ the linear recurrence

In[22]:= **rec = a[n + 1] − (n + 1)a[n] == 0;**

Given this recurrence, we can compute a linear differential equation for $A(x) = \sum_{n=0}^{\infty} a_n x^n$ by

In[23]:= **de = RE2DE[rec, a[n], A[x]]**

Out[23]= $−A[x] − (−1 + 3x)A'[x] − x^2 A''[x] == 0$

Conversely, given the linear differential equation `de` for $A(x)$, we can rediscover the linear recurrence for $a_n$:

In[24]:= **DE2RE[de, A[x], a[n]]**

Out[24]= $(n + 1)a[n + 1] − (n + 1)^2 a[n] == 0$

after removing the common factor $n + 1$.


# Lecture from June 20, 2023

In some instances one can derive as starting point a linear differential equation for a power series. In other instances one finds algebraic (polynomial) relations.

**Definition 7.6.** *A formal power series $y(x) \in \mathbb{K}[[x]]$ is called algebraic if there are polynomials $p_0(x), \ldots, p_d(x) \in \mathbb{K}[x]$ with $y_d(x) \neq 0$ such that*

$$p_d(x)y^d(x) + p_{d-1}(x)y(d)^{d-1} + \cdots + p_0 y(x)^0 = 0.$$

**Example 7.7.** The generating function $t(x) = \sum_{n=0}^{\infty} t_n x^n$ with (14) is algebraic; it satisfies the linear differential equation (15):

$$x\, t(x)^2 - t(x) + 1 = 0.$$

**Example 7.8.** The power series

$$y(x) = \sqrt{1+x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n \in \mathbb{Q}[[x]]$$

is algebraic; we have

$$y(x)^2 - (x+1)y(x)^0 = 0.$$

**HW 53.** Show that the power series $e^x$ is not algebraic.

We are now ready to present closure properties for holonomic power series.

**Theorem 7.9.** *Let $A(x) = \sum_{n=0}^{\infty} a_n\, x^n, B(x) = \sum_{n=0}^{\infty} b_n\, x^n \in \mathbb{K}[[x]]$ be holonomic. Then the following formal power series in $\mathbb{K}[[x]]$ are also holonomic:*

1. *$\alpha A(x) + \beta B(x) = \sum_{n=0}^{\infty}(\alpha a_n + \beta b_n)x^n$ for $\alpha, \beta \in \mathbb{K}$ [addition];*

2. *$A(x)B(x) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k\, b_{n-k}$ [Cauchy product];*

3. *$A(x) \cdot B(x) = \sum_{n=0}^{\infty}(a_n\, b_n)x^n$ [Hadamard product];*

4. *$A(B(x)) = \sum_{n=0}^{\infty} a_n B(x)^n$ if $B(x)$ is algebraic with $B(0) = 0$ [substitution];*

5. *$D_x A(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$;*

6. *$\int_x A(x)dx = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}x^n$;*

7. *$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_n (= A(x)\frac{1}{1-x})$ [indefinite summation];*

8. *$\sum_{n=0}^{\infty} \frac{a_n}{n!}x^n$ [Borel transform];*

9. *$\sum_{n=0}^{\infty} n!\, a_n x^n$ [inverse Borel transform].*

   *In addition,*

10. *$(a_{dn+k})_{n\geq 0}$ with $d, k \in \mathbb{N}$ is holonomic [dissection];*

11. *if $(a_n^{(0)})_{n\geq 0}, (a_n^{(1)})_{n\geq 0}, \ldots, (a_n^{(r-1)})_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ are holonomic, then its interlacing of the $r$ sequences, i.e.,*

$$(\underbrace{b_0^{(0)}, b_0^{(1)}, b_0^{(2)}, \ldots, b_0^{(r-1)}}_{\substack{\text{first entry} \\ \text{of each sequence}}}, \underbrace{b_1^{(0)}, b_1^{(1)}, b_1^{(2)}, \ldots, b_1^{(r-1)}}_{\substack{\text{second entry} \\ \text{of each sequence}}}, \underbrace{b_2^{(0)}, b_2^{(1)}, b_2^{(2)}, \ldots, b_2^{(r-1)}}_{\substack{\text{third entry} \\ \text{of each sequence}}}, \ldots)$$

   *is holonomic.*

The underlying proofs follow the tactic as carried out in the proof for item (3) of Theorem 6.8; see page 71. They are all algorithmic and the corresponding algorithms are implemented within the package `GeneratingFunctions`. In the following we will illustrate this powerful toolbox by numerous examples.

**Example 7.10.** Compute a linear differential equation for

$$e^x + \sum_{n=0}^{\infty} n! x^n \in \mathbb{K}[[x]].$$

We get immediately a linear differential equation for $e^x$, namely,

In[25]:= **de1 = y′[x] − y[x] == 0**

Next, we compute a linear differential equation for $\sum_{n=0}^{\infty} n! x^n$ (compare Example 7.5):

In[26]:= **rec = a[n + 1] − (n + 1)a[n] == 0;**

In[27]:= **de2 = RE2DE[rec, a[n], y[x]]**

Out[27]= $-y[x] − (−1 + 3x)y′[x] − x^2 y''[x] == 0$

Finally, we compute a linear differential equation for

$$y(x) = e^x + \sum_{n=0}^{\infty} n! x^n$$

by executing

In[28]:= **DEPlus[de1, de2, y[x]]**

Out[28]= $-(3 + 5x + x^2)y[x] − (−3 − 8x + 10x^2 + 3x^3)y′[x] − (3x − 11x^2 + x^4)y''[x] + (3x^3 + x^4)y'''[x] == 0$

Similarly, we compute a linear differential equation for

$$y(x) = e^x \sum_{n=0}^{\infty} n! x^n \in \mathbb{K}[[x]]$$

by executing

In[29]:= **DECauchy[de1, de2, y[x]]**

Out[29]= $(2 − 3x + x^2)y[x] − (1 − 3x + 2x^2)y′[x] + x^2 y''[x] == 0$

**Example 7.11.** Show that the following formal power series is holonomic:

$$Y(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}^3 \right)^2 x^n \in \mathbb{K}[[x]]. \tag{46}$$

(i) By the Hadamard product (item (3) of Theorem 7.9) $y(x)$ is holonomic if

$$A(x) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{n=0}^{\infty} n!^3 x^n \sum_{k=0}^{n} \frac{1}{k!^3} \frac{1}{(n-k)!^3}$$

is holonomic.

(ii) Using the inverse Borel transform (item (9) of Theorem 7.9) three times it follows that $A(x)$ is holonomic if

$$B(x) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{1}{k!^3} \frac{1}{(n-k)!^3} = \left( \sum_{n=0}^{\infty} \frac{1}{n!^3} x^n \right)^2$$

is holonomic.

(iii) Using the Cauchy product (item (2) of Theorem 7.9)) it follows that $B(x)$ is holonomic if

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n!^3} x^n$$

is holonomic.

(iv) By applying the Borel transform three times (item (8) of Theorem 7.9)) it follows that $C(x)$ is holonomic if

$$D(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is holonomic, which is clearly the case. Thus going backwards, also $Y(x)$ is holonomic.

Any algebraic formal power series is also holonomic, i.e., from an algebraic relation one can determine a linear differential equation.

