Varieties

Task 1 Find a general criteria for tropical lines, given by $P = a\,y + b\,z + c \in K[y, z]$ with $a, b, c \in K$, when they intersect transversally. What do you obtain for the cases $K = \mathbb{C}, K = \mathbb{C}\langle\langle x\rangle\rangle$ and $K = \mathbb{Q}_p$?

Task 2 Compute $\mathbb{V}(\mathrm{trop}(P)) = \mathrm{trop}(\mathbb{V}(P))$ for

$$P = x^2y^2 + yz + (x^2 + x^3)z^2 + (1 + x^3)y + x^{-1}z + x^3 \in \mathbb{C}\langle\langle x\rangle\rangle[y, z].$$

Task 3 Let $I = \langle P_1 = y + z + 1,\ P_2 = y + 2z\rangle \subset \mathbb{C}\langle\langle x\rangle\rangle[y, z]$. Compute

a) $\mathrm{trop}(\mathbb{V}(I))$;

b) $\mathrm{trop}(\mathbb{V}(P_1)) \cap \mathrm{trop}(\mathbb{V}(P_2))$;

c) A tropical basis of I.

Task 4 Verify the Fundamental Theorem of Tropical Geometry for the following polynomials.

a) $P = y^3 + z^3 - 1 \in \mathbb{C}\langle\langle x\rangle\rangle[y, z]$;

b) $P = 3y + x^2z + 2x \in \mathbb{C}\langle\langle x\rangle\rangle[y, z]$.

Task 5 Show that the assumption of non-trivial valuation in the Fundamental Theorem can not be dropped by considering $P = y^3 + z^3 - 1 \in \mathbb{C}[y, z]$.

For simplification of the description of tropical varieties we define *polyhedrons* (in $\mathbb{R}^n$) as the intersection of finitely many half-spaces written as

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

for given $d \times n$ matrix A and $b \in \mathbb{R}^d$, a *cone* as a polyhedron with $b = 0$ and a *face* of a polyhedron P as

$$face_w(P) = \{x \in P : w \cdot x \leq w \cdot y \text{ for all } y \in P\}.$$

Task 6 Find a representation of the unit square in $\mathbb{R}^2$ and its faces in aboves notation.