Valuations

Let $a \in \mathbb{Z}$ and p be a prime number. Then the *p-adic order* of a is defined as

$$\nu_p(a) = \begin{cases} \max_{k\in\mathbb{N}}(p^k \mid a), & a \neq 0 \\ \infty, & a = 0 \end{cases}$$

For $a = \frac{p}{q} \in \mathbb{Q}$ this map can be extended by setting $\nu_p(a) = \nu_p(p) - \nu_p(q)$.

Task 1 Show that the p-adic order is a valuation on $\mathbb{Q}$.

Given a valuation val, the *tropicalization* of a polynomial is defined by exchanging every multiplication and addition with its tropical counterparts and by replacing the coefficients with its valuations, i.e.:

$$P = \sum_{I\in\mathbb{N}^n} a_I x^{i_1} \cdots x^{i_n} \quad \longleftrightarrow \quad \operatorname{trop}(P) = \bigoplus_{I\in\mathbb{N}^n} \operatorname{val}(a_I) \odot x^{i_1} \odot \cdots \odot x^{i_n}.$$

Task 2 Compute the graph of the function defined by the tropicalization of the polynomial

$$P(x,y) = 6x^2 + 5xy + 10y^2 + 3x - y + 4$$

by using

a) the trivial valuation on $\mathbb{Q}$;

b) the 2-adic order as valuation;

c) the usual valuation on $\mathbb{Q}\langle\langle x\rangle\rangle$.

Task 3 Compute the first terms of the roots of the polynomial $P(y)$ from task 2 in $\mathbb{C}\langle\langle x\rangle\rangle$ by using Newton polygons.
Check the solution by solving $P(y) = 0$ symbolically with the formular for quadratic equations and expanding the solution around $x = 0$.

Let us introduce the field of *generalized power series* which includes, for a universal index set I, formal series of the form $\varphi(x) = \sum_{i\in I} c_i\, x^i$, where $\operatorname{supp}(\varphi) = \{i \in I \mid c_i \neq 0\}$ is a well-ordered set. Here the usual addition and multiplication, for $\varphi(x) = \sum_{i\in I} c_i\, x^i, \psi(x) = \sum_{i\in I} d_i\, x^i$ we set

$$\varphi(x) + \psi(x) = \sum_{i\in I} (c_i + d_i)\, x^i, \;\; \varphi(x)\cdot\psi(x) = \sum_{k\in I}\left(\sum_{i+j=k} c_i d_j\right) x^k,$$

are used.

Task 4 Show that the generalized power series indeed define a field.

Task 5 a) Find generalized power series that are not Puiseux series.

b) Is the series $\sum_{i\in\mathbb{Z}} x^i$ a generalized power series? Do you obtain something special about this series?

The field of (formal) Puiseux series $K\langle\langle x\rangle\rangle$ might not be algebraically closed in the case when K has positive characteristics as the following example shows.

Task 6 Let $P(y) = y^2 - y - x^{-1}$.

a) Compute the roots of $P = 0$ in $\mathbb{C}\langle\langle x\rangle\rangle$;

b) Show that the roots in any coefficient field K of characteristic 2 are

$$x^{-1/2} + x^{-1/4} + x^{-1/8} + \cdots\,,\;\; 1 + x^{-1/2} + x^{-1/4} + x^{-1/8} + \cdots\,.$$

Conclude that $K\langle\langle x\rangle\rangle$ is not algebraically closed.