**Corollary 7.12.** *If $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is algebraic, then $y(x)$ is holonomic.*

*Proof.* Define $B(x) := y(x) - y(0)$ which again is algebraic with $B(0) = 0$. Thus with $A(x) = x$ and item (4) of Theorem 7.9 it follows that $A(B(x)) = B(x)$ is holonomic. In particular, $y(x) = B(x) + y(0)$ is holonomic. □

**Example 7.13.** We show that

$$y(x) = \exp(\frac{x}{\sqrt{1-4x}}) \in \mathbb{K}[[x]]$$

is holonomic. More precisely, we start with the inner power series

$$B(x) = \frac{x}{\sqrt{1-4x}} \in \mathbb{K}[[[x]]$$

and observe that it is algebraic. More precisely, we have

$$B(x)^2 = \frac{x^2}{1-4x} \quad \Leftrightarrow \quad (1-4x)B(x)^2 - x^2 B(x)^0 = 0.$$

Since $A(x) = \exp(x) = e^x \in \mathbb{K}[[x]]$ is holonomic with

$$A'(x) - A(x) = 0,$$

it follows with part (4) of Theorem 7.9 that also $y(x) = A(B(x))$ is holonomic. Using again the package `GeneratingFunctions` we can compute a linear differential equation for $y(x)$ by executing

In[30]:= **de = y′[x] − y[x] == 0;**

In[31]:= **ACompose[de, (1 − 4x)y[x]² − x² == 0, y[x]]**

Out[31]= $(-1 + 6x - 12x^2 + 8x^3)y[x] + 4(-1 + 9x - 24x^2 + 16x^3)y'[x] + (1 - 14x + 72x^2 - 160x^3 + 128x^4)y''[x] == 0$

Together with parts (10) and (11) of Theorem 7.9 we obtain the following generalization[16] of Theorem 6.8:

**Corollary 7.14.** *Let* $(a_n)_{n\geq 0}, (b_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ *be holonomic. Then the following sequences are holonomic:*

1. $(\alpha\, a_n + \beta\, b_n)_{n\geq 0}$ *for* $\alpha, \beta \in \mathbb{K}$;

2. $(\sum_{k=0}^{n} a_k\, b_{n-k})_{n\geq 0}$ *[Cauchy product];*

3. $(\sum_{k=0}^{n} a_k)_{n\geq 0}$ *[indefinite summation];*

4. $(a_n \cdot b_n)_{n\geq 0}$ *[Hadamard product].*

*Proof.* If $(a_n)_{n\geq 0}, (b_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ are holonomic, then $A(x) = \sum_{n=0}^{\infty} a_n\, x^n, B(x) = \sum_{n=0}^{\infty} b_n\, x^n \in \mathbb{K}[[x]]$ are holonomic by part (1) of Theorem 7.3. Thus $\alpha\, A(x) + \beta\, B(x)$, $A(x)\, B(x)$, $A(x) \cdot B(x)$ and $\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k$ are holonomic power series. Thus by part (2) of Theorem 7.3 their coefficients are holonomic which proves items (1)–(4) of this theorem. $\qquad \square$

**Example 7.15.** We present an alternative tactic to show that (46), i.e.,

$$Y(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}^3 \right)^2 x^n$$

is holonomic.
(i) We apply part (1) of Theorem 7.3 and it follows that $Y(x)$ is holonomic if

$$a_n = \left( \sum_{k=0}^{n} \binom{n}{k}^3 \right)^2$$

is holonomic.
(ii) Using the Hadamard operation (item (4) of Corollary 7.14) it follows that this is the case if

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^3 = n!^3 \sum_{k=0}^{n} \frac{1}{k!^3} \frac{1}{(n-k)!^3}$$

is holonomic.
(iii) Using again the Hadamard operation (item (4) of Corollary 7.14) it follows that this is the case if

$$c_n = \sum_{k=0}^{n} \frac{1}{k!^3} \frac{1}{(n-k)!^3}$$

---

[16]A $c$-finite sequence is holonomic.

is holonomic.

(iv) Using the Cauchy product (item (2) of Corollary 7.14) it follows that this is the case if

$$d_n = \frac{1}{n!^3}$$

is holonomic, which is clearly the case. Hence going backwards, also $y(x)$ is holonomic. More precisely, we can carry out the following calculations:

Step (iv): We insert a linear recurrence for $d_n = y[n]$:

In[32]:= $\mathbf{recD = (n+1)^3 y[n+1] - y[n] == 0;}$

and compute a recurrence for $c_n = y[n]$:

In[33]:= $\mathbf{recC = RECauchy[rec, rec, y[n]]}$

Out[33]= $-8y[n] - (1+n)(16 + 21n + 7n^2)y[1+n] + (1+n)(2+n)^5 y[2+n] == 0$

Step (iii): Next, we compute a recurrence for $b_n = y[n]$:

In[34]:= $\mathbf{recH = y[n+1] - (n+1)^3 y[n] == 0;}$

In[35]:= $\mathbf{recB = REHadamard[recC, recH, y[n]]}$

Out[35]= $-8(1+n)^2 y[n] + (-16 - 21n - 7n^2)y[1+n] + (2+n)^2 y[2+n] == 0$

Step (ii): Now we are ready to compute a recurrence for $a_n = y[n]$:

In[36]:= $\mathbf{recA = REHadamard[recB, recB, y[n]]}$

Out[36]= $512(1+n)^4(44 + 35n + 7n^2)y[n] - 8(16 + 21n + 7n^2)(832 + 1740n + 1347n^2 + 456n^3 + 57n^4)y[1+n] - (44 + 35n + 7n^2)(832 + 1740n + 1347n^2 + 456n^3 + 57n^4)y[2+n] + (3+n)^4(16 + 21n + 7n^2)y[3+n] == 0$

Finally, we can compute a linear differential equation for $Y(x) = \sum_{n=0}^{\infty} y[n]x^n$:

In[37]:= $\mathbf{de = RE2DE[recA, y[n], Y[x]]}$

Out[37]= $45056Y[x] + 4096(-52 + 1021x)Y'[x] + 256(-143 - 17450x + 121600x^2)Y''[x] + 16(27 - 21534x - 940504x^2 + 3712256x^3)Y'''[x] + 4(596x - 162403x^2 - 3978656x^3 + 10685056x^4)Y^{(4)}[x] + 3(925x^2 - 137452x^3 - 2270272x^4 + 4567552x^5)Y^{(5)}[x] + (1073x^3 - 105702x^4 - 1298880x^5 + 2076160x^6)Y^{(6)}[x] + 14(11x^4 - 798x^5 - 7752x^6 + 10240x^7)Y^{(7)}[x] + 7(x^5 - 57x^6 - 456x^7 + 512x^8)Y^{(8)}[x] == 0$

**HW 54.** Compute a holonomic recurrence for

$$a_n = [x^n]e^x \sum_{n=0}^{\infty} H_n x^n.$$

**HW 55.** Compute a holonomic recurrence for

$$a_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$

**HW 56.** Compute a holonomic differential equation for

$$a(x) = \sin(x) \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} k! \right) x^n.$$

If a formal power series is invertible, then in most instances the inverse power series is not holonomic. However, finding such a proof is highly non-trivial. But some instances the following theorem is helpful.

**Theorem 7.16.** *Let $y(x) \in \mathbb{C}[[x]]$ be holonomic with $y(0) \neq 0$. Then $\frac{1}{y(x)}$ is holonomic if and only if $\frac{y'(x)}{y(x)}$ is algebraic.*

**HW 57.** Show that $y(x) = \frac{1}{\cos(x)}$ is not holonomic.
[Hint: one may use that fact that $\tan(x)$ is not algebraic.]

