Initials and Gröbner Bases

Task 1 Let $K = \mathbb{Q}_p$ be the field of p-adic numbers for some prime number p. Compute the valuation ring $\mathcal{R}$ of K, the maximal ideal $\mathfrak{m}$ and the residue field $\mathcal{R}/\mathfrak{m}$.

Task 2 Show that if K is an algebraically closed valued field, then the residue field k is algebraically closed.

Task 3 Let $f = 2x^2 + xy + 6y^2 + 5x - 3y + 4 \in \mathbb{Q}_2[x, y]$ and $w_1 = (2, 2), w_2 = (-2, -1)$.

a) Compute $\mathrm{in}_{w_1}(f), \mathrm{in}_{w_2}(f)$.

b) Let $I = \langle f \rangle$. Compute $\mathrm{in}_{w_1}(I)$ and $\mathrm{in}_{(0,0)}(I)$.

Task 4 Compute tropical Gröbner bases of the following ideals I with respect to w.

a) $I = \langle x + x^2 \rangle \subset \mathbb{C}[x], w = (1)$;

b) $I = \langle x_1 + 2x_2 - 3x_3, 3x_2 - 4x_3 + 5x_4 \rangle \subset \mathbb{Q}_2[x_1, \ldots, x_4]$ and $w_1 = (0, 0, 0, 0), w_2 = (1, 0, 0, 1)$.

Task 5 a) Compute the tropical variety of $\mathbb{V}(I)$ where $I = \langle f \rangle$ from Task 3 by using first $\mathrm{trop}(f)$ and then by using $\mathrm{in}_w(I)$.

b) Label $\mathrm{trop}(\mathbb{V}(I))$ by the corresponding initial ideals. The result is called the *Gröbner complex* of I.