Exercise sheet 5

meeting on **02/06/2020**

Exercise 25 [Theorem 5.2.1] Let $f, g \in R[x_1, \dots, x_n]$, F, G forms in $R[x_1, \dots, x_n, x_{n+1}]$. Then show the following:

a) $(F \cdot G)_* = F_* \cdot G_*$ and $(f \cdot g)^* = f^* \cdot g^*$.

b) If r is the highest power such that $x_{n+1}^r \mid F$, then $x_{n+1}^r \cdot (F_*)^* = F$, whereas $(f^*)_* = f$.

c) $(F+G)_* = F_* + G_*$ and $x_{n+1}^t (f+g)^* = x_{n+1}^{\max(0,r-s)} f^* + x_{n+1}^{\max(0,s-r)} g^*$, where $r = \deg(g)$, $s = \deg(f)$, $t = \max(r,s) - \deg(f+g)$.

Exercise 26 a) Show that every homogenous polynomial $f \in \mathbb{C}[x,y]$ of degree d can be factored into linear homogenous polynomials, i.e.

$$f(x,y) = \prod_{i=1}^{d} (a_i x + b_i y).$$

b) Let $g(x) = x^3 - 2x^2 - x + 2$. Compute the homogenization g^* and a decomposition of $V_p(g)$. Visualize $V_a(g)$, $V_p(g)$ and the cone $C(V_p(g))$.

Exercise 27 a) Let $V = V(x^2+1) \subset \mathbb{R}[x]$. Compute $\Gamma(V)$ and $K(V)$ defined in the way as for algebraically closed fields.

b) Show that $\mathbb{R}[x]/\langle x^2 - 4x + 3 \rangle$ is not an integral domain. How does this correspond with the reducibility of $V(x^2 - 4x + 3)$?

c) Let $V = V(y^5 - x^2) \subset \mathbb{C}^2$. Show that every element in $\Gamma(V)$ can be uniquely written as $a(y) + x\,b(y)$ with $a, b \in \mathbb{C}[y]$. Conclude that V and $\mathbb{C}$ cannot be isomorphic as a variety by showing that there is no ring isomorphism between $\Gamma(V)$ and $\mathbb{C}[t]$.

Exercise 28 Compute and classify all (including projective) singularities of the curves defined by the following polynomials.

a) $f_1 = x^4 + y^4 - x^2 y^2$.

b) $f_2 = y^2 + (x^2 - 5)(4x^4 - 20x^2 + 25)$.

c) $f_3 = x^4 + x^2(2 + y^2) - 2x^3 y - 6xy + 1$.

Plot the corresponding real components.

Exercise 29 a) Compute the intersection points and its multiplicity of the curves given by the polynomials $f = (x^2+y^2)^3 - 4x^2y^2$ and $g = x^2 + y^2 - 1$. Verify your result by Bézout's Theorem.

b) Find a polynomial defining an irreducible cubic curve with one singularity, namely a non-ordinary double point at the origin with tangent $x + y$, and $(1:1:0)$ and $(0:1:0)$ as curve points at infinity.

Exercise 30 Let $\mathcal{C}$ be the irreducible projective curve defined by a form $F \in K[x,y,z]$ such that $\mathcal{C}$ is not a line and let ψ be the ring homomorphism from $K[x,y,z]$ to $\Gamma(\mathcal{C})$ such that

$$\psi(x) = \left[\frac{\partial F}{\partial x}\right], \ \psi(y) = \left[\frac{\partial F}{\partial y}\right], \ \psi(z) = \left[\frac{\partial F}{\partial z}\right].$$

a) Show that $\ker(\psi)$ is a homogeneous prime ideal. This implies that $V(\ker(\psi))$, the so-called *dual curve* of $\mathcal{C}$, is irreducible.

b) Prove that $V(\ker(\psi))$ is the algebraic closure of

$$\left\{ \left(\frac{\partial F}{\partial x}(P) : \frac{\partial F}{\partial y}(P) : \frac{\partial F}{\partial z}(P) \right) \mid P \in \mathcal{C} \textit{ is simple} \right\}.$$

c) Form the previous item it can be deduced that the dual curve is defined by the generator of

$$K[u,v,w] \cap \left\langle F,\ u - \frac{\partial F}{\partial x},\ v - \frac{\partial F}{\partial y},\ w - \frac{\partial F}{\partial z} \right\rangle. \tag{1}$$

Compute the dual curves of $F_1 = 4x^2 + 9y^2 - z^2$ and $F_2 = x^3 + y^3 - z^3$. Plot the curves and its corresponding duals.

Optionally you might explain why the generator of (1) indeed defines the dual curve.