Lecture from June 27, 2023

# 8   Symbolic Summation (a short introduction)

In the previous lectures we have seen already applications in symbolic summation. For instance, we discovered and proved the identity

$$\sum_{k=0}^{A} \frac{\binom{2k}{k}\binom{-2-2k+2n}{-1-k+n}}{(1+k)(-k+n)} = \frac{(-1+2n)\binom{-2+2n}{-1+n}}{n(1+n)} + \frac{(1+2A)(2+2A-n)\binom{2A}{A}\binom{-2-2A+2n}{-1-A+n}}{(1+A)n(1+n)}$$

for $n \geq 0$ and $0 \leq A \leq n$ on page 37. In particular, the specialization to $A = n$ lead to (18), i.e.,

$$\frac{1}{n+1}\binom{2n}{n} = \sum_{k=0}^{n-1} \frac{1}{k+1}\binom{2k}{k}\frac{1}{n-k}\binom{2n-2k-2}{n-k-1}.$$

Similarly, we focused on the verification of such identities in Remark 4.4.

In the following, we will give some more ideas how such identities can be discovered and will link it to various examples coming from the last lectures. We start with the following telescoping problem:

**Given** an expression $f(k)$ that evaluates to a sequence.
**Find** an expression $g(k)$ such that the telescoping equation

$$f(k) = g(k+1) - g(k) \tag{47}$$

holds.

Suppose we find such an expression $g(k)$. Then we can follow the idea given in HW 3. Namely, summing (47) over $k$ from $a$ to $b$ (and assuming that no poles arise during the evaluation) yields

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a). \tag{48}$$

We note that we could always choose

$$g(k) = \sum_{i=a}^{k-1} f(i) \tag{49}$$

which would turn (48) to the trivial identity $\sum_{k=a}^{b} f(k) = \sum_{k=a}^{b} f(k)$.
Thus we should refine our problem from above:

**Find** an expression $g(k)$ with (47) where $g(k)$ is simpler than the trivial solution (49).

## 8.1  Indefinite summation of polynomials

We start with one of the most simplest cases: the summand is a polynomial, i.e., $f(x) \in \mathbb{K}[x]$. The following questions arise:

1. What is the domain of expressions in which we search $g(k)$?

2. How can we calculate a solution $g(k)$ in this solution domain?

As it turns out, the first question can be answered nicely: a solution $g(x)$ exists always in $\mathbb{K}[x]$. For the second question, we will consider two different tactics that are often used in summation packages.

• *Tactic 1: the classical approach.* In the following it will be convenient to use also the difference operator. For any sequence (expression) $g(k)$ we define

$$\Delta g(k) := g(k+1) - g(k).$$

Note that for indefinite integration of polynomials one can utilize the following well known property: for any $m \in \mathbb{N}$ we have

$$D_x x^m = m\, x^{m-1} \tag{50}$$

which implies

$$\int_a^b x^m dx = \frac{x^{m+1}}{m+1}\Big|_a^b = \frac{b^{m+1} - a^{m+1}}{m+1}. \tag{51}$$

Thus by linearity we can integrate any polynomial by

$$\int_a^b \sum_{m=0}^d c_m\, x^m dx = \sum_{m=0}^d c_m \int_a^b x^m dx = \sum_{m=0}^d \frac{c_m\left(b^{m+1} - a^{m+1}\right)}{m+1}. \tag{52}$$

For indefinite summation of polynomials we can follow precisely the same classical strategy which is based on the following lemma.

**Lemma 8.1.** *For $m \in \mathbb{N}$ we have*

$$\boxed{\Delta x^{\underline{m}} = m\, x^{\underline{m-1}}}.$$

*Proof.* We have

$$\Delta x^{\underline{m}} = (x+1)^{\underline{m}} - x^{\underline{m}}$$
$$= (x+1)x(x-1)\ldots(x-m+2) - x(x-1)\ldots(x-m+1)$$
$$= ((x+1) - (x-m+1))x(x-1)\ldots(x-m+2)$$
$$= m\, x^{\underline{m-1}}.$$

$\square$

As a consequence we get

$$\Delta \frac{x^{\underline{m+1}}}{m+1} = x^{\underline{m}}, \quad m \in \mathbb{N}$$

and summing this equation over $k$ from $a$ to $b$ yields

$$\sum_{x=a}^{b} x^{\underline{m}} = \frac{(b+1)^{\underline{m+1}} - a^{\underline{m+1}}}{m+1}.$$

Note that this is nothing else than the discrete version given in (50) and (51). In particular, for given

$$f(x) = \sum_{m=0}^{d} c_m \, x^{\underline{m}} \in \mathbb{K}[x] \tag{53}$$

with $d \in \mathbb{N}$ it follows that

$$g(x) = \sum_{m=0}^{d} \frac{c_m x^{\underline{m+1}}}{m+1}$$

is a telescoping solution of (47). Furthermore, analogously to (52) we obtain

$$\sum_{x=a}^{b} f(x) = \sum_{m=0}^{d} c_m \sum_{x=a}^{b} x^{\underline{m}} = \sum_{m=0}^{d} \frac{c_m((b+1)^{\underline{m+1}} - a^{\underline{m+1}})}{m+1}.$$

The only problem is that in many cases one does not have a polynomial given in the representation (53) for some $d \in \mathbb{N}$ but in the form

$$\sum_{m=0}^{d} \bar{c}_m \, x^m \in \mathbb{K}[x].$$

Luckily one can rewrite a polynomial written in the basis

$$1, x, x^2, \dots, x^d$$

to the representation written in the basis

$$x^{\underline{0}} = 1, x^{\underline{1}} = x, x^{\underline{2}} = x(x-1), \dots, x^{\underline{d}} = x(x-1)\dots(x-d+1)$$

by using the formula

$$x^m = \sum_{k=0}^{m} S(m,k) x^{\underline{k}}$$

where $S(n,k)$ denotes the Stirling numbers of second kind. They can be computed by

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n;$$

alternatively, there is a recurrence formula.

**Example 8.2.** Consider the polynomial

$$f(x) = x^4.$$

Using the formulas from above, we get

$$f(x) = x^4 = \sum_{k=0}^{4} S(4, k)x^{\underline{k}} = 0x^{\underline{0}} + 1x^{\underline{1}} + 7x^{\underline{2}} + 6x^{\underline{3}} + 1x^{\underline{4}}.$$

Thus we get

$$g(x) = \frac{1}{2}x^{\underline{2}} + \frac{7}{3}x^{\underline{3}} + \frac{3}{2}x^{\underline{4}} + \frac{1}{5}x^{\underline{5}}$$
$$= \frac{1}{30}(x - 1)x(2x - 1)(3x^2 - 3x - 1).$$

such that

$$g(x + 1) - g(x) = f(x)$$

holds. In particular we get

$$\sum_{k=1}^{n} k^4 = \sum_{k=1}^{n} f(k) = g(n + 1) - g(1) = \frac{1}{30}n(n + 1)(2n + 1)(3n^2 + 3n - 1). \tag{54}$$

• *Tactic 2: linear algebra.* We use the following property: for $f(x) \in \mathbb{K}[x]$ there is a $g(x) \in \mathbb{K}[x]$ with (47) where

$$\deg(g) \leq \deg(f) + 1.$$

Thus setting $d := \deg(f) + 1$ for given $f \in \mathbb{K}[x]$ the desired solution has the form

$$g(x) = \sum_{m=0}^{d} g_m\, x^m$$

and we can determine the unknowns $g_0, \dots, g_d \in \mathbb{K}$ by linear algebra as follows.

