cschneid@risc.jku.at

Exercises

Lecture for March 7, 2023

HW 1. Try to apply the Gauß-method to sum

1. $\sum_{k=0}^{n}(2k+1)$
2. $\sum_{k=1}^{n} k^2$
3. $\sum_{k=1}^{n} k^3$

Find and prove a formula for (a), (b) and (c).

HW 2. Prove for all $n \in \mathbb{N}$ that

$$\sum_{k=0}^{n-1} \frac{k}{(k+1)(k+2)} = H_n - \frac{2n}{n+1}.$$

HW 3. Let $f : \mathbb{Z} \to \mathbb{C}$ and $a, b \in \mathbb{Z}$ with $a \leq b$.

1. For
$$S(a,b) := \sum_{k=a}^{b}(f(k+1) - f(k))$$
show that
$$S(a,b) = f(b+1) - f(a).$$
2. Suppose in addition that $f(k) \neq 0$ for all k with $a \leq k \leq b$. For
$$P(a,b) := \prod_{k=a}^{b}(f(k+1) - f(k))$$
show that
$$P(a,b) = \frac{f(b+1)}{f(a)}.$$

HW 4. Use the previous homework to find a closed form for

$$a_n := \prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right).$$

BP 1. Consider the function $\exp : \mathbb{R} \to \mathbb{R}$ defined by

$$x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Prove: there is no rational function $r(x) \in \mathbb{R}(x)$ (i.e., $r(x) = \frac{p(x)}{q(x)}$ for polynomials $p, q \in \mathbb{R}[x]$) such that

$$\exp(x) = r(x) \quad \forall x \in U$$

where $U \subseteq \mathbb{R}$ is some non-empty open interval.

HW 5. Given a tower of n discs, initially stacked in decreasing size on one of three pegs. Transfer the entire tower to one of the other pegs, moving only one disc at each step and never moving a larger one onto a smaller one. Find a_n, the minimal number of moves ($n \geq 0$).

HW 6. How many slices of pizza can a person maximally obtain by making n straight cuts with a pizza knife. Let P_n ($n \geq 0$) be that number.

BP 2. Prove that there is no rational function $r(x) \in \mathbb{C}(x)$ such that

$$H_n = r(n)$$

holds for all $n \in \mathbb{N}$ with $n \geq \lambda$ for some $\lambda \in \mathbb{N}$.

Lecture from March 14, 2023

HW 7. Show that $H_n \sim \log(n)$.

Lecture from March 21, 2023

HW 8. Let $P(n)$ be defined by $P(1) = 1$ and

$$P(n) = \sum_{i=0}^{n-1} \frac{1}{n}\Big(1 + \frac{i}{n} P(i) + \frac{n-i-1}{n} P(n-i-1)\Big).$$

Show that $P(n) = 1 + \frac{2}{n^2} \sum_{i=0}^{n-1} i\, P(i)$.

HW 9. Let $P(n)$ be the sequence from the previous homework. Show for $n \geq 2$ that

$$n^2\, P(n) - (n-1)(n+1)P(n-1) = 2n - 1.$$

HW 10. Find a representation of $P(n)$ in terms of H_n.

HW 11. Show that

1. $P(n) \in O(\log(n))$;

2. $P(n) \sim 2\log(n)$.

Lecture from March 28, 2023

BP 3. Show that $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ as defined in the lecture is a vector space over $\mathbb{K}$.

BP 4. Show that $(\mathbb{K}^{\mathbb{N}}, +, \odot)$ with the Hadamard product $\odot$ is a commutative ring with 1, but not an integral domain.

BP 5. Show that $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ with the Cauchy product $\cdot$ is a commutative ring with 1.

HW 12. Show that $(\mathbb{K}^{\mathbb{N}}, +, \cdot)$ with the Cauchy product $\cdot$ has no zero-divisors.

HW 13. Show: For $\lambda \in \mathbb{K}$ and $m \in \mathbb{N}$ we have

$$(\lambda\, x^m) \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} \lambda a_n x^{n+m} = \sum_{n=m}^{\infty} \lambda a_{n-m} x^n;$$

here the multiplication on the left hand side is the standard Cauchy product.

