Questions on Tropcial Semirings

Let us consider the tropical semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$.

Task 1 Prove *Freshman's Dream* in $\mathbb{T}$: For every $n \in \mathbb{N}$ it holds that $(x \oplus y)^n = x^n \oplus y^n$.

Task 2 Compute the solutions of $x \oplus a = 0$ for given $a \in \mathbb{R}$.

Let us define matrices with entries in $\mathbb{R} \cup \{\infty\}$ endowed with the component-wise addition

$$A = (a_{i,j}), B = (b_{i,j}) \in Mat(n,m) : \ A \oplus B = (a_{i,j} \oplus b_{i,j}).$$

and the matrix multiplication

$$A = (a_{i,j}) \in Mat(n,m), B = (b_{i,j}) \in Mat(m,\ell) : \ A \odot B = \left(\bigoplus_{k=1}^{m} a_{i,k} \odot b_{k,j} \right).$$

Task 3 Which algebraic structure does $(Mat(n,n), \oplus, \odot)$ define?

Task 4 Can you say something about the image of $\{A \odot x \mid x \in \mathbb{R}^m\}$ for given $A \in Mat(n,m)$?