**Example 8.3.** Take $f(x) = x^4 \in \mathbb{Q}[x]$. With $d = \deg(f) + 1 = 5$ the ansatz

$$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + g_4 x^4 + g_5 x^5$$

for the unknowns $g_0, g_1, g_2, g_3, g_4, g_5 \in \mathbb{Q}$ is in place. This gives

$$\begin{aligned}
x^4 = \Delta g(x) &= g(x + 1) - g(x) \\
&= 0x^5 \\
&+ 5g_5 x^4 \\
&+ (4g_4 + 10g_5)x^3 \\
&+ (3g_3 + 6g_4 + 10g_5)x^2 \\
&+ (2g_2 + 3g_3 + 4g_4 + 5g_5)x \\
&+ (g_1 + g_2 + g_3 + g_4 + g_5)x^0.
\end{aligned}$$

By coefficient comparison this yields the linear system

$$
\begin{aligned}
[x^4] \qquad & 1 = 5g_5 \\
[x^3] \qquad & 0 = 4g_4 + 10g_5 \\
[x^2] \qquad & 0 = 3g_3 + 6g_4 + 10g_5 \\
[x^1] \qquad & 0 = 2g_2 + 3g_3 + 4g_4 + 5g_5 \\
[x^0] \qquad & 0 = g_1 + g_2 + g_3 + g_4 + g_5
\end{aligned}
$$

which is already in triangular form. Thus we can read off the solution

$$
g_5 = \frac{1}{5}, \qquad g_4 = -\frac{1}{2}, \qquad g_3 = \frac{1}{3}, \qquad g_2 = 0, \qquad g_1 = -\frac{1}{30}, \qquad g_0 = c
$$

with $c \in \mathbb{Q}$. In particular, we can choose $c = 0$ and obtain

$$
g(x) = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30} = \frac{1}{30}(x-1)x(2x-1)\left(3x^2 - 3x - 1\right).
$$

To this end, we continue as in the previous example and get (54).

## 8.2 More general summation objects for indefinite and definite summation

### 8.2.1 Telescoping

Clearly, the first tactic is very elegant, but it works only for the special case of polynomial summation. For the second tactic one has to work more (i.e., has to solve in addition a linear system), but it turns out to be more general. More precisely, one can carry over these ideas to a rather general setting that works not only for the polynomial ring $\mathbb{Q}[x]$ but in more general rings called $R\Pi\Sigma$-difference rings that have been implemented within the summation package `Sigma`. In the following all technical details are omitted and we proceed with a concrete example.

**Example 8.4.** We want to sum

$$
\sum_{k=0}^{n} H_k;
$$

compare Remark 4.4. In order to accomplish this task, we take

$$
f(k) = H_k
$$

and search for

$$
g(k) \in \mathbb{Q}(k)[H_k]
$$

with

$$
f(k) = g(k+1) - g(k). \tag{55}
$$

Here we can use a similar tactic as used in the case of polynomial summation. Namely, summation theory tells us that any such solution $g(k)$ has the property

$$
\deg(g) \le \deg(f) + 1 = 1 + 1 = 2.
$$

As a consequence we can make the ansatz

$$g(k) = g_0(k)H_k^0 + g_1(k)H_k^1 + g_2(k)H_k^2$$

with $g_0(k), g_1(k), g_2(k) \in \mathbb{Q}(k)$. Using recursive algorithms and linear system solving (details are skipped here) we find

$$g_0(k) = -k$$
$$g_1(k) = k$$
$$g_2(k) = 0,$$

i.e.,

$$g(k) = -k + kH_k + 0H_k^2.$$

Hence summing (55) over $k$ from 0 to $n$ gives

$$\sum_{k=0}^{n} H_k = g(n+1) - g(0) = (n+1)H_{n+1} - (n+1) = -n + (1+n)H_n.$$

The above machinery can be carried out within the summation package `Sigma`. After loading it into Mathematica

In[38]:= **<< Sigma.m**

> Sigma - A summation package by Carsten Schneider © RISC-JKU

one can insert the above sum

In[39]:= **mySum = SigmaSum[SigmaHNumber[k], {k, 0, 1}]**

Out[39]= $\sum_{k=0}^{n} H_k$

and can apply the command

In[40]:= **SigmaReduce[mySum]**

Out[40]= $-n + (1 + n)H_n$

In general one can insert, e.g., a sum of the form

$$\sum_{k=l}^{n} f(k) \tag{56}$$

with $l \in \mathbb{N}$ where $f(k)$ itself is given in terms of indefinite nested sums defined over hypergeometric products.

**Definition 8.5.** *Let $\mathbb{K}$ be a field. A product $\prod_{j=l}^{k} f(j)$, $l \in \mathbb{N}$, is called* hypergeometric in $k$ *over $\mathbb{K}$ if $f(x) \in \mathbb{K}(x)$ is a rational function where the numerator and denominator of $f(j)$ are nonzero for all $j \in \mathbb{Z}$ with $j \geq l$. An* expression in terms indefinite of nested sums over hypergeometric products in $k$ over $\mathbb{K}$ *is composed recursively by the three operations $(+, -, \cdot)$ with*

- *elements from the rational function field $\mathbb{K}(k)$,*

- *hypergeometric products in k over $\mathbb{K}$,*

- *and sums of the form $\sum_{j=l}^{k} f(j)$ with $l \in \mathbb{N}$ where $f(j)$ is an expression in terms of indefinite nested sums over hypergeometric products in $j$ over $\mathbb{K}$; here it is assumed that the evaluation of $f(j)$ for all $j \geq l$ does not introduce any poles.*

Restricting to the case that $f(k)$ in (56) is only a hypergeometric product, one may use Gosper's algorithm (1978). For the general class of indefinite nested sums defined over hypergeometric products the difference field approach built into `Sigma` can be used. The underlying algorithms are inspired by Karr's summation algorithm (1981) and contain many enhancements and generalizations. In the following we present examples that illustrate some of these features.

Inserting the sum arising on page 14

In[41]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n-1}} \frac{\mathbf{2k}}{\mathbf{(k+1)(k+2)}};$

we get the result

In[42]:= **SigmaReduce[mySum]**

Out[42]= $-\dfrac{2(-1+\mathbf{n})(1+2\mathbf{n})}{\mathbf{n}(1+\mathbf{n})} + 2\sum_{\mathbf{i_1}=1}^{-1+\mathbf{n}} \dfrac{1}{\mathbf{i_1}}$

which yields the identity given in (3). Internally, `Sigma` takes the summand

$$f(k) = \frac{2k}{(k+1)(k+2)} \in \mathbb{Q}(k)$$

and computes a solution $g(k)$ of the telescoping equation (47). Note that $g(k)$ does not exist in $\mathbb{Q}(k)$ but in the more general domain

$$g(k) = -\frac{4k}{1+k} + 2\sum_{i=1}^{k} \frac{1}{i} \in \mathbb{Q}(k)\Big[\sum_{i=1}^{k} \frac{1}{i}\Big] (= \mathbb{Q}(k)[H_k])$$

which has been derived automatically by `Sigma`.

Other examples are

In[43]:= $\mathbf{mySum = mySum = SigmaSum[SigmaPower[-1, k]SigmaBinomial[n, k]SigmaHNumber[k], \{k, a, b\}]}$

Out[43]= $\sum_{\mathbf{k=a}}^{\mathbf{b}} (-1)^{\mathbf{k}} \binom{\mathbf{n}}{\mathbf{k}} H_{\mathbf{k}}$

In[44]:= **SigmaReduce[mySum]**

Out[44]= $\big(\dfrac{(\mathbf{a-n})(-1+\mathbf{a-n})}{\mathbf{an^2}} + \dfrac{(-1+\mathbf{a-n})H_{\mathbf{a}}}{\mathbf{n}}\big)(-1)^{1+\mathbf{a}}\binom{\mathbf{n}}{-1+\mathbf{a}} + \big(\dfrac{-\mathbf{b+n}}{\mathbf{n^2}} + \dfrac{(-\mathbf{b+n})H_{\mathbf{b}}}{\mathbf{n}}\big)(-1)^{\mathbf{b}}\binom{\mathbf{n}}{\mathbf{b}}$

or

In[45]:= $\mathbf{mySum = SigmaSum[SigmaSum[SigmaBinomial[n, k], \{k, 0, r\}]^2, \{r, 0, b\}]}$

