Tropical Linear Algebra

Task 1 Let us consider the following weighted graph $\mathfrak{G}$ given by its adjacency matrix

$$G = \begin{pmatrix} 0 & 1 & 3 & 7 \\ 2 & 0 & 1 & 3 \\ 4 & 5 & 0 & 1 \\ 6 & 3 & 1 & 0 \end{pmatrix}.$$

a) Draw the graph $\mathfrak{G}$.

b) Compute the shortest path from any node i to j and compare the result for some cases with a graphical solution.

Task 2 Let

$$G = \begin{pmatrix} 4 & 4 & 5 \\ 1 & 3 & 2 \\ 1 & 3 & 4 \end{pmatrix}.$$

a) Determine the eigenvalue and eigenspace of G.

b) Compute the tropical determinant of G.

c) Compute the image of the tropical linear map defined by G.

There are several notions for the rank of a matrix A in the tropical setting, which are equivalent in classical linear algebra over a field:

- A matrix has rank 1 iff it is the product of a column vector and a row vector. A has rank r if it can be written as the sum of r many rank 1 matrices.
- The dimension of the column space.
- The rank of A is the largest integer r such that A has a nonsingular $r \times r$ submatrix.

The tropical counterparts for a tropical $(d \times n)$-matrix G are the following.

- *Barvinok rank:* A matrix has rank 1 iff it is the (tropical) product of a column vector and a row vector. G has rank r if it can be written as the (tropical) sum of r many rank 1 matrices.
- *Kapranov rank:* the smallest integer r such that there exists a valued field K and a linear subspace of K^d of dimension r whose tropicalization contains the columns in G.
- *Tropical rank:* the largest integer r such that G has a nonsingular $(r \times r)$-submatrix, where a matrix is singular iff the minimum in the tropical determinant is attained at least twice.

Task 3 Compute the Kapranov and tropcial rank of

$$G = \begin{pmatrix} 0 & 0 & 4 & 4 & 4 & 4 \\ 0 & 0 & 2 & 4 & 1 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \\ 2 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 2 & 4 & 1 & 4 & 0 & 0 \end{pmatrix}.$$

Task 4 Find the maximal Barvinok rank of any (5×5)-matrix with entries equals 0 or 1.