HW 14. For $k \in \mathbb{N}$ and $a(x), b(x) \in \mathbb{K}[[x]]$ show

$$[x^k](a(x) + b(x)) = [x^k]a(x) + [x^k]b(x),$$
$$[x^k](\lambda\, a(x)) = \lambda [x^k] a(x).$$

HW 15. In $(\mathbb{K}[x], +, \cdot)$ prove

1. $\left(\sum_{n=0}^{\infty} c^n x^n\right)(1 - cx) = 1 \qquad (c \in \mathbb{K})$
2. $\left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n\right) = 1.$

Lecture from April 18, 2023

HW 16. Show for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}$ that

$$\binom{z+1}{k} = \binom{z}{k} + \binom{z}{k-1}.$$

HW 17. Consider $D_x : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$ with

$$D_x(\sum_{n=0}^{\infty} a_n x^k) = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n.$$

Show that $(\mathbb{K}[[x]], D_x)$ forms a differential ring, i.e., the following two rules hold: for all $a, b \in \mathbb{K}[[x]]$,

1. $D_x(a + b) = D_x(a) + D_x(b)$
2. $D_x(a \cdot b) = D_x(a)b + aD_x(b)$.

HW 18. Let $\exp(c\,x) := \sum_{n=0}^{\infty} \frac{c^n}{n!} x^n$. For $a, b \in \mathbb{K}$ show:

$$\exp(a\,x)\exp(b\,x) = \exp((a+b)x).$$

HW 19. Find a closed form for the coefficients in the multiplicative inverse of $(1-2x)^2 \in \mathbb{K}[[x]]$.

HW 20. Find a closed form for the coefficients in the multiplicative inverse of $(1-x)^3 \in \mathbb{K}[[x]]$.

HW 21. Find a closed form for the coefficients in the multiplicative inverse of $\exp(2x) \in \mathbb{K}[[x]]$.

Lecture from April 25, 2023

HW 22. Consider the formal power series $f(x) = \frac{1}{(1-x)^2}\log(1-x) \in \mathbb{Q}[[x]]$. Express the coefficients $f_n \in \mathbb{Q}$ of $f(x) = \sum_{k=0}^{\infty} f_n x^n$ in terms of the harmonic numbers H_n.

HW 23. Let $g(x) \in \mathbb{K}[[x]]$ with $g(0) = 1$. Show that there is an $f(x) \in \mathbb{K}[[x]]$ with $f(x)^2 = g(x)$ and $f(0) = 1$. (Hint: adapt the construction to invert a formal power series.) Further, conclude (during your construction of $f(x)$) that there is exactly one other solution, namely $-f(x)$.

HW 24. Show that

$$\frac{(-1)^n}{2}\binom{\frac{1}{2}}{n+1}4^{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

Lecture from May 2, 2023

HW 25. Simplify

1. $\sum_{k=0}^{n} k\, k!$;
2. $\sum_{k=0}^{n}(-1)^k\binom{m}{k}$;
3. $\sum_{k=0}^{n}(-1)^k\binom{m}{k}H_k$.

HW 26. Simplify

1. $\sum_{k=0}^{n} H_k^2$;
2. $\sum_{k=0}^{n}(H_{m+k})^2$;
3. $\sum_{k=0}^{n} H_k^3$.

HW 27. Prove correctness for the found simplification of $\sum_{k=0}^{n} H_k^2$ (part (a) in HW 26).

HW 28. Given the sequence $a(n)$ defined by

$$-2(2n+1)a(n) + (n+2)a(n+1) = 0$$

and $a(0) = 1$. Show that $a(n) = \frac{1}{n+1}\binom{2n}{n}$ holds.

HW 29. Consider the quicksort recurrence

$$(n+1)F_{n+1} - (n+2)F_n = 2n, \quad n \geq 0$$

and transform it to a homogeneous recurrence (of higher order).
Hint use the trick from the lecture (shift and subtract) twice.

HW 30. Compute a differential equation for the generating function $Q(x) = \sum_{n=0}^{\infty} F_n x^n$ where F_n are the average comparisons to quicksort an array with n elements.
Hint: use, e.g., the homogeneous recurrence from HW 29.