Out[45]= $\sum_{\mathbf{r=0}}^{\mathbf{b}} \big(\sum_{\mathbf{k=0}}^{\mathbf{r}} \binom{\mathbf{n}}{\mathbf{k}}\big)^2$

In[46]:= **SigmaReduce[mySum]**

Out[46]= $(-\mathbf{b+n})\binom{\mathbf{n}}{\mathbf{b}}\sum_{\mathbf{i_1}=0}^{\mathbf{b}} \binom{\mathbf{n}}{\mathbf{i_1}} + \dfrac{1}{2}(2+2\mathbf{b-n})\big(\sum_{\mathbf{i_1}=0}^{\mathbf{b}} \binom{\mathbf{n}}{\mathbf{i_1}}\big)^2 - \dfrac{1}{2}\mathbf{n}\sum_{\mathbf{i_1}=0}^{\mathbf{b}} \binom{\mathbf{n}}{\mathbf{i_1}}^2$

### 8.2.2 Recurrence finding (creative telescoping)

We can play this fun-game for many summation problems. However, sooner or later one ends up at examples with the following behavior:

In[47]:= **mySum = SigmaSum[SigmaBinomial[n, k], {k, 0, a}]**

Out[47]= $\sum_{k=0}^{a} \binom{n}{k}$

In[48]:= **SigmaReduce[mySum]**

Out[48]= $\sum_{i_1=0}^{a} \binom{n}{i_1}$

This means that `Sigma` (but also, e.g., Gosper's algorithm which is applicable for this input) fails. In short, there does not exist a solution $g(k)$ of

$$\binom{n}{k} = g(k+1) - g(k)$$

where $g(k)$ is simpler than the input sum $\sum_{i=0}^{k} \binom{n}{i}$.

Such sums seem hopeless. However, in this particular example we have not used the fact that the sum contains an extra discrete parameter $n$. Due to Zeilberger's powerful creative telescoping paradigm (introduced 1990) such sums (more precisely definite sums where we specialize the summation bound $a$, e.g., to $n$) can be handled in the following way.

**Example 8.6.** We consider the sum

$$S(n) = \sum_{k=0}^{n} f(n, k)$$

with

$$f(n, k) = \binom{n}{k}.$$

As mentioned above, one first tries to find an expression $g(n, k)$ being simpler than $\sum_{i=0}^{k} \binom{n}{i}$ such that

$$g(n, k+1) - g(n, k) = f(n, k)$$

holds; as observed above, we fail with this tactic. Precisely here the creative telescoping tactic starts: one proceeds by looking for such an expression $g(n, k)$ together with a constant $c_1(n)$, free of $k$, such that

$$g(n, k+1) - g(n, k) = f(n, k) + c_1(n) f(n+1, k) \tag{57}$$

holds. In this particular instance we find

$$c_1(n) = -1/2, \qquad g(n, k) = \frac{1}{2} \frac{k}{n-k+1} \binom{n}{k}, \qquad g(n, k+1) = \frac{1}{2} \binom{n}{k}.$$

Now we repeat the game as for the telescoping case. We sum (57) over $k$ from 0 to $n$ and get

$$\underbrace{g(n, n+1)}_{\frac{1}{2}} - \underbrace{g(n, 0)}_{0} = \underbrace{\sum_{k=0}^{n} f(n, k)}_{S(n)} - \frac{1}{2} \underbrace{\sum_{k=0}^{n} f(n+1, k)}_{\underbrace{\sum_{k=0}^{n+1} f(n+1, k)}_{S(n+1)} - \underbrace{f(n+1, n+1)}_{1}}$$

$$\Updownarrow$$

$$S(n+1) - 2S(n) = 0.$$

Using this recurrence together with the initial value $S(0) = 1$ we discover the simplification

$$S(n) = \sum_{n=0}^{n} \binom{n}{k} = 2^n \tag{58}$$

which is nothing else than the binomial theorem. Within `Sigma` this mechanism can be carried out as follows. We insert our sum

In[49]:= **mySum = SigmaSum[SigmaBinomial[n, k], {k, 0, n}]**

Out[49]= $\sum_{k=0}^{n} \binom{n}{k}$

and compute a linear recurrence with the function call

In[50]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[50]= $2\text{SUM}[n] - \text{SUM}[n+1] == 0$

**Example 8.7.** Similarly, we can compute a recurrence for the definite sum

In[51]:= **mySum = SigmaSum[SigmaBinomial[n, k]$^3$, {k, 0, n}]**

Out[51]= $\sum_{k=0}^{n} \binom{n}{k}^3$

with the function call

In[52]:= **GenerateRecurrence[mySum, n][[1]]**

Out[52]= $8(1+n)^2\text{SUM}[n] + (16 + 21n + 7n^2)\text{SUM}[1+n] - (2+n)^2\text{SUM}[2+n] == 0$

Note that this is precisely the recurrence in Out[36] on page 80. In other words, symbolic summation provides a short-cut that enables one to skip the technical calculation steps (iv) and (iii) in Example 7.15.

In general, we can insert within the summation package `Sigma` sums of the form

$$S(n) = \sum_{k=l}^{n} f(n, k)$$

with $l \in \mathbb{N}$ and where $f(n, k)$ is an expression in terms of indefinite nested sums defined over hypergeometric products w.r.t. $k$; here the sums and products may involve also the extra parameter $n$. Then fixing $d \in \mathbb{N}$, one can search for an expression $g(n, k)$ (which is simpler than the input sum $S(n)$) together with constants $c_1(n), \ldots, c_d(n)$, which may depend on $n$ but which are free of $k$, such that the following creative telescoping equation holds:

$$g(n, k + 1) - g(n, k) = f(n, k) + c_1(n)f(n + 1, k) + \cdots + c_d(n)f(n + d, k).$$

Here one starts usually with $d = 0$ (which is telescoping) and increases $d = 0, 1, 2, 3, \ldots$ incrementally until one finds a solution for a given $d \in \mathbb{N}$. If one is successful, one can sum the found creative telescoping equation over $k$ and obtains a linear recurrence of order $d$ of the form

$$S(n) + c_1(n)S(n + 1) + \cdots + c_d(n)S(n + d) = h(n). \tag{59}$$

### 8.2.3 Recurrence solving (in terms of d'Alembertian solutions)

In most cases the found recurrence (59) is built by $c_1(n), \ldots, c_d(n)$ which are rational functions in $n$ and $h(n)$ can be simplified again to indefinite nested sums defined over hypergeometric products. At this point we utilize another strong tool of Sigma

Given such a recurrence, Sigma can find all solutions that can be expressed again within the class of indefinite nested sums defined over hypergeometric products (such solutions are also called d'Alembertian solutions). Internally, Petkovšek's Hyper algorithm (1992) and a generalized version of it has been implemented.

**Example 8.8.** [Cont. of Example 8.6] Given the recurrence rec from Out[50] one can solve it with the function call

In[53]:= **recSol = SolveRecurrence[rec, SUM[n]]**

Out[53]= $\{\{0, 2^n\}, \{1, 0\}\}$

This means that $2^n$ is a solution of the homogeneous version of the recurrence. Finally, using the initial value $S(0) = 1$, one gets the simplification (58). This last step can be also carried out with the function call

In[54]:= **FindLinearCombination[recSol, mySum, n, 1]**

Out[54]= $2^n$

where 1 specifies the number of initial values that should be used. To obtain the correct result this number must be at least the order of the recurrence; if it is higher, extra values are used which gives extra robustness tests. In other words, we have derived the identity (58) fully automatically by means of computer algebra.

This recurrence solver can be also used to solve $c$-finite recurrences – the solutions (see Theorem 6.6) are contained within the class of hypergeometric products.

