Exercise sheet 4

meeting on **12/05/2020**

Exercise 19 a) Let $f, g \in K[x_1, \ldots, x_n]$ and $g = g_1 g_2$ for some non-trivial factors g_1, g_2. Show that $V(f, g) = V(f, g_1) \cup V(f, g_2)$.

b) Show that $V(y - x^2, xz - y^2) = V(y - x^2, xz - x^4) =: V$. Decompose V into irreducible components and visualize it.

Exercise 20 In the proof of Theorem 4.2.6 the following is needed:
Let $P_1, \ldots, P_r \in \mathbb{A}^n(K)$ be pairwise different. Construct polynomials $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$ such that

$$f_j(P_i) = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases} .$$

Exercise 21 Let I be an ideal and J be a primary ideal of R, where R is a commutative ring with 1.

a) Let I be primary and $\sqrt{I} = \sqrt{J}$. Show that then $I \cap J$ is primary.

b) Show that if I is prime and $J \subseteq I$, then $\sqrt{J} \subseteq I$.

Exercise 22 Let I be a primary ideal and $f \in k[x_1, \ldots, x_n]$. Show the following:

a) If $f \in I$, then $I : \langle f \rangle = \langle 1 \rangle$;

b) If $f \notin I$, then $I : \langle f \rangle$ is primary with $\sqrt{I : \langle f \rangle} = \sqrt{I}$;

c) If $f \notin \sqrt{I}$, then $I : \langle f \rangle = I$.

Exercise 23 Let $f = x^2 y - z^2$ and $I = \langle xz - y^2, x^3 - yz \rangle$.

a) Compute $I : \langle f \rangle$ and show that it is prime.

b) Show that I has the decomposition $\langle x, y \rangle \cap \langle xz - y^2, x^3 - yz, x^2 y - z^2 \rangle$.

c) Show that also $\langle xz - y^2, x^3 - yz, x^2 y - z^2 \rangle$ is prime by computing a (polynomial) parametrization of the corresponding variety.

Based on the previous two examples it is worth mentioning that for every radical ideal $I \subset k[x_1, \ldots, x_n]$ with prime decomposition

$$I = \bigcap_{i=1}^{r} P_i,$$

the P_i's can be found in the set $\{I : \langle f \rangle \mid f \in k[x_1, \ldots, x_n]\}$. You may think about what this means for the algebraic set $V(I)$ and its decomposition.

Exercise 24 The relations $y - x^2 = 0,\ z - x^3 = 0$ define a space curve W in $\mathbb{R}^3$.

a) Consider the corresponding projective variety $V = V(wy - x^2,\ w^2 z - x^3)$ and varify $W = V \cap V(w - 1)$. Which elements of V are missed by the affine variety W in projective space? Use this to find two proper subvarieties of V.

b) Show that $W = \tilde{V} \cap V(w - 1)$, where $\tilde{V} = V(wy - x^2,\ w^2 z - x^3,\ x\,z - y^2)$. Decompose $\tilde{V}$ into two proper subvarieties.

You may give an explanation what this mean for the computation of the projective closure of W.