HW 31. Compute a differential equation for the generating function $H(x) = \sum_{n=0}^{\infty} H_n x^n$ (e.g., with RE2DE) and solve it (e.g., with DSolve). Compare your result with $H(x) = -\frac{1}{1-x}\log(1-x)$ from the lecture notes.

HW 32. For the function $f(x) = \frac{1+2x}{1-2x}$ there exists a complex series expansion. Find it.

HW 33. For the function $f(x) = \left(\frac{1+x}{1-x}\right)^2$ there exists a complex series expansion. Find it.

HW 34. For the function $f(x) = \sqrt{\frac{1+x}{1-x}}$ there exists a complex series expansion. Find it.

HW 35. For the function $f(x) = \log(\frac{1+x}{1-x})$ there exists a complex series expansion. Find it.

BP 6. For the above functions $f(x)$ and complex series expansions find (the maximal) $r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} f_n\, x^n \quad |x| < r.$$

Lecture from May 9, 2023

HW 36. Verify that the real function $A :]-1, 1[\to \mathbb{R}$ with $x \mapsto \frac{e^{-x}}{1-x}$ satisfies

$$A'(x) = \frac{x}{1-x} A(x), \quad A(0) = 1.$$

Note: By the same rules it follows that A (as complex function with inputs inside of the unit circle) satisfies this differential equation.

BP 7. Prove the identity

$$(n+1)\sum_{k=0}^{n+1} \frac{(-1)^k}{k!} = \sum_{k=0}^{n-1}\sum_{i=0}^{k} \frac{(-1)^i}{i!}$$

without analysis arguments (e.g., with symbolic summation).

HW 37. Show that the ring of formal Laurent series $(\mathbb{K}((x)), +, \cdot)$ is a field.

HW 38. Let $f(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{K}[[x]]$ and define its truncated version $F_n(x) = f_0 + f_1\, x + \dots + f_n\, x^n \in \mathbb{K}[[x]]$. Show that

$$f(x) = \lim_{n\to\infty} F_n(x).$$

BP 8. Suppose that $(a_k(x))_{k\geq 0}$ and $(b_k(x))_{k\geq 0}$ from $\mathbb{K}[[x]]$ are convergent. Show that $(a_k(x) + b_k(x))_{k\geq 0}$ is convergent.

Lecture from May 16, 2023

HW 39. Let $b_n \in \mathbb{K}[[x]]$ for all $n \in \mathbb{N}$ and define $a_N(x) := \sum_{n=0}^{N} b_n(x) \in \mathbb{K}[[x]]$ for $N \in \mathbb{N}$. Suppose that $(a_N(x))_{N\geq 0}$ converges to $b(x) \in \mathbb{K}[[x]]$, i.e.,

$$b(x) = \sum_{n=0}^{\infty} b_n(x) (= \lim_{N\to\infty} \sum_{n=0}^{N} b_n(x)).$$

For $k \in \mathbb{N}$ show that

$$[x^k]b(x) = \sum_{n=0}^{\infty} [x^k]b_n(x) (= \lim_{N\to\infty} \sum_{n=0}^{N} [x^k]b_n(x)).$$

HW 40. With the assumptions from the previous homework show that

$$D_x b(x) = \sum_{n=0}^{\infty} D_x b_n(x) (= \lim_{N\to\infty} \sum_{n=0}^{N} D_x b_n(x)).$$

HW 41. Show that for the sequence $(b_n(x))_{n\geq 0}$ with $b_n(x) = \frac{(1+x)^n}{n!} \in \mathbb{K}[[x]]$ the limit $\sum_{n=0}^{\infty} b_n(x) (= \lim_{N\to\infty} \sum_{n=0}^{N} b_n(x))$ does not exist.

HW 42. Let $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{1-x} - 1$. Calculate the first 20 coefficients of $f(g(x))$.

HW 43. Let $T(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$ where $\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-1)^n 4^n x^n \in \mathbb{K}[[x]]$. Find $S(x) \in \mathbb{K}[[x]]$ such that $S(T(x)) = T(S(x)) = x$.