**Example 8.9.** In[55]:= **RSolve[M[h] − 2M[h − 1] + M[h − 3] == 0, M[h], h]//FullSimplify**

Out[55]= $\left\{\left\{\mathrm{M}[\mathrm{h}] \to c_1 \left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^{\mathrm{h}} + c_2 \left(\frac{1}{2}\left(1 + \sqrt{5}\right)\right)^{\mathrm{h}} + c_3\right\}\right\}$

This means that

$$\left\{c_1 \left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^{h} + c_2 \left(\frac{1}{2}\left(1 + \sqrt{5}\right)\right)^{h} + c_3 \mid c_1, c_2, c_3 \in \mathbb{C}\right\}$$

provides the full solution space and thus there exist $c_1, c_2, c_3 \in \mathbb{C}$ such that

$$M(h) = c_1 \left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^{h} + c_2 \left(\frac{1}{2}\left(1 + \sqrt{5}\right)\right)^{h} + c_3$$

holds for all $h \in \mathbb{N}$. Finally, looking at $h = 0, 1, 2$ we get the linear system

$$1 = c_1 + c_2 + c_3$$
$$2 = c_1 \frac{1}{2}\left(1 - \sqrt{5}\right) + c_2 \frac{1}{2}\left(1 + \sqrt{5}\right) + c_3$$
$$4 = c_1 \left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^{2} + c_2 \left(\frac{1}{2}\left(1 + \sqrt{5}\right)\right)^{2} + c_3.$$

Solving it (using the Gauss algorithm) yields

$$c_1 = \frac{1}{5}(5 - 2\sqrt{5}),$$
$$c_2 = \frac{1}{5}(5 + 2\sqrt{5}),$$
$$c_3 = -1.$$

Remark: Providing in addition the initial values to `RSolve`, this calculation can be carried out in one stroke:

In[56]:= **RSolve[M[h] − 2M[h − 1] + M[h − 3] == 0**
          **&&M[0] == 1&&M[1] == 2&&M[2] == 4, M[h], h]//FullSimplify**

Out[56]= $\frac{1}{5}\left(\left(5 - 2\sqrt{5}\right)\left(\frac{1}{2}\left(1 - \sqrt{5}\right)\right)^{\mathrm{h}} + \left(5 + 2\sqrt{5}\right)\left(\frac{1}{2}\left(1 + \sqrt{5}\right)\right)^{\mathrm{h}} - 5\right)$

More precisely, we get

$$M(h) = -1 + \frac{5 - 2\sqrt{5}}{5} r_-^{h} + \frac{5 + 2\sqrt{5}}{5} r_+^{h}. \tag{60}$$

Instead of using the Mathematica-command `RSolve` in In[56] on page 92 one can execute the command

In[57]:= **recSol3 = SolveRecurrence[M[h] − 2M[h − 1] + M[h − 3] == 0, M[h]]**

Out[57]= $\left\{\{0, 1\}, \left\{0, \frac{(-1)^{\mathrm{h}}(\sqrt{5} - 1)^{\mathrm{h}}}{2^{\mathrm{h}}}\right\}, \left\{0, \frac{2^{\mathrm{h}}}{(\sqrt{5} - 1)^{\mathrm{h}}}\right\}, \{1, 0\}\right\}$

to find the full solution space:

$$\left\{c_1 + c_2 \frac{(-1)^{h}(\sqrt{5} - 1)^{h}}{2^{h}} + c_3 \frac{2^{h}}{(\sqrt{5} - 1)^{h}} \mid c_1, c_2, c_3 \in \mathbb{C}\right\}.$$

Finally we use the first three initial values $M(0) = 1$, $M(1) = 2$, $M(2) = 4$ to derive the corresponding constants $c_1, c_2, c_3$. This can be carried out with the `Sigma`-routine

In[58]:= **FindLinearCombination[recSol3, {0, {1, 2, 4}}, h, 3]**

Out[58]= $-1 + \dfrac{5 - 2\sqrt{5}}{5} \dfrac{(-1)^h (\sqrt{5} - 1)^h}{2^h} + \dfrac{5 + 2\sqrt{5}}{5} \dfrac{2^h}{(\sqrt{5} - 1)^h}$

One can easily check (which would be another little HW) that this output is equivalent to the found representation (60).

Alternatively, one can also insert directly the inhomogeneous recurrence from (40) and gets

In[59]:= **recSol2 = SolveRecurrence[M[h] − M[h − 1] − M[h − 2] == 1, M[h]]**

Out[59]= $\left\{0, \dfrac{(-1)^h (\sqrt{5} - 1)^h}{2^h}\right\}, \left\{0, \dfrac{2^h}{(\sqrt{5} - 1)^h}\right\}, \{1, -1\}\}$

Here we get two linearly independent solutions of the homogeneous version (indicated by a 0 in the first entry of the list), namely,

$$H_1(h) = \frac{(-1)^h (\sqrt{5} - 1)^h}{2^h} \left(= \left(\tfrac{1}{2}\left(1 - \sqrt{5}\right)\right)^h\right),$$

$$H_2(h) = \frac{2^h}{(\sqrt{5} - 1)^h} \left(= \left(\tfrac{1}{2}\left(1 + \sqrt{5}\right)\right)^h\right)$$

and one particular solution of the recurrence itself (indicated by a 1 in the first entry of the list):

$$P(h) = -1.$$

More precisely, we get the full solution by taking all linear combinations of the homogeneous solutions plus the derived particular solution:

$$V = \{P(h) + c_1\, H_1(h) + c_2\, H_2(h) \mid c_1, c_2 \in \mathbb{C}\}$$

$$= \left\{-1 + c_1 \frac{(-1)^h (\sqrt{5} - 1)^h}{2^h} + c_2 \frac{2^h}{(\sqrt{5} - 1)^h} \mid c_1, c_2 \in \mathbb{C}\right\}.$$

Finally, we use the first two initial values that uniquely determine the constants $c_1, c_2$ and thus yield the closed form of $M(h)$. Again this can be carried out by executing

In[60]:= **FindLinearCombination[recSol2, {0, {1, 2}}, h, 2]**

Out[60]= $-1 + \dfrac{5 - 2\sqrt{5}}{5} \dfrac{(-1)^h (\sqrt{5} - 1)^h}{2^h} + \dfrac{5 + 2\sqrt{5}}{5} \dfrac{2^h}{(\sqrt{5} - 1)^h}$

**Example 8.10.** Consider the recurrence (2), i.e.,

$$(n + 1)F(n + 1) - (n + 2)F(n) = 2n, \quad n \geq 0$$

with the initial value $F(0) = 0$ that we derived for the analysis of QuickSort. Instead of using the "variation of constants" method, one can also activate the recurrence solver of `Sigma` to the recurrence

In[61]:= **rec = (n + 1)F[n + 1] − (n + 2)F[n] == 2n;**

and one gets the solution

In[62]:= **recSol = SolveRecurrence[rec, F[n]];**

Out[62]= $\{\{0, 1 + n\}, \{1, 4 + 2(1 + n) \sum_{i_1=1}^{n} \frac{1}{i_1}\}\}$

This means that the set of solutions (that produce sequences with entries from $\mathbb{Q}$) is

$$\underbrace{\{c\,(n+1) + 4 + 2(1+n) \sum_{i_1=1}^{n} \frac{1}{i_1} \mid c \in \mathbb{Q}\}}_{F_c(n)}.$$

Finally, we determine $c \in \mathbb{Q}$ such that $F_c(0) = 0$ holds. In our case we have to choose $c = -4$ which implies

$$F(n) = F_{-4}(n), \quad n \in \mathbb{N}.$$

This last step can be carried out with

In[63]:= **FindLinearCombination[recSol, {0, {0}}, n, 1]**

Out[63]= $-4n + 2(1 + n) \sum_{i_1=1}^{n} \frac{1}{i_1}$

As a consequence, we rediscovered with computer algebra the representation (4).

## 8.3   Definite summation using `Sigma`

We conclude this section with variations of the identity (58) (see Examples 8.6 and 8.8) by utilizing the different tools of `Sigma`:

1. recurrence finding with the function command `GenerateRecurrence`;

2. recurrence solving with the function command `SolveRecurrence`;

3. and combining the solutions (in case that sufficiently many solutions are found) with the function command `FindLinearCombination`.

More precisely, we will simplify definite sums to alternative representations which are given in terms of indefinite nested sums defined over hypergeometric products

Let us consider as warm up example the sum

In[64]:= **mySum = SigmaSum[SigmaHNumber[k]SigmaBinomial[n, k], {k, 0, n}]**

Out[64]= $\sum_{k=0}^{n} \binom{n}{k} H_k$

First, we apply the creative telescoping summation paradigm (see Example 8.6 for the sum (58)) in `Sigma` and compute a linear recurrence of order 2:

In[65]:= **rec = rec = GenerateRecurrence[mySum, n][[1]]**

Out[65]= $-4(1 + n)\text{SUM}[n] + 2(3 + 2n)\text{SUM}[1 + n] + (-2 - n)\text{SUM}[2 + n] == -1$

Second, we solve the recurrence in terms of all solutions that are expressible in terms of indefinite nested sums defined over hypergeometric products. In this case, we find `all` solutions. Namely,

2 linearly independent solutions of the homogeneous version (indicated by a 0 in the first entries of the lists) and one particular solution (indicated by a 1 in the first entry of the list):

In[66]:= **recSol = SolveRecurrence[rec[[1]], SUM[n]]**

Out[66]= $\{\{0, 2^n\}$

$\{0, 2^n \sum_{i_1=1}^{n} \frac{1}{i_1}\}$

$\{1, -2^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}\}\}$

Third, we combine all these solutions (by considering the first 2 initial values of the input sum) and obtain a closed-form representation:

In[67]:= **sol = FindLinearCombination[recSol, mySum, n, 2]**

Out[67]= $2^n \sum_{i_1=1}^{n} \frac{1}{i_1} - 2^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}$

In other words, we obtain the simplification

$$\sum_{k=0}^{n} \binom{n}{k} H_k = 2^n \sum_{i_1=1}^{n} \frac{1}{i_1} - 2^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}.$$

Here the definite sum

$$\sum_{k=0}^{n} \binom{n}{k} H_k \tag{61}$$

(the summand and the summation range depend on $n$) is simplified to an expression in terms of the two indefinite nested sums:

$$\sum_{i_1=1}^{n} \frac{1}{i_1}, \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}.$$

Raising the power of the harmonic numbers in (61), we can repeat this game to more and more complicated calculations. Consider, e.g., the sum

In[68]:= **mySum = SigmaSum[SigmaHNumber[k]$^3$SigmaBinomial[n, k], k, 0, n]**

Out[68]= $\sum_{k=0}^{n} \binom{n}{k} H_k^3$

Again we apply the creative telescoping summation paradigm in `Sigma`. This time we compute a linear recurrence of order 6:

In[69]:= **rec = rec = GenerateRecurrence[mySum, n][[1]]**

Out[69]= $-16(1 + n)(2 + n)(3 + n)(5 + n)^2(14 + 18n + 3n^2)\text{SUM}[n] +$

$8(2 + n)(3 + n)(4258 + 12918n + 9753n^2 + 3088n^3 + 444n^4 + 24n^5)\text{SUM}[1 + n]$

$- 4(3 + n)(30368 + 145056n + 147724n^2 + 66050n^3 + 15043n^4 + 1716n^5 + 78n^6)\text{SUM}[2 + n] +$

$4(65124 + 531256n + 668609n^2 + 373678n^3 + 112942n^4 + 19256n^5 + 1749n^6 + 66n^7)\text{SUM}[3 + n]$

$- (4 + n)(15448 + 315504n + 323794n^2 + 135202n^3 + 28273n^4 + 2952n^5 + 123n^6)\text{SUM}[4 + n] +$

$(4 + n)(5 + n)(-452 + 19704n + 15835n^2 + 4700n^3 + 615n^4 + 30n^5)\text{SUM}[5 + n] -$

$(4 + n)(5 + n)(6 + n)^3(-1 + 12n + 3n^2)\text{SUM}[6 + n] == 968 + 395n - 35n^2 - 30n^3 - 3n^4$

Next, we solve the recurrence in terms of all solutions that are expressible in terms of indefinite nested sums defined over hypergeometric products. Again we find `all` solutions: 6 linearly

independent solutions of the homogeneous version (indicated by a 0 in the first entries of the lists) and one particular solution (indicated by a 1 in the first entry of the list):

In[70]:= **recSol = SolveRecurrence[rec[[1]], SUM[n]]**

Out[70]= $\{\{0, 2^n\},$

$\{0, 2^n \sum_{i_1=1}^{n} \frac{1}{i_1}\},$

$\{0, 2^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}\},$

$\{0, (2 \sum_{i_1=1}^{n} \frac{1}{i_1^2} + 2(\sum_{i_1=1}^{n} \frac{1}{i_1})^2 - 2 \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{2^{i_2}}{i_2}}{i_1}) 2^n\},$

$\{0, (-6 \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1^2} - 6(\sum_{i_1=1}^{n} \frac{1}{i_1}) \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1} + 8 \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{1}{i_2}}{i_1}) 2^n\},$

$\{0, (\frac{10}{3} \sum_{i_1=1}^{n} \frac{1}{i_1^3} - 2 \sum_{i_1=1}^{n} \frac{2^{i_1} \sum_{i_2=1}^{i_1} \frac{2^{-i_2}}{i_2}}{i_1^2} - 6 \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{2^{i_2}}{i_2}}{i_1^2} + (6 \sum_{i_1=1}^{n} \frac{1}{i_1^2} - 6 \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{2^{i_2}}{i_2}}{i_1}) \sum_{i_1=1}^{n} \frac{1}{i_1}$

$+ \frac{14}{3} (\sum_{i_1=1}^{n} \frac{1}{i_1})^3 + 2(\sum_{i_1=1}^{n} \frac{2^{i_1}}{i_1^2}) \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1} - 8(\sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}) \sum_{i_1=1}^{n} \frac{2^{i_1} \sum_{i_2=1}^{i_1} \frac{1}{i_2}}{i_1}$

$- 8 \sum_{i_1=1}^{n} \frac{(\sum_{i_2=1}^{i_1} \frac{1}{i_2})^2}{i_1} + 8 \sum_{i_1=1}^{n} \frac{2^{i_1} (\sum_{i_2=1}^{i_1} \frac{1}{i_2}) \sum_{i_2=1}^{i_1} \frac{2^{-i_2}}{i_2}}{i_1} + 8 \sum_{i_1=1}^{n} \frac{2^{-i_1} (\sum_{i_2=1}^{i_1} \frac{1}{i_2}) \sum_{i_2=1}^{i_1} \frac{2^{i_2}}{i_2}}{i_1}) 2^n\},$

$\{1, (-3 \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1^3} + 8 \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{1}{i_2}}{i_1^2} + (-6 \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1^2} + 9 \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{1}{i_2}}{i_1}) \sum_{i_1=1}^{n} \frac{1}{i_1} + \frac{5}{2}(\sum_{i_1=1}^{n} \frac{1}{i_1^2}) \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}$