HW 44. Let $(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$. Show: $(a_n)_{n\geq 0}$ satisfies the c-finite recurrence

$$a_{n+r} + c_{r-1} a_{n+r-1} + \cdots + c_0\, a_n = 0 \quad \forall n \in \mathbb{N}$$

with $c_i \in \mathbb{K}$ and $c_0 \neq 0$ if and only if

$$\sum_{n=0}^{\infty} a_n\, x^n = \frac{p(x)}{1 + c_{r-1}x + \cdots + c_0 x^r}$$

for some $p(x) \in \mathbb{K}[x]$ with $\deg(p(x)) \leq r-1$.

Lecture from May 23, 2023

HW 45. Let r_+ and $r_- \in \mathbb{C}$ be the roots of $q(x) = x^2 - x - 1 \in \mathbb{C}[x]$. Check that

$$A\,(r_+)^n + B\,(r_-)^n$$

with $A, B \in \mathbb{C}$ are solutions of the c-finite recurrence $a_{n+2} - a_{n+1} - a_n = 0$. In particular, show that $((r_+)^n)_{n\geq 0}$ and $((r_-)^n)_{n\geq 0}$ are linearly independent over $\mathbb{C}$.

HW 46. Define

$$V = \{(a_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}} \mid a_{n+2} - a_{n+1} - a_n = 0 \quad \forall n \in \mathbb{N}\}.$$

Show that $V = \{A\,((r_+)^n)_{n\geq 0} + B\,((r_-)^n)_{n\geq 0} \mid A, B \in \mathbb{C}\}$.

BP 9. Prove Theorem 5.4 for the special case $r = 2$.

BP 10. Prove Theorem 5.4 for the special case $r = 3$.

Lecture from June 6, 2023

HW 47. Let $M(h)$ be the minimal number of nodes in an AVL tree with height h. From

$$M(h) = -1 + \frac{5-2\sqrt{5}}{5}(r_-)^h + \frac{5+2\sqrt{5}}{5}(r_+)^h$$

with $r_+ = \frac{1}{2}(1+\sqrt{5})$ and $r_- = \frac{1}{2}(1-\sqrt{5})$ conclude that $h \le 1.44\,\mathrm{ld}(n) + c$.

HW 48. Use GeneratingFunctions.m (or another computer algebra package) to derive a c-finite recurrence for $(a_n)_{n\ge 0}$ with $a_n = F_{2n} - 2F_nF_{n+1} + F_n^2$.

HW 49. Prove $F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$ for $n \ge 1$.

HW 50. Prove $\sum_{n=0}^{\infty} F_n^3 x^n = \frac{x(1-2x-x^2)}{(1-4x-x^2)(1+x-x^2)}$.

HW 51. Use GeneratingFunctions.m (or another computer algebra package) to derive a c-finite recurrence for Kepler's identity:

$$F_{n+1}F_{n-1} - F_n = (-1)^n.$$

Verify the correctness of the identity.

HW 52. Find/prove $\sum_{k=0}^{n} F_k = F_{n+2} - 1$ for $n \in \mathbb{N}$.

Lecture from June 13, 2023

BP 11. Prove Kepler's identity (often also called Cassini's identity; compare the lecture notes)

$$F_{n+1}F_{n-1} - F_n = (-1)^n$$

directly from the Fibonacci recurrence.

HW 53. Implement an algorithm in your favorite computer algebra system (or programming language) that computes F_n in $O(\log(n))$ time.
[Hint: use, e.g., $\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = A^n$ with $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.]

Lecture from June 20, 2023

HW 54. Show that e^x is not algebraic.

HW 55. Compute a holonomic recurrence for

$$a_n = [x^n]e^x \sum_{n=0}^{\infty} H_n x^n.$$

HW 56. Compute a holonomic recurrence for

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}.$$

HW 57. Compute a holonomic differential equation for

$$a(x) = \sin(x) \sum_{n=0}^{\infty} (\sum_{k=0}^{n} k!) x^n.$$

HW 58. Show that $y(x) = \frac{1}{\cos(x)}$ is not holonomic.
[Hint: one may use that fact that $\tan(x)$ is not algebraic.]