$- \frac{7}{2}(\sum_{i_1=1}^{n} \frac{1}{i_1})^2 \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1} + 3(\sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1})^2 \sum_{i_1=1}^{n} \frac{2^{i_1}}{i_1} - 6 \sum_{i_1=1}^{n} \frac{2^{-i_1} (\sum_{i_2=1}^{i_1} \frac{1}{i_2})^2}{i_1} - 3 \sum_{i_1=1}^{n} \frac{2^{i_1} (\sum_{i_2=1}^{i_1} \frac{2^{-i_2}}{i_2})^2}{i_1}$

$- 3(\sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}) \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{2^{i_2}}{i_2}}{i_1} + \sum_{i_1=1}^{n} \frac{(\sum_{i_2=1}^{i_1} \frac{1}{i_2}) \sum_{i_2=1}^{i_1} \frac{2^{-i_2}}{i_2}}{i_1}) 2^n\}\}$

Finally, we combine all these solutions (by considering the first 6 initial values of the input sum) and obtain the closed-form representation:

In[71]:= **sol = FindLinearCombination[recSol, mySum, n, 6]**

Out[71]= $\frac{5}{3} 2^n \sum_{i_1=1}^{n} \frac{1}{i_1^3} - 32^n \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1^3} + 2^{3+n} \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{1}{i_2}}{i_1^2} - 2^n \sum_{i_1=1}^{n} \frac{2^{i_1} \sum_{i_2=1}^{i_1} \frac{2^{-i_2}}{i_2}}{i_1^2} - 32^n \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{2^{i_2}}{i_2}}{i_1^2} + 32^n (\sum_{i_1=1}^{n} \frac{1}{i_1^2}) \sum_{i_1=1}^{n} \frac{1}{i_1} -$

$32^{1+n}(\sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1^2}) \sum_{i_1=1}^{n} \frac{1}{i_1} + \frac{7}{3} 2^n (\sum_{i_1=1}^{n} \frac{1}{i_1})^3 + 52^{-1+n}(\sum_{i_1=1}^{n} \frac{1}{i_1^2}) \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1} + 2^n (\sum_{i_1=1}^{n} \frac{2^{i_1}}{i_1^2}) \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1} - 72^{-1+n}(\sum_{i_1=1}^{n} \frac{1}{i_1})^2 \sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1} +$

$32^n(\sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1})^2 \sum_{i_1=1}^{n} \frac{2^{i_1}}{i_1} + 92^n(\sum_{i_1=1}^{n} \frac{1}{i_1}) \sum_{i_1=1}^{n} \frac{2^{-i_1} \sum_{i_2=1}^{i_1} \frac{1}{i_2}}{i_1} - 2^{2+n}(\sum_{i_1=1}^{n} \frac{2^{-i_1}}{i_1}) \sum_{i_1=1}^{n} \frac{2^{i_1} \sum_{i_2=1}^{i_1} \frac{1}{i_2}}{i_1} - 2^{2+n} \sum_{i_1=1}^{n} \frac{(\sum_{i_2=1}^{i_1} \frac{1}{i_2})^2}{i_1} -$

$32^{1+n} \sum_{i_1=1}^{n} \frac{2^{-i_1} (\sum_{i_2=1}^{i_1} \frac{1}{i_2})^2}{i_1} + 2^n \sum_{i_1=1}^{n} \frac{(\sum_{i_2=1}^{i_1} \frac{1}{i_2}) \sum_{i_2=1}^{i_1} \frac{2^{-i_2}}{i_2}}{i_1} + 2^{2+n} \sum_{i_1=1}^{n} \frac{2^{i_1} (\sum_{i_2=1}^{i_1} \frac{1}{i_2}) \sum_{i_2=1}^{i_1} \frac{2^{-i_2}}{i_2}}{i_1} - 32^n \sum_{i_1=1}^{n} \frac{2^{i_1} (\sum_{i_2=1}^{i_1} \frac{2^{-i_2}}{i_2})^2}{i_1} -$

$$32^n \Big(\sum_{i_1=1}^{n}\frac{1}{i_1}\Big)\sum_{i_1=1}^{n}\frac{2^{-i_1}\sum_{i_2=1}^{i_1}\frac{2^{i_2}}{i_2}}{i_1} - 32^n \Big(\sum_{i_1=1}^{n}\frac{2^{-i_1}}{i_1}\Big)\sum_{i_1=1}^{n}\frac{2^{-i_1}\sum_{i_2=1}^{i_1}\frac{2^{i_2}}{i_2}}{i_1} + 2^{2+n}\sum_{i_1=1}^{n}\frac{2^{-i_1}\Big(\sum_{i_2=1}^{i_1}\frac{1}{i_2}\Big)\sum_{i_2=1}^{i_1}\frac{2^{i_2}}{i_2}}{i_1}$$

We note that the arising sums

$$\sum_{i_1=1}^{n}\frac{1}{i_1^3}, \sum_{i_1=1}^{n}\frac{1}{i_1^2}, \sum_{i_1=1}^{n}\frac{1}{i_1}, \sum_{i_1=1}^{n}\frac{2^{-i_1}}{i_1^3}, \sum_{i_1=1}^{n}\frac{2^{-i_1}}{i_1^2}, \sum_{i_1=1}^{n}\frac{2^{-i_1}}{i_1}, \sum_{i_1=1}^{n}\frac{2^{i_1}}{i_1^2}, \sum_{i_1=1}^{n}\frac{2^{i_1}}{i_1}, \sum_{i_1=1}^{n}\frac{2^{-i_1}\sum_{i_2=1}^{i_1}\frac{1}{i_2}}{i_1^2},$$

$$\sum_{i_1=1}^{n}\frac{2^{-i_1}\sum_{i_2=1}^{i_1}\frac{1}{i_2}}{i_1}, \sum_{i_1=1}^{n}\frac{2^{i_1}\sum_{i_2=1}^{i_1}\frac{1}{i_2}}{i_1}, \sum_{i_1=1}^{n}\frac{\Big(\sum_{i_2=1}^{i_1}\frac{1}{i_2}\Big)^2}{i_1}, \sum_{i_1=1}^{n}\frac{2^{-i_1}\Big(\sum_{i_2=1}^{i_1}\frac{1}{i_2}\Big)^2}{i_1}, \sum_{i_1=1}^{n}\frac{2^{i_1}\sum_{i_2=1}^{i_1}\frac{2^{-i_2}}{i_2}}{i_1^2},$$

$$\sum_{i_1=1}^{n}\frac{\Big(\sum_{i_2=1}^{i_1}\frac{1}{i_2}\Big)\sum_{i_2=1}^{i_1}\frac{2^{-i_2}}{i_2}}{i_1}, \sum_{i_1=1}^{n}\frac{2^{i_1}\Big(\sum_{i_2=1}^{i_1}\frac{1}{i_2}\Big)\sum_{i_2=1}^{i_1}\frac{2^{-i_2}}{i_2}}{i_1}, \sum_{i_1=1}^{n}\frac{2^{i_1}\Big(\sum_{i_2=1}^{i_1}\frac{2^{-i_2}}{i_2}\Big)^2}{i_1}, \sum_{i_1=1}^{n}\frac{2^{-i_1}\sum_{i_2=1}^{i_1}\frac{2^{i_2}}{i_2}}{i_1^2},$$

$$\sum_{i_1=1}^{n}\frac{2^{-i_1}\sum_{i_2=1}^{i_1}\frac{2^{i_2}}{i_2}}{i_1}, \sum_{i_1=1}^{n}\frac{2^{-i_1}\Big(\sum_{i_2=1}^{i_1}\frac{1}{i_2}\Big)\sum_{i_2=1}^{i_1}\frac{2^{i_2}}{i_2}}{i_1}$$

are again indefinite nested while the input sum

$$\sum_{k=0}^{n}\binom{n}{k}H_k^3$$

is definite: the summand and the summation range depend on $n$.